Chapter 2

Quantum field theory in curved spacetime

2.1 Introduction

Einstein's General Theory of Relativity is revolutionary in the sense that a new concept on spacetime structure and gravitation has been put forth. But there is a major drawback for the theory as it is not based on the principles of quantum theory. There are two reasons which compell us to look for a quantum theory of gravity. Advances made in grand unified theories make us to believe that it may be possible to unify all the four forces of basic interactions. Interestingly, the natural length scale which arises in grand unified theory is only a few orders of magnitude $< L_p$. Thus it is possible that a quantum theory of gravity may even play an important role in the unification of the strong and electromagnetic interactions. A unified theory of all forces might predict many new phenomena, and observations of them would justify the unification.
scheme [29]. The second reason arises directly from general theory of relativity. As mentioned earlier spacetime singularities occur in the solutions of classical general theory of relativity relevant to gravitational collapse and cosmology. In these singularities the classical description of spacetime structure breaks down. Thus it appears that the development of a quantum theory of gravitation will be an essential requirement for our understanding of the initial state of the Universe [10, 29]. The formulation of quantum gravity will be a great achievement as far as theoretical physics is concerned. All the attempts so far made to have a quantum theory of gravitation ran into difficulties. The lack of a satisfactory quantum theory of gravity does not mean that one can not perform any reliable calculations of quantum effects occurring in strong gravitational fields. A complete satisfactory theory exists for a free quantum matter field propagating in a fixed background spacetime. In this approach, the spacetime metric is treated classically and is coupled to the matter field which is treated quantum mechanically. Such a programme is known as semiclassical approximation [33]. Though this method is only an approximation to a full quantum theory of gravity this procedure can at least give a good indication of the types of quantum effects which might have occurred in strong gravitational fields. This programme is being used to study the effect of quantum gravity on other phenomena like creation of particles, black hole evaporation, etc.

The phenomena of particle creation can be understood by studying the matter field in a background metric. Therefore the essential ideas of formulation of filed theory in curved spacetime and particle creation mechanism are briefly discussed below.
2.2 Quantum fields in curved spacetime and particle creation

Advances made in Grand Unified Theories make us to believe that it may be possible to correlate observational data with quantum process in the early Universe [34, 35, 36, 37]. This has caused increasing interests in the study of quantum theory in curved spacetime.

A great deal of the formalism of quantum field theory in Minkowski spacetime can be extended to curved spacetime with little modifications. In flat spacetime, Lorentz invariance plays an important role in each of the basic ingredients in the construction of a quantum field theory. In flat spacetime the Lorentz invariance allows us to identify a unique vacuum state for the theory. However in curved spacetime, we do not have Lorentz symmetry. The formulation of a classical field theory and its formal quantization may be carried through in an arbitrary spacetime. The real difference between flat space and curved space arises in the characterisation of the quantum states and the physical interpretations of the states [38].

In view of current quantum concepts, the physical vacuum (i.e. the state without real particles) is a quite complex entity. According to the formulation of quantum field theory virtual (short-lived) particles are constantly created, they interact with one another, and are annihilated in the vacuum. The vacuum is stable and real particles (long-lived) are not produced. But we can see that in the presence of external fields virtual particles may acquire sufficient energy for becoming real. The result is that
quantum creation of particles from vacuum is possible in the presence of an external field [33, 38].

In general, there does not exit a unique vacuum state in a curved spacetime. As a result, the concept of particles becomes ambiguous, and the physical interpretations of particles become much more difficult. This issue can be realised by considering the formulation of quantum field theory in Minkowski spacetime and in curved spacetime.

First we discuss quantum field theory in Minkowskai spacetime.

Consider a real scalar field in Minkowski spacetime satisfying the equation of motion:

\[
\left(\eta^{\mu\nu} \nabla_\mu \nabla_\nu - m^2\right) \phi(x) = 0,
\]

(2.1)

Let \( \{u_k\} \) be a set of solutions of this equation, which are positive frequency modes with respect to some time-like Killing vector \( \zeta_1 \), that is

\[
L_{\zeta_1} u_k = -i\omega_1 u_k
\]

(2.2)

where \( \omega > 0 \) and \( L \) denotes the Lie derivatives. Assuming that the \( u_k \)'s are complete and orthonormal, we have:

\[
(u_i, u_k) = \delta_{ik} = - (u^*_i, u^*_k)
\]

(2.3)

\[
(u_i, u^*_i) = 0
\]

where

\[
(\phi_1, \phi_2) = i \int_t \phi_1^* \partial_t \phi_2 d^{n-1}x
\]

(2.4)
and $t$ denotes a spacelike hyperplane of simultaneity at instant $t$. Now let us choose a solution in the following form

$$u_k = \left[ (2\pi)^{n-1} 2\omega \right]^{\frac{1}{2}} e^{i k x - \omega t}$$  \hspace{1cm} (2.5)$$

where

$$\omega = \left( k^2 + m^2 \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.6)$$

Now the scalar field may be expanded in terms of these modes,

$$\phi = \sum_k \left( a_k u_k + a_k^\dagger u_k^\dagger \right)$$  \hspace{1cm} (2.7)$$

and the quantization of the theory is implemented by imposing canonical commutation relations:

$$[a_i, a_k^\dagger] = \delta_{ik}, \quad [a_i, a_k] = [a_i^\dagger, a_k^\dagger] = 0$$  \hspace{1cm} (2.8)$$

The vacuum state $| 0_i \rangle$ is defined as

$$a_k | 0_i \rangle = 0.$$  \hspace{1cm} (2.9)$$

In Minkowski spacetime there is a natural set of modes, namely, as given by (2.5), that are closely associated with the rectangular coordinate system $(t,x,y,z)$. In turn, these coordinates are associated with the Poincare group, the action of which leaves the Minkowski line element invariant. Thus vacuum is invariant under the action of Poincare group. Therefore the solutions contain only positive frequencies with respect to the Minkowski time coordinate. But the situation is quite different in curved spacetime.
Now consider a real scalar field in a curved manifold without horizons [38]. The field must satisfy the generally covariant Klein-Gordon equation:

\[(g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2) \phi(x) = 0,\] (2.10)

Let \(\{u_k\}\) be a set of solutions of this equation, which are positive frequency modes with respect to some time-like Killing vector \(\zeta_1\), that is

\[L_{\zeta_1} u_k = -i \omega_1 u_k\] (2.11)

where \(\omega > 0\) and \(L\) denotes the Lie derivatives. Assume that the \(u_k\)'s are complete and orthonormal then we have, as before the Klein-Gordon scalar product (generalized to curved space),

\[(u_i, u_k) = \delta_{ik} = -(u_i^*, u_k^*)\] (2.12)

\[(u_i, u_k^*) = 0\]

where

\[(\phi_1, \phi_2) = i \int_\Sigma \phi_1^* \partial_\mu \phi_2 \sqrt{-g} d\Sigma^\mu\] (2.13)

and \(\Sigma\) is a three-dimensional space like hypersurface. The scalar field may be expanded then in terms of these modes,

\[\phi = \sum_k (a_k u_k + a_k^* u_k^*)\] (2.14)

and the quantization of the theory is implemented as usual by imposing canonical commutation relations (2.8).
Defining a vacuum state \( |0_I\rangle \) such that
\[
a_k |0_I\rangle = 0,
\]
(2.15)
then we can construct the Fock space by the action of the creation operators \( a_k^\dagger \).

Suppose now to have a different family of solutions of the covariant wave equation (2.10), \( \{v_k\} \), with only positive frequencies with respect to another Killing vector \( L_{\zeta_i} \),
\[
L_{\zeta_i} v_k = i\omega_2 v_k
\]
(2.16)
(\( \omega > 0 \)), and \( v_k \) form a complete orthonormal set:
\[
(v_i, v_k) = \delta_{ik} = -(v_i^*, v_k^*),
\]
(2.17)
\[
(v_i, v_k^*) = 0.
\]
(2.18)
Then \( \phi \) may be expanded in this set also:
\[
\phi = \sum_k (b_k v_k + b_k^\dagger v_k^*)
\]
(2.19)
and in this decomposition the canonical quantization commutation relations are:
\[
[b_i, b_k^\dagger] = \delta_{ik}, \quad [b_i, b_k] = [b_i^\dagger, b_k^\dagger] = 0.
\]
Then \( b_k |0_{II}\rangle = 0 \)
(2.20)
which yields a new Fock space.
The annihilation and creation operators of the two quantization schemes, viz \( a \) and \( b \), can be related by the Bogolubov transformation. For this by considering the scalar product \( (u_i, \phi) \), and using (2.14) and (2.18) we get

\[
a_i = \sum_k \left( \alpha_{ik} b_k + \beta_{ik} b_k^\dagger \right)
\]

where

\[
\alpha_{ik} = (u_i, v_k), \beta_{ik} = (u_i, v_k^*)
\]

are known as Bogolubov transformation coefficients. In the same way, the scalar product \( (v_i, \phi) \) gives the following inverse relation

\[
b_i = \sum_k \left( \alpha_{ik}^* a_k - \beta_{ik}^* a_k^\dagger \right)
\]

Impose the compatibility of (2.21) and (2.23) we find that the Bogoliubov coefficients satisfy the conditions

\[
\sum_k \left( \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^* \right) = \delta_{ij}
\]

\[
\sum_k \left( -\alpha_{ik} \beta_{jk} + \beta_{ik} \alpha_{jk}^* \right) = 0.
\]

Using these coefficients we can expand \( u_k \) in terms of \( v_k \) and vice versa (as both set are complete). We find, e.g.,

\[
v_i = \sum_k \left( -\alpha_{ki} u_k + \beta_{ki}^* u_k^* \right).
\]

It is evident that, as long as \( \beta_{ik} \neq 0 \), the Bogolubov transformations can induce a mixing up of positive and negative frequency modes, and that \( v_i \) is not a positive
frequency mode with respect to the Killing vectors $\zeta_1$. This means, in other words, that the $|0\rangle_1$ vacuum is not annihilated by $b_k$ and the two vacua are not equivalent.

In particular, the state $|0\rangle_1$ is not empty for an observer who defines positive frequencies with respect to $\zeta_2$: the state $|0\rangle_1$ contains particles and in the mode $v_k$ the expectation value of their number operator $b_k^\dagger b_k$ is, according to (2.23)

$$
\langle0_1| b_k^\dagger b_k | 0_1\rangle = \sum_{ij} \beta_{ik} \beta_{jk} \langle0_1| a_i a_j^\dagger | 0_1\rangle = \sum_i |\beta_{ik}|^2.
$$

Now we are in a position to describe the physical phenomenon of particle creation by time varying gravitational field. Let us assume that no particles were present before the gravitational field is turned on. If the Heisenberg picture is adopted to describe the quantum dynamics, then $|0\rangle_{in}$ is the state of the system for all times. However, the physical number operator which counts particles in the out-region is $N_k = b_k^\dagger b_k$. Thus the mean number of particles created in the mode $k$ is

$$
\langle N_k \rangle_{in} = \langle 0| b_k^\dagger b_k |0\rangle_{in} = \sum_j |\beta_{jk}|^2.
$$

If any of the $\beta_{jk}$ coefficient are non-zero, i.e., if any mixing of positive and negative frequency solutions occur, then particles are created by gravitational field.

The formalism may be extended, with little complications, to describe particle creation in the presence of horizons. In that case the two Killing vectors $\zeta_1$ and $\zeta_2$, defining the equivalent vacua, may correspond to observers using different coordinate systems to cover the same manifold: for example Minkowski and Rindler coordinates if we have a uniform accelerated observer in flat space or Kruskal or Schwarzschild coordinates
in the so called eternal black hole model.

If we have a field in the vacuum state $|0\rangle$, the probability amplitude to find the field in an excited state containing $n$ particles in a given mode $v_k$, for an observer associated to the other vacuum $|0_{II}\rangle$, is given by $A_k(n) = \langle n_k | 0_I \rangle$, where

$$| n_k \rangle_{II} = \frac{1}{\sqrt{n!}} (b_k)^2 | 0_{II} \rangle$$

The probability distribution, $P_k(n) = |A_k(n)|^2$, can be computed explicitly using the relation connecting the two vacua, i.e., expanding $|0_I\rangle$ as a superposition of states belonging to the Fock space constructed from $|0_{II}\rangle$. To simplify the formalism, let us consider spatial homogeneity. In this case the Bogolubov transformation is diagonal, because both set of modes $u_k$ and $v_k$ have the spatial dependence and the coefficient becomes

$$a_{ik} = \alpha_i \delta_{ik}, \beta_{ik} = \beta_i \delta_{ik}$$

(no summation over $i$) From (2.21) we have then

$$a_k = a_k b_k + \beta_k b_k^\dagger$$

and the condition (2.24) reduces to

$$| \alpha_i |^2 - | \beta_i |^2 = 1.$$ 

The special feature of Minkowski space is that the conventional vacuum state is the same for all initial measuring device throughout the spacetime. This is because the
vacuum is invariant under the Poincare group and so are the set of inertial observers in Minkowski space.

In curved spacetime the definition of vacuum is associated with the quantum measurement processes used to detect the quanta present. The state of motion of the measuring device can affect whether particles are observed or not. For example, a free-falling detector will not always register the same particle density as a noninertial accelerating detector does. This means that particle concepts does not generally have a universal significance and is observer dependent.

In many problems of interest the spacetime can be treated as asymptotically Minkowskian in the remote past and or in the remote future and they are respectively referred to as ‘in’ and ‘out’ regions and in Minkowskian quantum field theory it is assumed that as \( t \to \pm \infty \), all the field interactions approach zero. The analogue situation here is that the ‘in’ and ‘out’ regions admit natural particle states and a privileged quantum vacuum. If the state of the quantum field in the ‘in’ region is chosen to be the vacuum state, it will remain in that state during its subsequent evolution. However at later times, outside the ‘in’ region, freely falling particle detectors may still register particles in the vacuum state. If there is also an ‘out’ region then the ‘in’ vacuum may not coincide with the ‘out’ vacuum and observers in the ‘out’ region will detect presence of particles. This phenomenon is now referred to as ‘particle creation by time-dependent gravitational field’[38].

Quantum field theory in curved spacetime reveals that quantum concepts are essential
to understand various problems in cosmology. Such studies naturally lead to quantum cosmology.

2.3 Quantum effects in cosmology

One of the greatest successes of classical cosmology is its ability to describe the important features of the evolution of the Universe by using some specific initial conditions. The observed Universe could have arisen from a much larger class of initial conditions than in the hot big bang model, it is certainly not true that it could have arisen from any initial state - one could have chosen an initial quantum state for the matter which do not have correct density perturbation spectrum, and indeed could choose initial conditions for which inflation does not occur and so on. In order to have complete explanation of the presently observed state of the Universe, therefore, it is necessary to face up to the initial conditions. This is one reason compelling us for replacing classical cosmology by quantum cosmology.

Another fundamental problem facing the standard cosmology is the occurrence of singularities in spacetime, examples of which are the initial singularity of cosmological models and the curvature singularities on the behavior of black holes. General theorems have been proved which demonstrate that singularities are inevitable in standard cosmology, provided that certain conditions are imposed on the energy momentum-tensor [11]. These conditions are reasonable for classical matter, but are not expected to hold in general for energy-momentum tensor associated with quantized matter fields.
This holds out the hope that quantum effects associated with the matter fields can lead to the avoidance of singularities.

It has been assumed that quantum zero-point fluctuations got amplified during inflationary period and produced density perturbations, rotational perturbations and gravitational waves. Density perturbations seeded the observed inhomogeneity in the Universe. These perturbations which eventually give rise to galaxy clusters began as quantum fluctuations that were enormously stretched during the inflationary expansion phase in the first $10^{-35}$ sec after big bang. The inflationary scenario requires that the mean density of the Universe may be very close to its closure value. In the standard model, most of the remaining dark matter is presumed to consist of weakly interacting particles whose thermal velocities would have been negligable in the epoch when structures started to develop. However the origin of the inflationary stage still remain as an unsolved problem. Also what kind of evolution the Universe experienced before the inflationary stage and how did the Universe itself originate still remain as unanswered questions. A frequently made assumption is that initially the Universe was filled with radiations and the inflation era was preceded by an essentialy quantum gravitational phenomenon called the spontaneous birth of the Universe. Gravitational waves seem to be the only source of impartial information about the very early Universe and the quantum birth of the Universe. For this one need quantum theory of gravity [15, 29].

The second motivation for quantum cosmology comes from quantum gravity. At a
deeper level, both the background geometry and matter fields are to be treated quantum mechanically, this is the realm of quantum cosmology. The main object in quantum cosmology is the introduction of a wavefunction of the Universe [15] which, in general, describes all degrees of freedom on an equal footing. But there is no unique wavefunction of the Universe. Presently, we do not know any guiding principle allowing one to prefer one cosmological wavefunction over others. We do not have a fully satisfactory and consistent quantum theory of gravity. But a viable programme is now accepted in which the quantum gravity effects can be ignored as they are likely to be small, but quantum mechanics plays a vital role in the behavior of matter fields. Thus we have a problem of defining a consistent scheme in which the spacetime metric is treated classically but is coupled to the matter fields which can be treated quantum mechanically except near the spacetime singularity. This formulation is now generally called the semiclassical theory of gravity. At the moment, classical general relativity still provides a most successful description of gravity and matter field is treated as quantum mechanically as the source of gravity. In semiclassical theory, Einstein equation takes the following form [33]:

\[ G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle. \]  

(2.33)

The right hand side of the equation is supposed to be the energy-momentum tensor of the matter field. This means that in semiclassical theory the source on the right hand side of Einstein equation is taken to be the expectation value of some suitably defined energy-momentum operator for the matter fields. Another satisfactory explanation is needed in cosmology for the mechanism of particle creation in the early Universe.
These observations emphasize the fact that quantum concept and quantum effects are needed to understand the various stages of evolution of the Universe, especially the very early Universe.

Recently in order to probe quantum effects in cosmology, quantum optics concepts like coherent states [39] and squeezed states [40, 41] have been found to be very useful. It is now believed that relic gravitons and other primordial perturbations, created from zero-point quantum fluctuations in the course of cosmological evolution exist in specific quantum states known as squeezed states [42]. Creating particles like gravitons and other primordial perturbations from the zero-point quantum fluctuations in the process of the cosmological evolution were studied by Grishchuk and Sidorov [42] using squeezed state formalism. Gasperini and Giovannini [43] have shown that the entropy growth in the cosmological process of pair creation is completely determined by the associated squeezing parameter. Albrecht et al. [44] analyzed inflationary cosmology in the light of squeezed states. Hu et al. [45] have given a systematic description of the dependence on the initial states in terms of squeezing parameter. Using the squeezed state formalism Grishchuk [46] have studied generation of rotational cosmological perturbations. Novello et al. [47] treated cosmological perturbations in the quantum framework by using squeezed states. Caves [48] suggested that the concept of squeezed states might have a role in increasing the sensitivity of a gravitational wave detectors.

At a first glance it seems that the two areas, viz, quantum optics and cosmology have no
direct connections but the mathematical formalism and physical concepts are similar, the quantum optics concepts are found to be very useful to study many problems in cosmology. The basic properties of coherent states and squeezed states are discussed below.

2.4 Coherent states and Squeezed states

Coherent states and squeezed states are two important classes of quantum states well known in quantum optics [40]. A more appropriate basis for many optical fields are coherent states. The coherent states have an indefinite number of photons which allow them to have a more precisely defined phase than a number state where phase is completely random. The variances of quadrature components in a coherent state are equal having the minimum value allowed by the uncertainty principle. In this sense they are quantum mechanical state close to classical description of the field.

A single mode coherent state (scs) [39] is defined as:

\[ |\lambda\rangle = D(\lambda) |0\rangle \] (2.34)

where \(D(\lambda)\) is the single mode displacement operator and is given by

\[ D(\lambda) = \exp (\lambda a^\dagger - \lambda^* a) . \] (2.35)

The displacement operator have the following properties:

\[ D^\dagger (\lambda) = D^{-1} (\lambda) = D(-\lambda) \] (2.36)
\[ D^\dagger(\lambda) a D(\lambda) = a + \lambda \]
\[ D^\dagger(\lambda) a^\dagger D(\lambda) = a^\dagger + \lambda^* \]

where \(a^\dagger\) and \(a\) are creation and annihilation operators respectively and coherent states are eigen states of \(a\):

\[ a \mid \lambda \rangle = \lambda \mid \lambda \rangle \quad (2.37) \]

Similarly two mode coherent states (tcs) are defined as:

\[ | \lambda_+ \lambda_- \rangle = D(\lambda_+ \lambda_- | 00 \rangle \quad (2.38) \]

where \(D(\lambda_+ \lambda_-)\) is the two mode displacement operator which is the product of two single mode displacement operators.

Now using these properties we can calculate the expectation values of the position and momentum operators for the harmonic oscillators in single mode coherent states.

\[ \langle q \rangle_{scs} = \sqrt{\frac{\hbar}{2\omega}}(\lambda + \lambda^*) \quad (2.39) \]
\[ \langle p \rangle_{scs} = \frac{1}{i} \sqrt{\frac{\hbar}{2\omega}}(\lambda - \lambda^*), \]
\[ \langle q^2 \rangle_{scs} = \frac{\hbar}{2\omega}(\lambda^2 + \lambda^*^2 + 2|\lambda|^2 + 1), \]
\[ \langle p^2 \rangle_{scs} = -\frac{\hbar}{2\omega}(\lambda^2 + \lambda^*^2 - 2|\lambda|^2 + 1), \]

where \(q = \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger)\) and \(p = \frac{1}{i} \sqrt{\frac{\hbar}{2\omega}}(a - a^\dagger)\)

The coherent states form a two-dimensional continuum of states and are, in fact, overcomplete. The completeness relation is:

\[ \frac{1}{\pi} \int \mid \lambda \rangle \langle \lambda \mid d^2\lambda = 1 \quad (2.40) \]
The coherent states have a physical significance that the field generated by a highly stabilized laser operating well above the threshold is a coherent state. They form a useful basis for expanding the optical field in laser physics and in nonlinear optics.

Another class of minimum uncertainty states is the squeezed states. Hollenhorst [49] first used the term squeezed states. These states are characterized by reduced quantum fluctuations in one quadrature component of the field at the expense of increased fluctuations in the other noncommuting component. This remarkable property of squeezed state field has no classical interpretation and makes sense only in models when the nonlinear medium and the radiation fields are treated quantum mechanically.

A single mode squeezed states (sss) (or displaced squeezed states) is defined as[40, 41]:

\[ | \lambda, \xi \rangle = D(\lambda)S(r, \varphi) | 0 \rangle \quad \text{(2.41)} \]

where \( D(\lambda) \) is the single mode displacement operator given by (2.35) and \( S(r, \varphi) \) is the single mode squeezing operator and is given as:

\[ S(r, \varphi) = \exp \frac{r}{2} (e^{-i\varphi} a^2 - e^{i\varphi} a^\dagger^2) \quad \text{(2.42)} \]

where \( r \) is the squeezing parameter which determines the strength of squeezing and \( \varphi \) is the squeezing angle which determines the distribution between conjugate variables and \( 0 \leq r < \infty \) and \(-\pi \leq \varphi \leq \pi\). While \( a \) is annihilation operator and \( a^\dagger \) is the creation operator for the single mode states and they obey the following commutation relation.

\[ [a, a^\dagger] = 1 \quad \text{(2.43)} \]
and all other commutation relations vanish.

When \( \lambda = 0 \) (2.41) reduces to single mode squeezed vacuum states (ssv).

\[
|\xi\rangle = S(r, \varphi) |0\rangle. \tag{2.44}
\]

The squeezing operator obey the following relation

\[
S^\dagger(r, \varphi) = S^{-}(r, \varphi) = S(-r, \varphi) \tag{2.45}
\]

and has the following properties

\[
S^\dagger a S = a \cosh r - \alpha^\dagger e^{i\varphi} \sinh r \tag{2.46}
\]

\[
S^\dagger a^\dagger S = a^\dagger \cosh r - \alpha e^{-i\varphi} \sinh r. \tag{2.47}
\]

Similarly two mode squeezed states (tss) are defined as

\[
|\lambda_+ , \lambda_- , \xi\rangle = D_+(\lambda_+ , \lambda_-) S_t(r, \varphi) |0, 0\rangle \tag{2.47}
\]

The two mode displacement operator is given by.

\[
D(\lambda_+ , \lambda_-) = \exp(\lambda a^\dagger b^\dagger - \lambda^* a b) \tag{2.48}
\]

and \( S_t(r, \varphi) \) is the two mode squeezed vacuum operator and is given by:

\[
S_t(r, \varphi) = \exp r(e^{-i\varphi} a b - e^{i\varphi} a^\dagger b^\dagger) \tag{2.49}
\]

where \( \lambda_+ , \lambda_- \) and \( \xi \) are complex numbers and \( a, b \) are annihilation operators for each mode and \( a^\dagger, b^\dagger \) are creation operators and

\[
[a, a^\dagger] = [b, b^\dagger] = 1 \tag{2.50}
\]
and all other commutation relations vanish.

When $\lambda_+ = \lambda_- = 0$ (2.48) reduces to two mode squeezed vacuum states (tsv).

$$|\xi\rangle = S(r, \varphi) |0, 0\rangle.$$ (2.51)

The most fundamental properties of two mode squeezed vacuum states are given below

$$S^\dagger a S = a \cosh r - b^\dagger e^{i\varphi} \sinh r$$ (2.52)

$$S^\dagger a^\dagger S = a^\dagger \cosh r - b e^{-i\varphi} \sinh r.$$

The squeezed vacuum states under considerations are many particle states, hence the resulting field can be called classical but the statistical property is highly different from that of the coherent states. From that point of view, squeezed vacuum is purely a quantum phenomenon having no analogue in classical physics.

Basically single mode and two mode squeezed states are two photon problems. The displacement operator adds a constant to $a$, thus changing the mean values of the position and momentum variables. The single mode squeezed operator mixes $a$ and $a^\dagger$. Consequently, it induces a correlation between the position and momentum variables that is independent of their mean values. The two mode squeezed operator mixes $a$ with $b^\dagger$ and $b$ with $a^\dagger$. Consequently, it induces correlation between the position and momentum of the different modes.

Theoretical predictions have shown that squeezing of quantum fluctuations can occur in a variety of nonlinear optical phenomena like, four wave mixing [50, 51], parametric
amplification [52, 53, 54], harmonic generation [55, 56, 57], multiphoton absorption process [58, 59], optical bistability [60], etc. Squeezed number states, squeezed coherent states and squeezed thermal states are some of the well known squeezed states. Over the past decade considerable efforts have been put into experiments aiming at the generation and detection of squeezed states of the field. Slusher et.al. [61] in 1985 first performed a four wave mixing experiment in sodium vapour to generate squeezing inside a resonant cavity.

In curved spacetime there is, in general, no unique choice of the \{v\}, and hence no unique vacuum state. This means that we cannot identify what constitutes a state without paricle content, and the notation of 'particle' becomes ambiguous. One possible resolution of this difficulty is to choose some quantities other than particle content to lable quantum states. Possible choice might include local expectation values of the field operator. In the particular case of asymptotically flat spacetime, we might use the particle content in an asymptototic region. Even this characterisation is not unique. However, this non-uniqueness is an essential feature of the theory with physical consequences, namely, the phenomenon of particle creation.

The phenomenon particle creation in a nonstationary background metric using squeezed state and coherent state formalisms are investigated in the next chapter.