Chapter 2

An (a,c,∞) Policy Bulk Service Queue with State Dependent Service Rates

2.1 Introduction

A single server bulk service queueing system with deterministic service time is considered by Kosten (1942), where the service commences only when there is a minimum of **a** units in the queue and the service capacity is infinite. Bailey(1954) has considered an usual bulk service (UBSR) model, in which the server serves a maximum of **b** units in a batch and the server is intermittently available. Neuts (1967) considered a bulk service model in which the server begins service only if there are at least **a** units in the queue and serves a maximum of **b** units in a batch, if after a service completion epoch the queue size is less than **a**, the server waits till the queue size reaches to the limit **a**. Such a rule is called general bulk service rule(GBSR). Borthakur(1971), Medhi(1975), Sim, Templeton, Chaudhry(1983) and others have studied similar bulk service models. Bhat(1964) Teghem et al.(1968), Cohen(1969) etc. considered bulk queues with varying batch sizes.

In this chapter, a bulk service queueing system under the policy (a,c,∞) with state dependent service rates is analysed. The arrival process is assumed to be Poisson with parameter λ and the arrivals are served by a single server in batches with two service rates according to the control limit policy (a,c,∞). In this model the server begins service only if the queue size is at least **c** and serves them altogether in a batch according to an exponential distribution with rate μ_1, (called usual service or type-I service). The server continue batch service even when the queue size is less than **c** and not less than a secondary limit **a** after a service completion epoch, but with a different service rate μ_2, (called special service or type-II service). The server becomes idle when the queue size is less than **a**, after a service completion epoch.

After a service completion epoch the server may find the system size(n) in any of the following cases. (i) $0 \leq n \leq a-1$, (ii) $a \leq n \leq c-1$, (iii) $n \geq c$. In case (i), the server stops service and becomes idle. In case (ii), the server serves them altogether in a batch exponentially with parameter μ_2 (type-II service). In case (iii), the server serves them in a batch according to an exponential distribution with parameter μ_1 (type-I service). The transient and steady state behaviour of the model are discussed and obtained explicit expressions for expected queue length, expected busy periods and expected waiting time in the queue. An optimality problem associated to the model is discussed by using expected cost function and a numerical illustration is also given.

2.2 Analysis of the Model

Let X(t) denote the queue size and Y(t) denote the state of the server at time t. The state of the server Y(t) can assume the values 0, 1 or 2 according as the server is idle, busy with type-I service or busy with type-II service. The two dimensional stochastic processes {Y(t), X(t), t $\geq$ 0} forms a Markov process with state space
$S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{(0,n), n = 0,1,2,\ldots,c-1\}$,
$S_2 = \{(1,n), n = 0,1,2,\ldots\}$ and $S_3 = \{(2,n), n = 0,1,2,\ldots\}$.
Let $P(i,n,t) = P\{Y(t) = i, X(t) = n\}, i = 0,1,2; n = 0,1,2,\ldots$
The following are the transitions that can be occurred during the time interval (t,t+h) with the indicated probabilities.

Transitions during the interval (t, t+h)	Probability
(0,n) $\to$ (0,n+1)	λ h+ o(h) , 0 $\leq$ n $\leq$ c-2
(0,c-1) $\to$ (1,0)	λ h+ o(h)
(1,n) $\to$ (0,n)	μ_1h+o(h), 0 $\leq$ n $\leq$ a-1
(1,n) $\to$ (2,0)	μ_1h+o(h), a $\leq$ n $\leq$ c-1
(1,n) $\to$ (1,0)	μ_1h+o(h), n$\geq$ c
(1,n) $\to$ (1,n+1)	λ h+ o(h), n = 0,1,2,...
(2,n) $\to$ (2,n+1)	λ h+ o(h), n = 0,1,2,...
(2,n) $\to$ (0,n)	μ_2h+o(h), 0 $\leq$ n$\leq$ a-1
(2,n) $\to$ (2,0)	μ_2h+o(h), a $\leq$ n$\leq$ c-1
(2,n) $\to$ (1,0)	μ_2h+o(h), n $\geq$ c

Consequently the forward difference differential equations are

$$P'(0,0,t) = -\lambda P(0,0,t) + \mu_1 P(1,0,t) + \mu_2 P(2,0,t) \quad (2.2.1)$$

$$\begin{aligned} P'(0,n,t) &= -\lambda P(0,n,t) + \lambda P(0,n-1,t) \\ &\quad + \mu_1 P(1,n,t) + \mu_2 P(2,n,t), 1 \leq n \leq a-1 \end{aligned} \quad (2.2.2)$$

$$P'(0,n,t) = -\lambda P(0,n,t) + \lambda P(0,n-1,t), a \leq n \leq c-1 \quad (2.2.3)$$

$$
\begin{aligned}
P'(1,0,t) &= -(\lambda+\mu_1)P(1,0,t)+\lambda P(0,c-1,t) \\
&\quad +\mu_1\sum_{n\geq c}P(1,n,t)+\mu_2\sum_{n\geq c}P(2,n,t) && (2.2.4) \\
P'(1,n,t) &= -(\lambda+\mu_1)P(1,n,t)+\lambda P(1,n-1,t), n=1,2,3,\ldots && (2.2.5) \\
P'(2,0,t) &= -(\lambda+\mu_2)P(2,0,t)+\mu_1\sum_{n=a}^{c-1}P(1,n,t)+\mu_2\sum_{n=a}^{c-1}P(2,n,t) && (2.2.6) \\
P'(2,n,t) &= -(\lambda+\mu_2)P(2,n,t)+\lambda P(2,n-1,t), n=1,2,3,\ldots && (2.2.7)
\end{aligned}
$$

2.3 Method of Solution

Let P*(i,n,s) denote the Laplace transform of P(i,n,t). Here, we assume that P(0,0,0) =1. Corresponding to the equations (2.2.1) to (2.2.7), the following are the Laplace transform of the transient probabilities.

$$
\begin{aligned}
(s+\lambda)P^*(0,0,s)-1 &= \mu_1P^*(1,0,s)+\mu_2P^*(2,0,s) && (2.3.1) \\
(s+\lambda)P^*(0,n,s) &= \lambda P^*(0,n-1,s)+\mu_1P^*(1,n,s) \\
&\quad +\mu_2P^*(2,n,s), 1\leq n\leq a-1 && (2.3.2) \\
(s+\lambda)P^*(0,n,s) &= \lambda P^*(0,n-1,s), a\leq n\leq c-1 && (2.3.3) \\
(s+\lambda+\mu_1)P^*(1,0,s) &= \lambda P^*(0,c-1,s) \\
&\quad +\mu_1\sum_{n\geq c}P^*(1,n,s)+\mu_2\sum_{n\geq c}P^*(2,n,s) && (2.3.4) \\
(s+\lambda+\mu_1)P^*(1,n,s) &= \lambda P^*(1,n-1,s), n=1,2,3,\ldots && (2.3.5) \\
(s+\lambda+\mu_2)P^*(2,0,s) &= \mu_1\sum_{n=a}^{c-1}P^*(1,n,s)+\mu_2\sum_{n=a}^{c-1}P^*(2,n,s) && (2.3.6) \\
(s+\lambda+\mu_2)P^*(2,n,s) &= \lambda P^*(2,n-1,s), n=1,2,3,\ldots && (2.3.7)
\end{aligned}
$$

Solving the above system of equations, we get the following Laplace transforms of the transient probabilities.

$$
P^*(0,0,s) = P^*(1,0,s)[e_6+e_7D_1]+\frac{1}{s+\lambda}, \quad (2.3.8)
$$

(2.3.9)

$$P^*(0,n,s) = P^*(1,0,s)\{e_5^n(e_6+e_7D_1)+e_6e_1\frac{e_5^n-e_1^n}{e_5-e_1}+e_7e_2D_1\frac{e_5^n-e_2^n}{e_5-e_2}\} + \frac{e_5^n}{s+\lambda}, 1\le n\le a-1 \quad (2.3.10)$$

$$P^*(0,n,s) = P^*(1,0,s)e_5^{n-a+1}D_2+\frac{e_5^n}{s+\lambda}, a\le n\le c-1 \quad (2.3.11)$$

$$P^*(1,n,s) = P^*(1,0,s)e_1^n, n\ge 1 \quad (2.3.12)$$

$$P^*(2,0,s) = P^*(1,0,s)D_1, \quad (2.3.13)$$

$$P^*(2,n,s) = P^*(1,0,s)D_1e_2^n, n\ge 1 \quad (2.3.14)$$

where $e_1=\frac{\lambda}{s+\lambda+\mu_1}$, $e_2=\frac{\lambda}{s+\lambda+\mu_2}$, $e_3=\frac{\mu_1}{s+\lambda+\mu_2}$

$e_4=\frac{\mu_2}{s+\lambda+\mu_2}$, $e_5=\frac{\lambda}{s+\lambda}$, $e_6=\frac{\mu_1}{s+\lambda}$, $e_7=\frac{\mu_2}{s+\lambda}$

$$D_1 = \left[1-e_4\frac{e_2^a-e_2^c}{1-e_2}\right]^{-1}.e_3\frac{e_1^a-e_1^c}{1-e_1},$$

$$D_2 = e_5^{a-1}[e_6+e_7D_1]+e_6e_1\frac{e_5^{a-1}-e_1^{a-1}}{e_5-e_1}+e_7e_2D_1\frac{e_5^{a-1}-e_2^{a-1}}{e_5-e_2},$$

The value of P*(1,0,S) can be obtained by using the normalizing condition,

$$\sum_{n=0}^{c-1}P^*(0,n,s)+\sum_{n\ge0}P^*(1,n,s)+\sum_{n\ge0}P^*(2,n,s)=\frac{1}{s} \quad \text{as}$$

$$P^*(1,0,s) = \{[e_6+e_7D_1]\frac{1-e_5^a}{1-e_5}+\frac{e_6e_1}{e_5-e_1}[\frac{e_5-e_5^a}{1-e_5}-\frac{e_1-e_1^a}{1-e_1}] +\frac{e_7e_2D_1}{e_5-e_2}[\frac{e_5-e_5^a}{1-e_5}-\frac{s_2-e_2^a}{1-e^2}]+D_2\frac{e_5-e_5^{c-a-1}}{1-e_5} +\frac{1}{1-e_1}+\frac{D_1}{1-e_2}\}^{-1}\frac{e_5^c}{s} \quad (2.3.15)$$

2.4 Steady State Probabilities

Using the final value theorem on Laplace transforms, the steady state probabilities can be obtained as

$P(i,n) = \lim_{t\to\infty} P(i,n,t) = \lim_{s\to 0} sP^*(i,n,s)$

Hence from (2.3.8) to (2.3.14), the steady state distribution is given by

$$P(0,0) = P(1,0)[\theta_5 + \theta_6 T_1] \tag{2.4.1}$$

$$P(0,n) = P(1,0)\{\theta_5 + \theta_6 T_1 + \theta_5\theta_1 \frac{1-\theta_1^n}{1-\theta_1} + \theta_6\theta_2 T_1 \frac{1-\theta_2^n}{1-\theta_2}\}, 1 \le n \le a-1 \tag{2.4.2}$$

$$P(0,n) = P(1,0)T_2, \quad a \le n \le c-1 \tag{2.4.3}$$

$$P(1,n) = P(1,0)\theta_1^n, n \ge 1 \tag{2.4.4}$$

$$P(2,0) = P(1,0)T_1 \tag{2.4.5}$$

$$P(2,n) = P(1,0)T_1\theta_2^n, n \ge 1 \tag{2.4.6}$$

where
$$\theta_1 = \frac{\lambda}{\lambda+\mu_1}, \qquad \theta_2 = \frac{\lambda}{\lambda+\mu_2}, \qquad \theta_3 = \frac{\mu_1}{\lambda+\mu_2},$$
$$\theta_4 = \frac{\mu_2}{\lambda+\mu_2}, \qquad \theta_5 = \frac{\mu_1}{\lambda}, \qquad \theta_6 = \frac{\mu_2}{\lambda}$$
$$T_1 = \left[1 - \theta_4 \frac{\theta_2^a - \theta_2^c}{1-\theta_2}\right]^{-1} \theta_3 \frac{\theta_1^a - \theta_1^c}{1-\theta_1},$$
$$T2 = \theta_5 + \theta_6 T_1 + \theta_5\theta_1 \frac{1-\theta_1^{a-1}}{1-\theta_1} + \theta_5\theta_2 T_1 \frac{1-\theta_2^{a-1}}{1-\theta_2}$$

and
$$P(1,0) = \{a(\theta_5+\theta_6T_1) + \frac{\theta_5\theta_2}{1-\theta_2}[a - \frac{\theta_2-\theta_2^a}{1-\theta_2}] + \frac{\theta_6\theta_2T_1}{1-\theta_2}[a - \frac{\theta_2-\theta_2^a}{1-\theta_2}]$$
$$+T_2(c-a-1) + \frac{1}{1-\theta_1} + \frac{T_1}{1-\theta_2}\}^{-1} \tag{2.4.7}$$

2.5 Expected Queue Length

The expected queue length is given by

$$\begin{aligned}
L_q &= \sum_{n=1}^{c-1} nP(0,n) + \sum_{n\geq 1} nP(1,n) + \sum_{n\geq 1} nP(2,n) \\
&= P(1,0)\{(\theta_5 + \theta_6 T_1)\frac{a(a-1)}{2} + \frac{\theta_5\theta_1}{1-\theta_1}[\frac{a(a-1)}{2} - K(a,\theta_1)] \\
&\quad + \frac{\theta_6\theta_2 T_1}{1-\theta_2}[\frac{a(a-1)}{2} - K(a,\theta_2)] + T_2\frac{c(c-1)-a(a-1)}{2} \\
&\quad + \theta_1(1-\theta_1)^{-2} + T_1\theta_2(1-\theta_2)^{-2}\}, \qquad (2.5.1) \\
\text{where } K(a,x) &= (1-x)^{-2}x(1-x^a) - ax^a(1-x)^{-1}
\end{aligned}$$

2.6 Busy Period Distribution

Let the random variable T denote the busy period of the server. The busy period of the server begins with the commencement of a service and ends when the server become idle for the first time. Here, the busy period of the server starts when at least **c** units arrived in the system and ends when the system size is less than **a**, (ie, when the system size n = 0,1,2,3,..., a-1), after a service completion epoch. Hence the distribution of busy period T can be obtained as follows.

Let $f_n(t) = P\{t \leq T < t+dt, Y(t+dt) = 0, N(t+dt) = n\}, n = 0,1,\ldots,a-1$

and let $f_n{}^*(s)$ be the Laplace transform of $f_n(t)$.

Then, $f_n(t) = \frac{d}{dt}P(0,n,t), n = 0,1,2,\ldots,a-1$ and $f_n{}^*(s) = sP^*(0,n,s)$

The Laplace transform of the busy period distribution is given by

$$\begin{aligned}
b^*(s) &= \sum_{n=0}^{a-1} f_n{}^*(s) \\
&= \sum_{n=0}^{a-1} sP^*(0,n,s) \qquad (2.6.1)
\end{aligned}$$

Now consider the Laplace transform of the transient probabilities after avoiding the states (0,n), n = 0,1,2,...,a-1 and assuming P(1,0,0) = 1.

$$\begin{aligned}
sP^*(0,0,s) &= \mu_1 P^*(1,0,s) + \mu_2 P^*(2,0,s) && (2.6.2)\\
sP^*(0,n,s) &= \mu_1 P^*(1,n,s) + \mu_2 P^*(2,n,s), &&\\
& \qquad\qquad 1 \le n \le a-1 && (2.6.3)\\
(s+\lambda+\mu_1)P^*(1,0,s) - 1 &= \mu_1 \sum_{n\ge c} P^*(1,n,s) + \mu_2 \sum_{n\ge c} P^*(2,n,s) && (2.6.4)\\
(s+\lambda+\mu_1)P^*(1,n,s) &= \lambda P^*(1,n-1,s), n \ge 1 && (2.6.5)\\
(s+\lambda+\mu_2)P^*(2,0,s) &= \mu_1 \sum_{n=a}^{c-1} P^*(1,n,s) + \mu_2 \sum_{n=a}^{c-1} P^*(2,n,s) && (2.6.6)\\
(s+\lambda+\mu_2)P^*(2,n,s) &= \lambda P^*(2,n-1,s), n \ge 1 && (2.6.7)
\end{aligned}$$

Thus we get the Laplace transforms of the transient probabilities are

$$\begin{aligned}
P^*(0,0,s) &= P^*(1,0,s)\frac{1}{s}(\mu_1 + \mu_2 D_1),\\
P^*(0,n,s) &= P^*(1,0,s)[\frac{\mu_1}{s}e_1^n + \frac{\mu_2}{s}D_1 e_2^n], 1 \le n \le a-1\\
P^*(1,n,s) &= P^*(1,0,s)e_1^n, n \ge 1\\
P^*(2,0,s) &= P^*(1,0,s)D_1\\
P^*(2,n,s) &= P^*(1,0,s)D_1 e_2^n, n \ge 1\\
\text{and} \quad P^*(1,0,s) &= [s+\lambda+\mu_1 - \frac{\mu_1 e_1^c}{1-e_1} - D_1\frac{\mu_2 e_2^c}{1-e_2}]^{-1}
\end{aligned}$$

Hence from (2.6.1) the Laplace transform of the busy period distribution is given by

$$b^*(s) = [s+\lambda+\mu_1 - \frac{\mu_1 e_1^c}{1-e_1} - D_1\mu_2\frac{e_2^c}{1-e_2}]^{-1}[\mu_1\frac{1-e_1^a}{1-e_2} + D_1\mu_2\frac{1-e_2^a}{1-e_2}] \qquad (2.6.8)$$

and the expected busy period of the server is given by

$$
\begin{aligned}
E(T) &= \frac{-d}{ds} b^*(s)/_{s=0} \\
&= T_3^2 T_4 \{1 - \mu_1 \theta_1^{c+1} \frac{c(\theta_1 - 1) - \theta_1}{\lambda(1-\theta_1)^2} - T_1 \mu_1 \theta_2^{c+1} \frac{c(\theta_2 - 1) - \theta_2}{\lambda(1-\theta_2)^2} \\
&\quad - T_5 \frac{\mu_2 \theta_1^c}{1-\theta_1}\} - T_3 \{\mu_1 \frac{\theta_1^{a+1}[a - \theta_1(a-1)] - \theta_1^2}{\lambda(1-\theta_1)^2} \\
&\quad + T_1 \mu_2 \frac{\theta_2^{a+1}[a - \theta_2(a-1)] - \theta_2^2}{\lambda(1-\theta_2)^2} + T_5 \mu_2 \frac{1-\theta_2^a}{1-\theta_2} \qquad (2.6.9) \\
\text{where } T_3 &= [1 - \frac{\mu_1 \theta_1^c}{1-\theta_1} - \mu_2 T_1 \frac{\theta_2^c}{1-\theta_2}]^{-1}, \\
T_4 &= [\mu_1 \frac{1-\theta_1^a}{1-\theta_1} - \mu_2 T_1 \frac{1-\theta_2^a}{1-\theta_2}], \\
T_5 &= \lim_{s \to 0} \frac{d}{ds} D_1 \\
&= T_1 \{ \frac{a\theta_1^{a+1} - c\theta_1^{c+1}}{\lambda(\theta_1^a - \theta_1^c)} - \frac{\theta_1^2}{1-\theta_1} - \frac{\theta_3}{\mu_1} \\
&\quad T_1 [\frac{\theta_4}{\lambda} \frac{a\theta_2^{a+1} - c\theta_2^{c+1}}{1-\theta_2} + \theta_4 \frac{\theta_2^a - \theta_2^c}{1-\theta_2} (\frac{1}{\mu_2} - \frac{\theta_2^2}{\lambda(1-\theta_2)})]\}
\end{aligned}
$$

2.7 Busy Period Distribution in Type-I Service

In the queueing model under the policy (a,c,∞), the busy period of the server in type-I service begins when the queue size is at least '**c**' and lasts until the number of units in the queue becomes less than '**c**' for the first time, at the service completion epoch of type-I service. In type-I service, the server servers the units in batches of infinite capacity and with rate μ_1.

Let T_{b_1} be the busy period of the server in type-I service. The distribution of the busy period T_{b_1} can be obtained by considering the states (0,n),n = 0,1,2,...,a-1 and

(2,0) as absorbing. Also let us assume that P(1,0,0)=1.

$$\begin{aligned}
\text{Let} \quad f_{0,n}(t) &= P\{t \leq T_{b_1} < t+dt, Y(t+dt)=0, X(t+dt)=n\}, \\
& \qquad\qquad n = 0,1,2,\ldots,a-1 \\
f_{2,0}(t) &= Pr.\{t \leq T_{b_1} < t+dt, Y(t+dt)=2, X(t+dt)=0\} \\
\text{Then} \quad f_{0,n}(t) &= \frac{d}{dt}P(0,n,t), n = 0,1,2,...,a-1. \\
\text{and} \quad f_{2,0}(t) &= \frac{d}{dt}P(2,0,t)
\end{aligned}$$

Let $f^*_{0,n}(s)$ and $f^*_{2,0}(s)$ be the Laplace Transform of $f_{0,n}(t)$ and $f_{2,0}(t)$.

Then, $f^*_{0,n}(s) = sP^*(0,n,s), n = 0,1,2..,a-1$

$f^*_{2,0}(s) = sP^*(2,0,s).$

Let $b^*_1(s)$ be the Laplace Transform of busy period distribution of the server in type-I service.

$$\begin{aligned}
b^*_1(s) &= \sum_{n=0}^{a-1} f^*_{1,n}(s) + f^*_{2,0}(s) \\
&= \sum_{n=0}^{a-1} sP^*(0,n,s) + sP^*(2,0,s) \qquad (2.7.1)
\end{aligned}$$

The Laplace transform of state probability are given by

$$\begin{aligned}
sP^*(0,n,s) &= \mu_1 P^*(1,n,s), 0 \leq n \leq a-1 \qquad (2.7.2) \\
(s+\lambda+\mu_1)P^*(1,0,s) - 1 &= \mu_1 \sum_{n \geq c} P^*(1,n,s) \qquad (2.7.3) \\
(s+\lambda+\mu_1)P^*(1,n,s) &= \lambda P^*(1,n-1,s), n \geq 1 \qquad (2.7.4) \\
sP^*(2,0,s) &= \mu_1 \sum_{n=a}^{c-1} P^*(1,n,s) \qquad (2.7.5)
\end{aligned}$$

Solving equation (2.7.4) we get

$$P^*(1,n,s) = P^*(1,0,s)e_1^n, n \geq 1 \qquad (2.7.6)$$

Hence,

$$P^*(0,n,s) = P^*(1,0,s)\frac{\mu_1}{s}e_1^n, 0 \leq n \leq a-1 \quad (2.7.7)$$

$$P^*(2,0,s) = P^*(1,0,s)\frac{\mu_1(e_1^a - e_1^c)}{s(1-e_1)} \quad (2.7.8)$$

and $P^*(1,0,s)$ can be obtained by using the normalization condition

$$\sum_{n=0}^{a-1} P^*(0,n,s) + \sum_{n=0}^{\infty} P^*(1,n,s) + P^*(2,0,s) = \frac{1}{s}$$

$$\text{we get} \quad P^*(1,0,s) = \frac{1-e_1}{s+\mu_1(1-e_1^c)} \quad (2.7.9)$$

$$\text{and} \quad b_1^*(s) = \frac{\mu_1(1-e_1^c)}{s+\mu_1(1-e_1^c)} \quad (2.7.10)$$

Hence the expected busy period of the server in type-I service (E_{b_1}) is given by

$$Eb_1 = \frac{-db_1^*(s)}{ds}/s = 0$$

$$= \frac{1}{\mu_1(1-e_1^c)} \quad (2.7.11)$$

Remarks: The expected busy period of the server in type-I service(2.7.11) is a decreasing function in '**c**' and independent of the secondary limit '**a**'. Hence the expected busy period E_{b_1} is influenced by the control limit **c**(the minimum queue size required to start type-I service) and not by the secondary limit **a**(the minimum size to continue type-II service at the completion of type-I service)

2.8 Busy Period Distribution in Type-II Service

In the model, the busy period of the server in type-II service with rate μ_2 begins if the queue size becomes less than **c** and not less than the secondary limit **a**, at the service completion epoch of type-I service.(ie, the queue size X(t)=n so that $a \leq n \leq c-1$). The busy period of the server in type-II service ends if the queue size becomes less than '**a**' or greater than '**c-1**' at the service completion epoch of type-II service. Let (T_{b_2}) be the busy period of the server in type-II service. Then, the distribution of (T_{b_2}) can be obtained by considering the states (0,n), n= 0,1,...,a-1 and (1,0) as absorbing with P(2,0,0) =1.

$$\begin{aligned}
\text{Let} \quad g_{0,n}(t) &= P\{t \leq T_{b_2} < t+dt, Y(t+dt)=0, X(t+dt)=n\}, \\
&\qquad\qquad n = 0,1,2,\ldots,a-1 \\
g_{1,0}(t) &= P\{t \leq T_{b_2} < t+dt, Y(t+dt)=1, X(t+dt)=0\} \\
\text{Then} \quad g_{0,n}(t) &= \frac{d}{dt}P(0,n,t), n=0,1,2,\ldots,a-1. \\
\text{and} \quad g_{1,0}(t) &= \frac{d}{dt}P(1,0,t)
\end{aligned}$$

Let $g^*_{0,n}(s)$ and $g^*_{1,0}(s)$ be the Laplace Transform of $g_{0,n}(t)$ and $g_{1,0}(t)$ respectively

$$\begin{aligned}
\text{Then} \quad g^*_{0,n}(s) &= sP^*(0,n,s), n=0,1,2..,a-1 \\
g^*_{1,0}(s) &= sP^*(1,n,s)
\end{aligned}$$

Let $b^*_2(s)$ be the Laplace transform of busy period distribution of the server in type-II service.

$$\begin{aligned}
b^*_2(s) &= \sum_{n=0}^{a-1} g^*_{0,n}(s) + g^*_{1,0}(s)) \\
&= \sum_{n=0}^{a-1} sP^*(0,n,s) + sP^*(1,0,s) \qquad (2.8.1)
\end{aligned}$$

The Laplace transform of transient state probabilities are

$$\begin{aligned}
sP^*(0,n,s) &= \mu_2 P^*(2,n,s), 0 \leq n \leq a-1 \qquad (2.8.2) \\
sP^*(1,0,s) &= \mu_2 \sum_{n \geq c} P^*(2,n,s), \qquad (2.8.3) \\
(s+\lambda+\mu_2)P^*(2,0,s) - 1 &= \mu_2 \sum_{n=a}^{c-1} P^*(2,n,s) \qquad (2.8.4) \\
(s+\lambda+\mu_2)P^*(2,n,s) &= \lambda P^*(2,n-1,s), n \geq 1 \qquad (2.8.5)
\end{aligned}$$

Solving the above system of equations, we get

$$\begin{aligned} P^*(0,n,s) &= P^*(2,0,s)\frac{\mu_2}{s}e_2^n, n=0,1,\ldots,a-1. \\ P^*(1,0,s) &= P^*(2,0,s)\frac{\mu_2}{s}\frac{(e_2^c)}{(1-e_2)}, \\ P^*(2,n,s) &= P^*(2,0,s)e_2^n, n\geq 1 \\ \text{where} \quad e_2 &= \frac{\lambda}{s+\lambda+\mu_2} \end{aligned}$$

The value of $P^*(2,0,s)$ can be obtained by using the normalization condition

$$\sum_{n=0}^{a-1} P^*(0,n,s) + \sum_{n=0}^{\infty} P^*(1,n,s) + \sum_{n=0}^{\infty} P^*(2,n,s) = \frac{1}{s}$$

$$\text{we get} \quad P^*(2,0,s) = \frac{1-e_2}{s+\mu_2(1-e_2^a+e_2^c)} \tag{2.8.6}$$

Hence,the Laplace transform of busy period distribution of the server in type-II service is

$$b_2^*(s) = \frac{\mu_2(1-e_2^a+e_2^c)}{s+\mu_2(1-e_2^a+e_2^c)} \tag{2.8.7}$$

Hence the expected busy period of the server in type-II service (E_{b_2}) is given by

$$\begin{aligned} Eb_2 &= \frac{-db_2^*(s)}{ds}/s=0 \\ &= \frac{1}{\mu_2(1-\theta_2^a+\theta_1^c)} \end{aligned} \tag{2.8.8}$$

2.9 Waiting Time Distribution

Let the random variable W_q denote the waiting time of an arriving unit in the queue. An arriving unit may find the system in any of the following cases.

(i) (0,n), $0 \leq n \leq c-2$ (ii) (0,c-1)

(iii) (1,n), $0 \leq n \leq a-2$ (iv) (1,n), $n \geq a$

(v) (2,n), $0 \leq n \leq a-2$ (vi) (2,n), $n \geq a-1$

In case (ii) the arriving customer does not have to wait and in all the other cases the arrival has to wait . Hence the probability of a no delay is given by,

$$\begin{aligned} P(W=0) &= P(0,c-1) \\ &= P(1,0)T_2 \end{aligned} \tag{2.9.1}$$

Therefore, the probability of a delay is 1 - P(0,c-1)

In case (i), the arriving unit has to wait for the arrival of (c-n-1) more units. It has a gamma distribution with parameter λ and (c-n-1).

In case (iii), the arriving unit has to wait for the completion of type-I service if (a-n-1) more units arrive before the service completion or **q** $(0 \leq q \leq a-n-2)$ units arrive during type-I service, the service is over and (c-n-q-1) units arrive. Let Z denote the random variable associated with case(iii). Then Z = Z_1+Z_2, where Z_1 denote the arrival of (a-n-1) units before the service completion of type-I service, Z_2 denote the random variable of q $(0 \leq q \leq a-n-2)$ arrivals during type-I service, the service is over and (c-n-q-1) units arrive and the variables Z_1 and Z_2 are mutually exclusive.

Let $f_z(t)$ $f_{1,z_1}(t)$ and $f_{2,z_2}(t)$ denote the probability density function of Z, Z_1 and Z_2 respectively. Hence the probability density function of Z is given by

$f_z(t) = f_{1,z_1}(t) + f_{2,z_2}(t)$.

$$\begin{aligned} f_{1,z_1}(t) &= f(\mu_1, 1; t)\Gamma_t(\lambda, a-n-1) \\ f_{2,z_2}(t) &= \sum_{q=0}^{a-n-2} \int_0^t \Gamma_s(\mu_1, 1) f(\lambda, q; s) f(\lambda, c-n-q-1; (t-s)ds \\ &= \sum_{q=0}^{a-n-2} \int_0^t (1-e^{-\mu_1 s}) \frac{\lambda^{c-n-1} e^{-\lambda t}}{\Gamma q \Gamma(c-n-q-1)} s^{q-1}(t-s)^{c-n-q-2} ds \\ &= (a-n-1) f(\lambda, c-n-1; t) - \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_1)^l \lambda^{c-n-1} e^{-\lambda t}}{l B(q,l) \Gamma(c-n+l-1)} t^{c-n+l-2} \end{aligned}$$

$$\text{where} \quad f(\lambda,k;t) = \frac{\lambda^k}{\Gamma K} e^{-\lambda t} t^{k-1}, k = 0,1,2,$$

$$\text{and} \quad \Gamma_x(\lambda,k) = \int_0^x f(\lambda,k;t), \text{is the incomplete gamma function.}$$

$$= 1 - \sum_{r=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!}$$

$$B(q,l) = \frac{\Gamma q . \Gamma l}{\Gamma q + \Gamma l}$$

Therefore the p.d.f of Z,

$$f_z(t) = f(\mu_1,1;t)\Gamma_t(\lambda, a-n-1) + (a-n-1)f(\lambda, c-n-1;t) - \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_1)^l \lambda^{c-n-1} e^{-\lambda t}}{l B(q,l) \Gamma(c-n+l-1)} t^{c-n+l-2} \tag{2.9.2}$$

Therefore the p.d.f of Z,

$$f_z(t) = f(\mu_1,1;t)\Gamma_t(\lambda, a-n-1) + (a-n-1)f(\lambda, c-n-1;t) - \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_1)^l \lambda^{c-n-1} e^{-\lambda t}}{l B(q,l) \Gamma(c-n+l-1)} t^{c-n+l-2} \tag{2.9.3}$$

In case (iv), the arriving unit has to wait for the completion of type-I service, which has an exponential distribution with parameter μ_1.

In case (v), the arriving unit has to wait for the completion of type-II service if (a-n-1) more units arrive during the service or **q** $(0 \leq q \leq a-n-2)$ units arrive during type-II service, the service is over and (c-n-q-1) units arrive. Let U denote the random variable associated with case(v). Then U = U_1+U_2, where U_1 denote the arrival of (a-n-1) units before the service completion of type-II service, U_2 denote the random variable of q $(0 \leq q \leq a-n-2)$ arrivals during type-II service, the service is over and (c-n-q-1) units arrive and the variables U_1 and U_2 are mutually exclusive.

Let $g_u(t)$, $g_{1,u_1}(t)$ and $g_{2,u_2}(t)$ denote the probability density function of U, U_1 and U_2

respectively. Hence the probability density function of U is given by

$$\begin{aligned}
g_u(t) &= g_{1,u_1}(t) + g_{2,u_2}(t) \\
g_{1,u_1}(t) &= f(\mu_2, 1; t)\Gamma_t(\lambda, a-n-1) \\
g_{2,u_2}(t) &= \sum_{q=0}^{a-n-2} \int_0^t \Gamma_s(\mu_2, 1) f(\lambda, q; s) f(\lambda, c-n-q-1; (t-s)ds \\
&= \sum_{q=0}^{a-n-2} \int_0^t (1-e^{-\mu_2 s}) \frac{\lambda^{c-n-1} e^{-\lambda t}}{\Gamma q \Gamma(c-n-q-1)} s^{q-1}(t-s)^{c-n-q-2} ds \\
&= (a-n-1) f(\lambda, c-n-1; t) - \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_2)^l \lambda^{c-n-1} e^{-\lambda t}}{l B(q,l) \Gamma(c-n+l-1)} t^{c-n+l-2}
\end{aligned}$$

Therefore the p.d.f of U,

$$\begin{aligned}
g_u(t) &= f(\mu_2, 1; t)\Gamma_t(\lambda, a-n-1) + (a-n-1) f(\lambda, c-n-1; t) \\
&\quad - \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_2)^l \lambda^{c-n-1} e^{-\lambda t}}{l B(q,l) \Gamma(c-n+l-1)} t^{c-n+l-2} \qquad (2.9.4)
\end{aligned}$$

In case (vi), the arriving unit has to wait for the completion of type-II service, which has an exponential distribution with parameter μ_2

Thus the p.d.f of the random variable W_q is,

$$\begin{aligned}
dw(t) &= \sum_{n=0}^{a-1} P(0,n) f(\lambda, c-n-1; t) + \sum_{n=a}^{c-2} P(0,n) f(\lambda, c-n-1; t) \\
&\quad + \sum_{n=0}^{a-2} P(1,n) f_z(t) + \sum_{n \geq a} P(1,n) \mu_1 e^{-\mu_1 t} \\
&\quad + \sum_{n=0}^{a-1} P(2,n) g_{z_2}(t) + \sum_{n \geq a} P(0,n) \mu_2 e^{-\mu_2 t} \qquad (2.9.5)
\end{aligned}$$

$$
\begin{aligned}
= \; & P(1,0)\{[\frac{\theta_5}{1-\theta_1}+\frac{T_1\theta_6}{1-\theta_2}]\lambda A_1(1,c-1,c-a-1) \\
& +\theta_1^{c-2}\lambda A_1(\theta_1,c-1,c-a-1)-\theta_1^{c-2}\lambda A_1(\theta_1,c-1,c-a-1) \\
& +T_2\lambda E(c-a-1,\lambda t)+\frac{\theta_1^{a-1}}{1-\theta_1}\mu_1 e^{-\mu_1 t}+T_1\frac{\theta_2^{a-1}}{1-\theta_2}\mu_2 e^{-\mu_2 t} \\
& +[(a-1)-\sum_{n=0}^{a-2}E(a-n-1,\lambda t)][\mu_1 e^{-\mu_1 t}+T_1\mu_2 e^{-\mu_2 t}] \\
& +\sum_{n=0}^{a-2}(\theta_1^n+\theta_2^n)(a-n-1)f(\lambda,c-n-1;t) \\
& -\sum_{n=0}^{a-2}\sum_{q=0}^{a-n-2}\sum_{l=0}^{\infty}\left[\frac{(-\mu_1)^l\theta_1^n+(-\mu_2)^l\theta_2^n}{lB(q,l)\Gamma(c-m+l-1)}\right]\lambda^{c-m-1}e^{-\lambda t}t^{c-m+l-2}\} \quad (2.9.6)
\end{aligned}
$$

$$
\begin{aligned}
\text{where } e(m,z) &= \sum_{n=0}^{m-1}\frac{z^n}{n!} \\
E(m,z) &= e^{-z}e(m,z)=e^{-z}\sum_{n=0}^{m-1}\frac{z^n}{n!} \\
A_1(x,m,n) &= e^{-\lambda t}[e(m,\frac{\lambda t}{x})-e(n,\frac{\lambda t}{x})] \\
A_1(1,m,n) &= E(m,\lambda t)-E(n,\lambda t)
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
B_1(x,m,n) &= \int_0^\infty A_1(x,m,n)tdt \\
&= \frac{mx^{n-1}-(m+1)x^n+(n+1)x^m-nx^{m-1}}{\lambda^2(x-1)^2x^{m+n-2}} \\
B_1(1,m,n) &= \int_0^\infty A_1(1,m,n)tdt=\frac{m(m+1)-n(n+1)}{2\lambda^2} \\
\int_0^\infty E(m,\lambda t)tdt &= \frac{m(m+1)}{2\lambda^2} \\
\int_0^\infty e^{-\lambda t}e(m,\frac{\lambda t}{x})tdt &= \frac{x^{m+1}-(m+1)x+m}{\lambda^2(x-1)^2x^{m-1}} \\
L_2(a-1,x) &= \sum_{m=0}^{a-2}x^m(a-m-1)(c-m-1) \\
&= (a-1)(c-1)\frac{x-x^a}{1-x}-(a+c-2)K(a-1,\theta)+K_1(a-1,x)
\end{aligned}
$$

$$
\begin{aligned}
K(a,x) &= \sum_{n=0}^{a-1} nx^n = (1-x)^{-2}(x-x^a) - ax^a(1-x)^{-1} \\
K_1(a,x) &= \sum_{m=0}^{a-1} m^2 x^m \\
L_3(a-1,x,\theta_{\mu_i}) &= \sum_{n=1}^{a-2} x^n \sum_{q=0}^{a-n-2} \sum_{l=0}^{\infty} \frac{(-\mu_i)^l (c-n+l-1)}{lB(q,l)\lambda^{l+1}} \\
&= \frac{1}{1-\theta_{\mu_i}} \{ \frac{x-x^{a-1}}{1-x} [c-1-\frac{\mu_i\theta_{\mu_i}}{\lambda(1-\theta_{\mu_i})}] \\
&\quad - \frac{x\theta_{\mu_i}^{a-1} - x^{a-1}\theta_{\mu_i}}{\theta_{\mu_i}-x} [c-1-\frac{\mu_i\theta_{\mu_i}}{\lambda}(a-\frac{1}{\theta_{\mu_i}})] \\
&\quad + \theta_{\mu_i}^{a-1} K(a-1,\frac{x}{\theta_{\mu_i}})(1-\frac{\mu_i\theta_{\mu_i}}{\lambda}) - K(a-1,x)\}, \quad \theta_{\mu_i} = \frac{\lambda}{\lambda+\mu_i}
\end{aligned}
$$

The expected waiting time in the queue is given by

$$
E(w) = \int_0^\infty t dw(t)
$$

$$
\begin{aligned}
&= P(1,0)\{[\frac{\theta_5}{1-\theta_1} + \frac{T_1\theta_6}{1-\theta_2}]\lambda B_1(1,c-1,c-a-1) \\
&\quad + \theta_1^{c-2}\lambda B_1(\theta_1,c-1,c-a-1) - \theta_1^{c-2}\lambda B_1(\theta_1,c-1,c-a-1) \\
&\quad + T_2\frac{(c-a)(c-a-1)}{\lambda} + (a-1)[\frac{1}{\mu_1} + \frac{T_1}{\mu_2}] \\
&\quad - \frac{\mu_1}{(1-\theta_1)^2(\lambda+\mu_1)^2} K_3(a,\theta_1) + \frac{1}{\lambda}[L_2(a-1,\theta_1) - L_3(a-1,\theta_1,\theta_{\mu_1})] \\
&\quad + \frac{(a-1)(c-1)}{\lambda}(1+T_1) - [\frac{(c-1)(1-\theta_4^{a-1})}{\lambda(1-\theta_4)} - \frac{\mu_1\theta_4}{\lambda^2}K(a-1,\theta_4)](1+T_1) \\
&\quad + \frac{\theta_1^{a-1}}{(1-\theta_1)\mu_1} + \frac{T_1\theta_2^{a-1}}{(1-\theta_2)\mu_2} + \frac{T_1}{\lambda}[L_2(a-1,\theta_2) - L_3(a-1,\theta_2,\theta_{\mu_2})] \\
&\quad - \frac{T_1\mu_2}{(1-\theta_3)^2(\lambda+\mu_2)^2} K_3(a,\theta_3) \qquad (2.9.7)
\end{aligned}
$$

$$
\text{where } K_3(a,\theta_i) = [a-1-\frac{\theta_i^a-\theta_i}{1-\theta_i}(\theta_i + a(1-\theta_i)) + \theta_i^{a-1}(1-\theta_i)K(a-1,\frac{1}{\theta_i})]
$$

2.10 Determination of the Control Limits a and c

The optimal values for **a** and **c** can be obtained by considering a cost function, defined in terms of the expected busy periods of server in type-I and type-II services, and the expected queue length. The busy period of the sever begins when at least **c** units are in the queue and lasts until the the queue size becomes less than **a** for the first time, at a service completion epoch of a batch. Hence, the busy period of the server may contain the busy period in type-I service or the busy periods in both type-I and type-II services depending on the queue size. Here we assume that cost is charged for holding customers in the queue, for the sever in type-I service and for the server in type-II service. Also, an over head cost is charged to initiate batch service.

Let C_0 be the over head cost to initiate a batch service, C_h be the holding cost per unit time for a customer in the queue, C_1 denote cost per unit time for the server in type-I service and C_2 be the cost per unit time for the server in the type-II service. Then the expected cost function can be defined by

$$E_{a,c}cost = C_0 + C_h L_q + C_1 E_{b_1} + C_2 E_{b_2} \qquad (2.10.1)$$

where L_q = expected queue length, E_{b_1}= expected busy period of server in type-I service and E_{b_2}= expected busy period of server in type-II service

2.11 Numerical Illustration

In this section numerical values for the steady state probabilities, performance measures and optimality problem for the arbitrary **set:** λ = **0.9**, μ_1 =**0.5**, μ_2 =**0.8** and for different values of **a** and **c** are discussed.

The steady state probabilities can be computed by using the equations (2.4.1) to (2.4.7). The steady state probabilities for the arbitrary set when a=4 and c=10 are given by

Table 2.11.1

Steady state probabilities

(for λ=0.9, μ_1=0.5, μ_2=0.8, a=4, c=10)

n	P(0,n)	P(1,n)	P(2,n)	n	P(0,n)	P(1,n)	P(2,n)
0	0.03938	0.06251	0.00524	9	0.09423	0.00075	9.063e-006
1	0.07983	0.02583	0.00147	10	0	0.00048	4.798e-006
2	0.08975	0.01661	0.00078	11	0	0.00031	9.063e-006
3	0.09604	0.01068	0.00041	12	0	0.00020	2.540e-006
4	0.10005	0.00686	0.00022	13	0	0.00013	1.345e-006
5	0.09423	0.00441	0.00012	14	0	8.273e-005	7.119e-007
6	0.09423	0.00284	6.108e-005	15	0	5.318e-005	3.769e-007
7	0.09423	0.00182	3.234e-005	16	0	3.419e-005	1.995e-007
8	0.09423	0.00117	1.712e-005	17	0	2.198e-005	1.056e-007

The table (2.11.2) gives expected queue length obtained by using the equation (2.5.1) for various values of **a** and **c**

Table-2.11.2

Expected queue length (for λ=0.9, μ_1=0.5, μ_2=0.8,)

c\a	0	1	2	3	4	5
7	2.23578	2.566	2.7923	2.95224	3.06142	3.13086
9	3.06618	3.43908	3.68636	3.86082	3.98295	4.06549
10	3.50002	3.89134	4.14752	4.32788	4.45491	4.54202
12	4.39266	4.81584	5.08723	5.27749	5.41243	5.5066
13	4.84845	5.28543	5.56334	5.75777	5.89597	5.99297
14	5.30916	5.75871	6.04253	6.24072	6.38181	6.48129
15	5.774	6.23505	6.52423	6.72581	6.86951	6.97118

Remarks: Here it can be seen that the expected queue length increases as **a** and **c** increases.

The expected busy period of the server in type-I service E_{b_1} computed for the arbitrary set and for different values of **a** and **c** by using the equation (2.7.11) are given by

Table-2.11.3

Expected busy period in type-I service (for λ=0.9, μ_1=0.5, μ_2=0.8,)

c\a	0	1	2	3	4	5
7	2.09506	2.09506	2.09506	2.09506	2.09506	2.09506
8	2.06009	2.06009	2.06009	2.06009	2.06009	2.06009
9	2.03822	2.03822	2.03822	2.03822	2.03822	2.03822
10	2.0244	2.0244	2.0244	2.0244	2.0244	2.0244
11	2.01562	2.01562	2.01562	2.01562	2.01562	2.01562
12	2.01001	2.01001	2.01001	2.01001	2.01001	2.01001
13	2.00643	2.00643	2.00643	2.00643	2.00643	2.00643
14	2.00413	2.00413	2.00413	2.00413	2.00413	2.00413
15	2.00265	2.00265	2.00265	2.00265	2.00265	2.00265

Remarks: Here we can observe that the expected busy period of the server in type-I service E_{b_1} decreases for increasing values of '**c**' but '**a**' has no influence on E_{b_1}. That is E_{b_1} depends only on the initial level required to begin the batch service(type-I service) and not on the secondary limit **a**.

The expected busy period of the server in type-II service E_{b_2} computed for the arbitrary set and for different values of **a** and **c** by using the equation (2.8.8) are given by

Table-2.11.4

Expected busy period in type-II service (for λ=0.9, μ_1=0.5, μ_2=0.8,)

c\a	0	1	2	3	4	5
7	2.59205	1.7091	1.44798	1.33962	1.28857	1.26309
8	2.62187	1.72201	1.45723	1.34754	1.2959	1.27013
9	2.63794	1.72893	1.46218	1.35177	1.29981	1.27389
10	2.64652	1.73262	1.46482	1.35402	1.30189	1.27588
11	2.65109	1.73457	1.46622	1.35522	1.303	1.27695
12	2.65352	1.73561	1.46696	1.35585	1.30358	1.27751
13	2.6548	1.73616	1.46735	1.35619	1.30389	1.27781
14	2.65548	1.73645	1.46756	1.35637	1.30406	1.27796
15	2.65584	1.73661	1.46767	1.35646	1.30414	1.27805

Remarks: Here it is clear that the expected busy period of the server in type-II service increases as '**c**' increases and decreases for increasing values of '**a**'. This shows that the secondary limit '**a**', introduced in the general bulk service rule makes the model more realistic.

The expected cost function $E_{a,c}(cost)$ obtained by using the equation(2.10.1) for different values of **a** and **c** and for the chosen arbitrary set are given by

Table-2.11.5

Expected cost for λ=0.9, μ_1=0.5, μ_2=0.8, C_0=100, C_h=20, C_1=80, C_2= 30

c\ a	1	2	3	4	5	6	7	8	9
1	602.55	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	438.80	430.31	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	397.13	384.14	381.81	0.00	0.00	0.00	0.00	0.00	0.00
4	384.75	368.42	365.45	365.13	0.00	0.00	0.00	0.00	0.00
5	382.74	364.35	361.07	**360.82**	361.12	0.00	0.00	0.00	0.00
6	385.23	365.78	362.42	362.26	362.76	363.19	0.00	0.00	0.00
7	390.08	370.19	366.89	366.83	367.49	368.11	368.50	0.00	0.00
8	396.33	376.39	373.19	373.25	374.03	374.80	375.34	375.65	0.00
9	403.51	383.70	380.65	380.82	381.71	382.58	383.24	383.66	383.89

Remarks: Here it can be noted that the expected cost is a concave function of **a** and **c**. Hence optimum values for **a** and **c** exist and can be determined. It is clear from the table the expected cost function attains the minimum at (a=4, c=5). The optimum cost for λ=0.9, μ_1=0.5, μ_2=0.8, C_0=100, C_h=20, C_1=80, C_2= 30 is **360.82**. So the optimal control limits are **a=4, c=5**