Chapter 6

M/M(a,c,d)/1 Bulk Service Queue with Accessible and Non-Accessible Batches

6.1 Introduction

Queueing systems with accessible to the on going service batch have been considered by Chaimsiri and Leonard(1981), Gross and Harris(1985), Sivasamy(1986,1990) and Baburaj(2000). Sivasamy analysed the model $M/M^{a,b}/1$ with accessible limit d($a \leq d \leq b$), where the arriving customers are allowed to enter the on going service batch till the accessible limit d is reached or the service is over. Baburaj(2000) considered a Markovian single and batch service queue with accessibility to the service batch. He has chosen an accessible limit in the batch service and allowed the late entries to join the service batch till the system size is less than the accessible limit or the service is over. In this chapter the concept of accessibility is introduced to the queueing model discussed in the chapter 3.

A state dependent single server bulk service queue with accessible and non accessible batches under the policy (a,c,d) is considered in this chapter. We assume that customers arrive according to a homogeneous Poisson process with rate λ and the service mechanism is assumed to operate as follows: The server initiate service if the queue size is at least **c** and serves a maximum of **d**units in a batch exponentially with rate μ_1 according to FCFS rule. This service may be called type-I service. The server continue batch service even if the number of units in the queue is less than **c** and not less than a secondary limit **a**, after a service completion epoch, but with different service rate μ_2. This is called type-II service and also assume that late entries are allowed to join the on going service batch of type-II service as long as the batch size is less than **c-1** or the service is over.This batch is called accessible batch (AB) with accessible limit **c-1**. In the case of type-I service, accessibility is not allowed to the on going service batch. This batch is called non-accessible batch (NAB).

At the service completion epoch of a batch the server may find the queue size(n) in any of the following cases.

(i) n, n = 0,1,...,a - 1, (ii) n, n = a,a+1,...,c-1 (iii) n, n = c,c+1,...

In case (i), the server becomes idle. In case (ii), the server provides type-II service by taking entire units for service and admit subsequent arrivals to the on going service batch till the current batch is over or it reaches the accessible limit c-1, whichever occur earlier. This batch is called accessible batch(AB). In case (iii), the server renders type-I service by taking a maximum of (n,d) units in a batch (with service rate μ_1), where **d** is the maximum service capacity of the system and the batch is non-accessible to the late entries(NAB). The steady state probabilities, busy period distributions, expected queue length, expected waiting time and expected cost function are discussed with a numerical illustration.

6.2 Analysis of the Model

Let P(0,n,t), n=0,1,...,c-1 denote the probability that the server is idle and **n** customers are in the system at time t, P(1,n,t), n= 0,1,... be the probability that the server is busy with non-accessible batch(NAB)(type-I service) and n customer are in the queue. P(2,n,t), n=a,a+1,a+2,... represents the probability that the server is busy with accessible batch(type-II service) and n customers are in the system at time t. Here the state space of the system is

S = $S_1 \cup S_2 \cup S_3$, where S_1 = {(0,n),n=0,1,2,...,c-1}, S_2 = {(1,n), n=0,1,2,...}
and S_3 = {(2,n) n=a,a+1,...}

The following are the transitions that can be occurred during the time interval (t,t+h].

Transitions during (t, t+h]			**Probabilities**
(0,n)	$\rightarrow$	(0,n+1)	λ h + O(h),n= 0,1,...,c-2
(0,c-1)	$\rightarrow$	(1,0)	λh + O(h)
(1,n)	$\rightarrow$	(1,n+1)	λh + O(h), n= 0,1,...
(1,n)	$\rightarrow$	(0,n)	μ_1h + O(h), n = 0,1,2,...,a-1
(1,n)	$\rightarrow$	(2,n)	μ_1h + O(h), n= a,a+1,a+2,...,c-1
(1,n)	$\rightarrow$	(1,0)	μ_1h + O(h), n= c,c+1,...,d
(1,n)	$\rightarrow$	(1,n-d)	μ_1h + O(h), n$\geq$d
(2,n)	$\rightarrow$	(0,0)	μ_2h + O(h), n = a,a+1,...,c-1
(2,n)	$\rightarrow$	(0,n-c)	μ_2h + O(h), n =c,c+1,...,c+a-2
(2,n)	$\rightarrow$	(1,0)	μ_2h + O(h),n =2c-1,2c,...,c+d
(2,n)	$\rightarrow$	(1,n-(c+d+1))	μ_2h + O(h) n$\geq$d+c+1
(2,n)	$\rightarrow$	(2,n-c+1)	μ_2h + O(h),c+a-1 $<$n$<$2c-2
(2,n)	$\rightarrow$	(2,n+1)	λh + O(h), n $\geq$a

Hence, we have the following system of difference differential equations governing the above transitions, with initial condition P(0,0,0)=1

$$
\begin{aligned}
P'(0,0,t) &= -\lambda P(0,0,t) + \mu_1 P(1,0,t) + \mu_2 \sum_{n=a}^{c-1} P(2,n,t) && (6.2.1) \\
P'(0,n,t) &= -\lambda P(0,n,t) + \lambda P(0,n-1,t) + \mu_1 P(1,n,t) \\
&\quad + \mu_2 P(2, c-1+n, t), n = 1,2,3,\ldots,a-1 && (6.2.2) \\
P'(0,n,t) &= -\lambda P(0,n,t) + \lambda P(0,n-1,t), n = a, a+1, \ldots, c-1 && (6.2.3) \\
P'(1,0,t) &= -(\lambda + \mu_1) P(1,0,t) + \lambda P(0,c-1,t) \\
&\quad + \mu_1 \sum_{n=c}^{d} P(1,n,t) + \mu_2 \sum_{n=c}^{d} P(2, c-1+n, t) && (6.2.4) \\
P'(1,n,t) &= -(\lambda + \mu_1) P(1,n,t) + \lambda P(1,n-1,t) + \mu_1 P(1, n+d, t) \\
&\quad + \mu_2 P(2, c-1+n+d, t), n = 1,2,\ldots && (6.2.5) \\
P'(2,a,t) &= -(\lambda + \mu_2) P(2,a,t) + \mu_1 P(1,a,t) + \mu_2 P(2, c-1+a, t) && (6.2.6) \\
P'(2,n,t) &= -(\lambda + \mu_2) P(2,n,t) + \lambda P(2,n-1,t) + \mu_1 P(1,n,t) \\
&\quad + \mu_2 P(2, c-1+n, t), n = a+1, a+2, \ldots, c-1 && (6.2.7) \\
P'(2,n,t) &= -(\lambda + \mu_2) P(2,n,t) + \lambda P(2,n-1,t), n \geq c && (6.2.8)
\end{aligned}
$$

6.3 Steady State Distribution

Let $P(i,n) = \lim_{t \to \infty} P(i,n,t)$, i=0,1,2 be the steady state probabilities. Then from equations (6.2.1) to (6.2.8)we get the steady state equations as,

$$
\begin{aligned}
\lambda P(0,0) &= \mu_1 P(1,0) + \mu_2 \sum_{n=a}^{c-1} P(2,n) && (6.3.1) \\
\lambda P(0,n) &= \lambda P(0,n-1) + \mu_1 P(1,n) \\
&\quad + \mu_2 P(2, c-1+n), n = 1,2,3,...,a-1 && (6.3.2) \\
P(0,n) &= P(0,n-1), n = a, a+1, ..., c-1 && (6.3.3)
\end{aligned}
$$

$$(\lambda+\mu_1)P(1,0) = \lambda P(0,c-1)+\mu_1\sum_{n=c}^{d}P(1,n)+\mu_2\sum_{n=c}^{d}P(2,c-1+n), \quad (6.3.4)$$

$$(\lambda+\mu_1)P(1,n) = \lambda P(1,n-1)+\mu_1 P(1,n+d) + \mu_2 P(2,c-1+n+d), n=1,2,... \quad (6.3.5)$$

$$(\lambda+\mu_2)P(2,a) = \mu_1 P(1,a)+\mu_2 P(2,c-1+a) \quad (6.3.6)$$

$$(\lambda+\mu_2)P(2,n) = \lambda P(2,n-1)+\mu_1 P(1,n) + \mu_2 P(2,c-1+n), n=a+1,a+2,...,c-1 \quad (6.3.7)$$

$$(\lambda+\mu_2)P(2,n) = \lambda P(2,n-1), n\geq c \quad (6.3.8)$$

Solving (6.3.8) recursively we get

$$P(2,n)=\theta_1^{n-c+d}P(2,c-1), n\geq c$$

Using Rouche's theorem and solving (6.3.5) we get,

$$P(1,n)=P(1,0)r^n-\frac{\theta_6\theta_1^{n+d}P(2,c-1)}{K(\theta_1)}, n\geq 1$$

where r is the unique positive real root less unity of the characteristic equation

$$K(z)=\mu_1 z^{d+1}-(\lambda+\mu_1)z+\lambda=0$$

From(6.3.6) $\quad P(2,a) = P(1,0)\theta_2 r^a+P(2,c-1)[\theta_3-\frac{\theta_6\theta_2\theta_1^d}{K(\theta_1)}]\theta_1^a$

From(6.3.7) $P(2,c-1) = P(1,0)T_1$,

where $\quad T_1=\theta_2\left[\frac{r^a\theta_1^{c-a}-r^c}{\theta_1-r}]\right]\left[1-(\theta_3-\frac{\theta_2\theta_6\theta_1^d}{K(\theta_1)})(c-a)\theta_1^{c-1}\right]^{-1}$

Hence the steady state probabilities are

$$
\begin{aligned}
P(0,n) &= P(1,0)\{T_2 + \frac{\mu_1}{\lambda}(\frac{r - r^{n+1}}{1-r}) \\
&\quad + T_1(\frac{\mu_2}{\lambda} - \frac{\mu_1\theta_6\theta_1^d}{\lambda K(\theta_1)})(\frac{\theta_1 - \theta_1^{n+1}}{1-\theta_1})\}, n = 0,1,2,\ldots,a-1 && (6.3.9) \\
P(0,n) &= P(10)T_3, n = a, a+1,\ldots,c-1 && (6.3.10) \\
P(1,n) &= P(1,0)[r^n - \frac{T_1\theta_1^{n+d}\theta_6}{K(\theta_1)}], n \geq 1 && (6.3.11) \\
P(2,n) &= P(1,0)\{\theta_2[\frac{r^a\theta_1^{n-a+1} - r^{n+1}}{\theta_1 - r}] \\
&\quad + T_1(\theta_3 - \frac{\theta_2\theta_6\theta_1^d}{K(\theta_1)})(n-a+1)\theta_1^n\}, a \leq n \leq c-1 && (6.3.12) \\
P(2,n) &= P(1,0)\theta_1^{n-c+1}T_1, n \geq c && (6.3.13)
\end{aligned}
$$

where P(1,0) can be obtained by using normalization condition $\sum_{n=0}^{c-1} P(0,n) + \sum_{n=0}^{\infty} P(1,n) + \sum_{n=a}^{\infty} P(2,n) = 1$ as

$$
\begin{aligned}
P(1,0) &= \{T + aT_2 + (c-a)T_3 + \frac{1}{1-r}\left[1 + \frac{r\mu_1}{\lambda}(a - \frac{1-r^a}{1-r})\right] \\
&\quad + \frac{T_1\theta_1}{1-\theta_1}\left[1 - \frac{\theta_1^d\theta_6}{K(\theta_1)} + \frac{1}{\lambda}(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K(\theta_1)})(a - \frac{1-\theta_1^a}{1-\theta_1})\right]\}^{-1} && (6.3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
T &= \frac{\theta_2 r^a}{\theta_1 - r}\left[\frac{(\theta_1 - \theta_1^{c-a+1})}{1-\theta_1} - \frac{r - r^{c-a}}{1-r}\right] + T_1(\theta_3 - \frac{\theta_1^d\theta_2\theta_6}{K(\theta_1)})(\frac{\theta_1^a - \theta_1^c}{(1-\theta_1)^2} - \frac{(c-a)\theta_1^c}{1-\theta_1}), \\
T_2 &= \frac{\mu_1}{\lambda} + \frac{\mu_2}{\lambda}\{L_1 + \frac{\theta_2 r^a}{\theta_1 - r}\left[\theta_1 - r + \frac{\theta_1^2 - \theta_1^{c-a+1}}{1-\theta_1} - \frac{r^2 - r^{c-a+1}}{1-r}\right]\}, \\
L_1 &= T_1\left[\theta_3 - \frac{\theta_1^d\theta_2\theta_6}{K_1(\theta_1)}\right]\left[\frac{\theta_1^a - \theta_1^c}{(1-\theta_1)^2} - \frac{(c-a)\theta_1^c}{1-\theta_1}\right], \\
T_3 &= T_2 + \frac{\mu_1}{\lambda}(\frac{r - r^a}{1-r}) + T_1(\frac{\mu_2}{\lambda} - \frac{\mu_1\theta_6\theta_1^d}{\lambda K(\theta_1)})(\frac{\theta_1 - \theta_1^a}{1-\theta_1}), \\
K(\theta_1) &= \mu_1\theta_1^{d+1} - (\lambda + \mu_1)\theta_1 + \lambda \quad \text{and} \\
\theta_1 &= \frac{\lambda}{\lambda + \mu_2}, \quad \theta_2 = \frac{\mu_1}{\lambda + \mu_2}, \quad \theta_3 = \frac{\mu_2}{\lambda + \mu_2}, \\
\theta_4 &= \frac{\lambda}{\lambda + \mu_1}, \quad \theta_5 = \frac{\mu_1}{\lambda + \mu_1}, \quad \theta_6 = \frac{\mu_2}{\lambda + \mu_1},
\end{aligned}
$$

6.4 Expected Queue Length

The expected length of the queue is given by

$$\begin{aligned}
Lq &= \sum_{n=1}^{c-1} nP(0,n) + \sum_{n=1}^{\infty} nP(1,n) + \sum_{n=a}^{\infty} nP(2,n) \\
L_q &= P(1,0)\{(T_2 - T_3)\frac{a(a-1)}{2} + T_3\frac{c(c-1)}{2} \\
&+ \frac{r}{1-r}\left[\frac{1}{1-r} + \frac{\mu_1}{\lambda}(\frac{a(a-1)}{2} - K(a,r))\right] \\
&+ \frac{T_1\theta_1}{1-\theta_1}[c + (\frac{\mu_2}{\lambda} - \frac{\mu_1\theta_6\theta_1^d}{\lambda K_1(\theta_1)})(\frac{a(a-1)}{2} - K(a,\theta_1)) + \frac{\theta_1^2}{(1-\theta_1)^2}(1 - \frac{\theta_1^{d-1}\theta_6}{K(\theta_1)})] \\
&+ \left[\frac{\theta_2 r^a}{\theta_1^{a-1}(\theta_1 - r)} - T_1(a-1)(\theta_3 - \frac{\theta_1^d\theta_2\theta_6}{K(\theta_1)})\right][K(c,\theta_1) - S_1(a,\theta_1)] \\
&- \frac{\theta_2 r}{\theta_1 - r}[K(c,r) - K(a,r)] + T_1(\theta_3 - \frac{\theta_1^d\theta_2\theta_6}{K(\theta_1)})[K_1(c,\theta_1) - K_1(a,\theta_1)]\} \quad (6.4.1)
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad K(a,x) &= \sum_{n=0}^{a-1} nx^n = (1-x)^{-2}x(1-x^a) - ax^a(1-x)^{-1} \\
K_1(a,x) &= \sum_{m=0}^{a-1} m^2 x^m
\end{aligned}$$

6.5 Busy Period Distribution

In this model, the busy period of the server begins when at least **c** units are in the queue and lasts until, for the first time the queue size becomes less than **a**, at the service completion epoch. Let the random variable T_b denote the busy period of the sever, Y(t) denote state of the server and N(t) denote the number of customers in the system at time t. The variable Y(t) assumes values 0, 1, and 2 according as the server is idle, busy with type-I service and busy with type-II service. The busy period distribution can be obtained by considering the states (0,n), n =0,1,2,. . . ,a-1 are absorbing. Also assume that P(1,0,0)=1

Then the distribution of busy period can be obtained as follows.

Let $f_n(t) = P\{t \leq T_b < t + dt, Y(t+dt) = 0, N(t+dt) = n\}, n = 0, 1, \ldots, a-1$

and let $f_n^*(s)$ be the Laplace transform of $f_n(t)$.

$$\begin{aligned} \text{Then } f_n(t) &= \frac{d}{dt}P(0,n,t), n = 0,1,2,a-1 \\ \text{and } f_n^*(s) &= sP^*(0,n,s), n = 0,1,2,\ldots,a-1 \end{aligned}$$

The Laplace transform of the busy period distribution is given by

$$\begin{aligned} b^*(s) &= \sum_{n=0}^{a-1} f_n{}^*(s) \\ &= \sum_{n=0}^{a-1} sP^*(0,n,s) \end{aligned} \tag{6.5.1}$$

Consider the Laplace transform of the transient probabilities of the system avoiding the state (0,n), n=0,1,2,...,a-1. Using standard arguments we get the following equations for the Laplace transform for the transient distribution.

$$sP^*(0,0,s) = \mu_1 P^*(1,0,s) + \mu_2 \sum_{n=a}^{c-1} P^*(2,n,s) \tag{6.5.2}$$

$$sP^*(0,n,s) = \mu_1 P^*(1,n,s) + \mu_2 P^*(2,c-1+n,s), n = 1,2,\ldots,a-1 \tag{6.5.3}$$

$$(s+\mu_1)P^*(1,0,s) - 1 = \mu_1 \sum_{n=c}^{d} P^*(1,n,s) + \mu_2 \sum_{n=c}^{d} P^*(2,c-1+n,s) \tag{6.5.4}$$

$$(s+\lambda+\mu_1)P^*(1,n,s) = \lambda P^*(1,n-1,s) + \mu_1 P^*(1,n+d,s) + \mu_2 P^*(2,c-1+n+d,s), n = 1,2,3,\ldots \tag{6.5.5}$$

$$(s+\lambda+\mu_2)P^*(2,a,s) = \mu_1 P^*(1,a,s) + \mu_2 P^*(2,c-1+a,s) \tag{6.5.6}$$

$$(s+\lambda+\mu_2)P^*(2,n,s) = \lambda P^*(2,n-1,s) + \mu_1 P^*(1,n,s) + \mu_2 P^*(2,c-1+n,s), a \leq n \leq c-1 \tag{6.5.7}$$

$$(s+\lambda+\mu_2)P^*(2,n,s) = \lambda P^*(2,n-1,s), n \geq c \tag{6.5.8}$$

Let $\sum_{n=a}^{c-1} P^*(2,n,s) = DP^*(1,0,s)$

$$\text{and} \quad e_1 = \frac{\lambda}{s+\lambda+\mu_2}, \quad e_2 = \frac{\mu_1}{s+\lambda+\mu_2}, \quad e_3 = \frac{\mu_2}{s+\lambda+\mu_2},$$
$$e_4 = \frac{\lambda}{s+\lambda+\mu_1}, \quad e_5 = \frac{\mu_1}{s+\lambda+\mu_1}, \quad e_6 = \frac{\mu_2}{s+\lambda+\mu_1},$$

Solving equation (6.5.8) recursively, we get

$$P^*(2,n,s) = P^*(2,c-1,s)e_1^{n-c+1}, n \geq c$$

Solving equation (6.5.5) and using Rouche's theorem ,we get

$$P^*(1,n,s) = P^*(1,0,s)R^n - \frac{e_6 e_1^{n+d} P^*(2,c-1,s)}{K(e_1)}$$

where R is the unique positive real root less than unity of the equation

$$K(z) = \mu_1 z^{d+1} - (s+\lambda+\mu_1)z + \lambda = 0$$

Hence from (6.5.7) we get,

$$\begin{aligned}
P^*(2,n,s) &= P(1,0,s)\{e_2[\frac{R^a e_1^{n-a+1} - R^{n+1}}{e_1 - r}] \\
&+ P^*(2,c-1,s)[(e_3 - \frac{e_2 e_6 e_1^d}{K(e_1)})(n-a+1)e_1^n], \\
&\qquad n = a+1, a+2, \ldots, c-1
\end{aligned}$$

$$\begin{aligned}
P(2,c-1,s) &= P(1,0,s)D_1 \\
\text{where,} \quad D_1 &= e_2\left[\frac{R^a e_1^{c-a} - R^c}{e_1 - r}\right]\left[1 - (e_3 - \frac{e_1^d e_2 e_6}{K(e_1)})(c-a)e_1^{c-1}\right]^{-1}
\end{aligned}$$

Using these results and equations (6.5.2) and (6.5.3), we get the Laplace transform of transient probabilities as

$$
\begin{aligned}
P^*(0,0,s) &= P^*(1,0,s)(\frac{\mu_1}{s}+\frac{\mu_2}{s}D) \\
P^*(0,n,s) &= P^*(1,0,s)[\frac{\mu_1}{s}R^n+\frac{D_1}{s}(\mu_2-\frac{\mu_1 e_1^d e_6}{K(e_1)})e_1^n], n=1,2,...,a-1 \\
P^*(1,n,s) &= P^*(1,0,s)[R^n-\frac{D_1 e_1^{n+d} e_6}{K(e_1)}], n=1,2,3,... \\
P^*(2,n,s) &= P(1,0,s)\{e_2[\frac{R^a e_1^{n-a+1}-R^{n+1}}{e_1-r}] \\
&+ P^*(2,c-1,s)[(e_3-\frac{e_2 e_6 e_1^d}{K(e_1)})(n-a+1)e_1^n], \\
&\qquad n=a,a+1,...,c-1 \\
P^*(2,n,s) &= P(1,0,s)D_1 e_1^{n-c+1}], n\geq c
\end{aligned}
$$

and the value of $P^*(1,0,s)$ can be obtained by using the normalization condition

$$\sum_{n=0}^{a-1}P^*(0,n,s)+\sum_{n=0}^{\infty}P^*(1,n,s)+\sum_{n=a}^{\infty}P^*(2,n,s)=\frac{1}{s}, \text{as}$$

$$P^*(1,0,s) = \{D_2+s[D+\frac{D_1 e_1}{1-e_1}(1-\frac{e_1^d e_6}{K(e_1)})+\frac{1}{1-R}]\}^{-1} \tag{6.5.9}$$

$$
\begin{aligned}
\text{where}\quad D_2 &= \mu_1\frac{1-R^a}{1-R}+D\mu_2+D_1(\mu_2-\frac{\mu_1 e_1^d e_6}{K(e_1)})\frac{e_1-e_1^a}{1-e_1} \\
\text{and}\quad D &= \frac{e_2 R^a}{e_1-R}\left[\frac{e_1-e_1^{c-a+1}}{1-e_1}-\frac{R-R^{c-a}}{1-R}\right] \\
&+D_1(e_3-\frac{e_1^d e_2 e_6}{K(e_1)})\left[\frac{e_1^a-e_1^c}{(1-e_1)^2}-\frac{(c-a)e_1^c}{1-e_1}\right]
\end{aligned}
$$

Hence the Laplace transform of the busy period is

$$
\begin{aligned}
b^*(s) &= P^*(1,0,s)D_2 \\
&= \left[1+\frac{s}{D_2}[D+\frac{D_1 e_1}{1-e_1}(1-\frac{e_1^d e_6}{K(e_1)})]\right]^{-1} \qquad (6.5.10)
\end{aligned}
$$

Then the expected busy period is given by

$$
\begin{aligned}
E(T_b) &= \frac{-d}{ds} b^*(s)/_{s=0} \\
&= T_4^{-1}\left[T + \frac{T_1\theta_1}{1-\theta_1}(1 - \frac{\theta_1^d\theta_6}{K_1(\theta_1)}\right] \qquad (6.5.11) \\
\text{where } T_4 &= \mu_1\frac{1-r^a}{1-r} + T\mu_2 + T_1(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K_1(\theta_1)})\frac{\theta_1-\theta_1^a}{1-\theta_1}
\end{aligned}
$$

6.6 Busy Period in Type-I Service

In the model, the busy period of the server in type-I service begins, if the queue size becomes at least **c** and ends when the size of the queue is less than **c** for the first time, at the service completion epoch of type-I service.

Let Y(t) denote the state of the server and N(t) denote system size at time t. Let T_{b_1} denote the busy period of the server in type-I service. Then the distribution of T_{b_1} can be obtained by considering the system states
$(Y(t) = 0, N(t) = n)$, n=0,1,2,...,a-1 and $(Y(t) = 2, N(t) = n)$, n=a,a+1, ...,c-1 as absorbing with P(1,0,0)=1.

$$
\begin{aligned}
\text{Let } f_{0,n}(t) &= P\{t \le T_{b_1} < t+dt, Y(t+dt) = 0, N(t+dt) = n\}, \\
&\qquad n = 0,1,2,\ldots,a-1 \\
f_{2,n}(t) &= Pr.\{t \le T_{b_1} < t+dt, Y(t+dt) = 2, N(t+dt) = n\}, \\
&\qquad n = a, a+1, \ldots, c-1. \\
\text{Then } f_{0,n}(t) &= \frac{d}{dt}P(0,n,t), n = 0,1,2,\ldots,a-1. \\
\text{and } f_{2,n}(t) &= \frac{d}{dt}P(2,n,t), n = a, a+1, \ldots, c-1.
\end{aligned}
$$

Let $f_{0,n}^*(s)$ and $f_{2,n}^*(s)$ be the Laplace transform of $f_{0,n}(t)$ and $f_{2,n}(t)$.

$$
\begin{aligned}
\text{Then, } f_{0,1}^*(s) &= sP^*(0,n,s), n = 0,1,2,\ldots,a-1 \\
f_{2,n}^*(s) &= sP^*(2,n,s), n = a, a+1, \ldots, c-1.
\end{aligned}
$$

Let $b_1^*(s)$ be the Laplace Transform of busy period distribution of the server in type-I service.

$$\begin{aligned} b_1^*(s) &= \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=a}^{c-1} f_{2,n}^*(s) \\ &= \sum_{n=0}^{a-1} sP^*(0,n,s) + \sum_{n=a}^{c-1} sP^*(2,n,s) \end{aligned} \tag{6.6.1}$$

The Laplace transform of state probability are given by

$$sP^*(0,0,s) = \mu_1 P^*(1,0,s), \tag{6.6.2}$$

$$sP^*(0,n,s) = \mu_1 P^*(1,n,s), 0 \leq n \leq a-1 \tag{6.6.3}$$

$$(s+\lambda+\mu_1)P^*(1,0,s) - 1 = \mu_1 \sum_{n=c}^{d} P^*(1,n,s) \tag{6.6.4}$$

$$(s+\lambda+\mu_1)P^*(1,n,s) = \lambda P^*(1,n-1,s) + \mu_1 P^*(1,n+d,s), n \geq 1 \tag{6.6.5}$$

$$sP^*(2,n,s) = \mu_1 P^*(1,n,s), n = a, a+1, \ldots, c-1. \tag{6.6.6}$$

Solving equation (6.6.5) by using Rouche's theorem we get

$$P^*(1,n,s) = P^*(1,0,s)R^n, n \geq 1 \tag{6.6.7}$$

where R is the unique positive real root less than unity of the equation

$$\mu_1 Z^{d+1} - (\lambda + \mu_1 + s)Z + \lambda = 0$$

Hence,

$$P^*(0,n,s) = P^*(1,0,s)\frac{\mu_1}{s}R^n, 0 \leq n \leq a-1 \tag{6.6.8}$$

$$P^*(2,n,s) = P^*(1,0,s)\frac{\mu_1}{s}, a \leq n \leq c-1 \tag{6.6.9}$$

Now , using normalization condition

$$\sum_{n=0}^{a-1} P^*(0,n,s) + \sum_{n=0}^{\infty} P^*(1,n,s) + P^*(2,0,s) = \frac{1}{s}$$

$$\text{we get} \quad P^*(1,0,s) = \frac{1-R}{s+\mu_1(1-R^c)} \tag{6.6.10}$$

$$\text{and} \quad b_1^*(s) = \frac{\mu_1(1-R^c)}{s+\mu_1-\mu_1 R^c} \tag{6.6.11}$$

Hence the expected busy period of the server in type-I service is given by

$$\begin{aligned} Eb_1 &= \frac{-db_1^*(s)}{ds}/s=0 \\ &= \frac{1}{\mu_1(1-r^c)} \end{aligned} \tag{6.6.12}$$

where r is the unique positive real root in (0,1) of the equation

$$\mu_1\theta_1^{d+1} - (\lambda+\mu_1)\theta_1 + \lambda = 0$$

6.7 Busy Period in Type-II Service

In the model, the busy period of the server in type-II service with rate μ_2 begins only when queue size becomes less than **c** but not less than the secondary limit **a**, at the service completion epoch of type-I service. The busy period ends when the queue size becomes below the level **a** or above the level **c**, at the service completion epoch of type-II service.

Let Y(t) denote the state of the server and X(t) denote the queue size at time t. Let (T_{b_2}) be the busy period of the server in type-II service. Then the distribution of (T_{b_2}) can be obtained by considering the states $(Y(t)=0, X(t)=n)$, n=0,1,2,...,a-1 and $(Y(t)=1, X(t)=n)$, n=0,1,2,..., as absorbing.

Also assume that P(2,c-1,0) =1.

$$\text{Let} \quad g_{0,n}(t) = P\{t \le T_{b_2} < t+dt, Y(t+dt)=0, X(t+dt)=n\},$$
$$n = 0,1,2,\ldots,a-1$$

$$g_{1,n}(t) = P\{t \le T_{b_2} < t+dt, Y(t+dt)=1, X(t+dt)=n\}$$
$$n = 0,1,2,\ldots$$

$$\begin{aligned} \text{Then} \quad g_{0,n}(t) &= \frac{d}{dt}P(0,n,t), n = 0,1,2,\ldots,a-1. \\ \text{and} \quad g_{1,n}(t) &= \frac{d}{dt}P(1,n,t), n = 0,1,2,\ldots \end{aligned}$$

Let $g^*_{0,n}(s)$ and $g^*_{1,n}(s)$ be the Laplace transform of $g_{0,n}(t)$.and $g_{1,n}(t)$ respectively

Then $\quad g^*_{0,n}(s) = sP^*(0,n,s), n = 0,1,2,\ldots,a-1$

$g^*_{1,n}(s) = sP^*(1,n,s), n = 0,1,2,\ldots$

Let $b^*_2(s)$ be the Laplace transform of busy period distribution of the server in type-II service.

$$\begin{aligned} b^*_2(s) &= \sum_{n=0}^{a-1} g^*_{0,n}(s) + \sum_{n=0}^{\infty} g^*_{1,n}(s)) \\ &= \sum_{n=0}^{a-1} sP^*(0,n,s) + \sum_{n=0}^{\infty} sP^*(1,n,s) \end{aligned} \quad (6.7.1)$$

The Laplace transform of transient state probabilities are

$$sP^*(0,0,s) = \mu_2 \sum_{n=a}^{c-1} P^*(2,n,s), \quad (6.7.2)$$

$$sP^*(0,n,s) = \mu_2 P^*(2,c-1+n,s), 0 \le n \le a-1 \quad (6.7.3)$$

$$sP^*(1,0,s) = \mu_2 \sum_{n=c}^{d} P^*(2,c-1+n,s), \quad (6.7.4)$$

$$sP^*(1,n,s) = \mu_2 P^*(2,c-1+n+d,s), n \ge 1 \quad (6.7.5)$$

$$(s+\lambda+\mu_2)P^*(2,n,s) = \mu_2 P^*(2,c-1+n,s), n = a,a+1,\ldots,c-2, \quad (6.7.6)$$

$$(s+\lambda+\mu_2)P^*(2,c-1,s) - 1 = \mu_2 P^*(2,2c-2,s), \quad (6.7.7)$$

$$(s+\lambda+\mu_2)P^*(2,n,s) = \lambda P^*(2,n-1,s), n \ge c \quad (6.7.8)$$

Solving the above system of equations, we get

$$
\begin{aligned}
P^*(0,0,s) &= P^*(2,c-1,s)\frac{\mu_2}{s}\left[\frac{1-e_1+e_1^{a+1}-e_1^c}{1-e_1}\right], && (6.7.9)\\
P^*(0,n,s) &= P^*(2,c-1,s)\frac{\mu_2}{s}e_1^n, n=1,2,\dots,a-1. && (6.7.10)\\
P^*(1,0,s) &= P^*(2,c-1,s)\frac{\mu_2}{s}\left[\frac{e_1^c-e_1^{d+1}}{1-e_1}\right], && (6.7.11)\\
P^*(1,n,s) &= P^*(2,c-1,s)\frac{\mu_2}{s}e_1^{n+d}, n\geq 1 && (6.7.12)\\
P^*(2,n,s) &= P^*(2,c-1,s)e_1^{n+1}, n=a,a+1,\dots,c-2, && (6.7.13)\\
P^*(2,n,s) &= P^*(2,c-1,s)e_1^{n-c+1}, n\geq c && (6.7.14)\\
\text{where}\quad e_1 &= \frac{\lambda}{s+\lambda+\mu_2}
\end{aligned}
$$

The value of $P^*(2,c-1,s)$ can be obtained by using normalization condition

$$\sum_{n=0}^{a-1}P^*(0,n,s)+\sum_{n=0}^{\infty}P^*(1,n,s)+\sum_{n=a}^{\infty}P^*(2,n,s) = \frac{1}{s}$$

we get $$P^*(2,c-1,s)=\frac{1-e_1}{s(1-e_1^c+e_1^{a+1})+\mu_2(1-e_1^a+e_1^{a+1})} \qquad (6.7.15)$$

Hence,the Laplace Transform of busy period distribution of the server in type-II service is

$$b_2^*(s) = \frac{\mu_2(1-e_1^a+e_1^{a+1})}{s(1-e_1^c+e_1^{a+1})+\mu_2(1-e_1^a+e_1^{a+1})} \qquad (6.7.16)$$

Hence the expected busy period of the server in type-II service is given by

$$
\begin{aligned}
Eb_2 &= \frac{-db_2^*(s)}{ds}/s=0\\
&= \frac{(1-e_1^c+e_1^{a+1})}{\mu_2^2(1-e_1^a+e_1^{a+1})^2} && (6.7.17)\\
\text{where}\quad \theta_1 &= \frac{\lambda}{\lambda+\mu_2}
\end{aligned}
$$

Remarks: Here we can see that the expected busy period of the server in type-II service (6.7.17) depends on both **a** and **c** and as **c** increases the expected busy period increases. On the other hand for the increasing values of a the expected busy period of the server in type-II service decreases.

6.8 Waiting Time Distribution

Let W_q denote the waiting time of an arriving unit in the queue and w(t)be the probability density function of W_q. The arriving customer may find the system in any of the following cases.

(i) $(0,n), n = 0,1,2,....c-2$

(ii) $(0,c-1)$

(iii) $(1,n), n = dk+m, k = 0,1,..., m = 0,1,2,..a-2$

(iv) $(1,n), n = dk+m, k = 0,1,..., m = a-1, a,, ...d-1$

(v) $(2,n), n = a, a+1, ..., c-2$

(vi) $(2,n), n = c-1+dk+m, k = 0,1,..., m = 0,1,2,..a-2$

(vii) $(2,n), n = c-1+dk+m, k = 0,1,2,...; m = a-1, a, ..., d-1$

The arriving unit does not have to wait if the system is in case (ii) or in case(v). Hence the probability of no delay is

$$P(W_q = 0) = P(0,c-1) + \sum_{n=a}^{c-1} P(2,n)$$

and the probability of a delay is

$$1 - P(W_q = 0) = 1 - \{P(0,c-1) + \sum_{n=a}^{c-1} P(2,n)\}$$

In case (i), the arriving unit has to wait for the arrival of (c-n-1) more units. It has a gamma distribution with parameter λ, and (c-n-1).

$$\text{ie, f}(\lambda,\text{c-n-1};t) = \frac{\lambda^{c-n-1}}{\Gamma(c-n-1)} e^{\lambda t} t^{c-n-2}, t > 0, \lambda > 0, 0 \leq n \leq c-2.$$

In case (iii), the arriving unit has to wait for the completion of the services of (k+1) batches if (a-m-1) units arrive before the service completion of the batches or **q** $(0 \leq q \leq a-m-2)$ arrivals occur during the batch service, service of the (k+1) batches

are over and (c-m-q-1) units arrive. Let Z denote the random variable associated with case(iii). Then Z = Z_1+Z_2, where Z_1 denote the arrival of (a-m-1) units before the service completion of (k+1)batches, Z_2 denote the random variable of q arrivals occur during the batch service, service of the (k+1) batches are over and (c-m-q-1) units arrive $(0 \le q \le a-m-2)$ and the variables Z_1 and Z_2 are mutually exclusive.
Let $f_z(t)$, $f_{1,z_1}(t)$ and $f_{2,z_2}(t)$ denote the probability density function of Z, Z_1 and Z_2 respectively. Hence the probability density function of Z is given by
$f_z(t) = f_{1,z_1}(t) + f_{2,z_2}(t)$.

$$\begin{aligned}
f_{1,z_1}(t) &= f(\mu_1, k+1; t)\Gamma_t(\lambda, a-m-1) \\
f_{2,z_2}(t) &= \sum_{q=0}^{a-m-2} \int_0^t \Gamma_s(\mu_1, k+1) f(\lambda, q; s) f(\lambda, c-m-q-1; (t-s)ds \\
&= \sum_{q=0}^{a-m-2} \int_0^t (1-e^{-\mu_1 s}) \frac{\lambda^{c-m-1} e^{-\lambda t}}{\Gamma q \Gamma(c-m-q-1)} s^{q-1}(t-s)^{c-m-q-2} ds \\
&= (a-m-1) f(\lambda, c-m-1; t) \\
&\quad - \sum_{q=0}^{a-m-2} \sum_{j=0}^{k} \sum_{l=0}^{\infty} \frac{\Gamma(l+j+q)}{l! j! \Gamma q} (-1)^l (\frac{\mu_1}{\lambda})^{l+j} f(\lambda, c-m+l+j-1; t)
\end{aligned}$$

Therefore the p.d.f of Z,

$$\begin{aligned}
f_z(t) &= f(\mu_1, k+1; t)\Gamma_t(\lambda, a-m-1) + (a-m-1) f(\lambda, c-m-1; t) \\
&\quad - \sum_{q=0}^{a-m-2} \sum_{j=0}^{k} \sum_{l=0}^{\infty} \frac{\Gamma(l+j+q)}{l! j! \Gamma q} (-1)^l (\frac{\mu_1}{\lambda})^{l+j} f(\lambda, c-m+l+j-1; t)
\end{aligned} \tag{6.8.1}$$

In case (iv), the arriving unit has to wait for the service completion of (k+1) batches with rate μ_1, which follows a gamma distribution with parameter μ_1 and k+1.
(ie, $f(\mu_1, k+1; t)$).
In case (vi), the arriving unit has to wait for the service of the current batch with rate μ_2 and service of k batches with rate μ_1 if (a-m-1) units occur before the completion of batches or after the arrival of the unit **q** $(0 \le q \le a-m-2)$ arrivals occur

during the batch service, service of the (k+1) batches are over and (c-m-q-1) units arrive. Let U be the random variable associate to the case(v), which is the sum of two mutually exclusive random variables.

ie, $U = U_1 + U_2$, where U_1 denotes the random variable of (a-m-1) units arrive before the service completion of batches, U_2 represents the random variable of **q** $(0 \le q \le a - m - 2)$ arrivals occur during the batch service, service of the (k+1) batches are over and (c-m-q-1) units arrive.

Therefore the p.d.f of U, $g_u(t) = g_{1,u_1}(t) + g_{2,u_2}(t)$

Using the theory of convolution, the distribution function of batch service is given by

$$H(t) = \Gamma_t(\mu_1, k) - (\frac{\mu_1}{\mu_1 - \mu_2})^k e^{-\mu_2 t}\Gamma_t(\mu_1 - \mu_2; k) \tag{6.8.2}$$

and the corresponding p.d.f is given by

$$h(t) = (\frac{\mu_1}{\mu_1 - \mu_2})^k \mu_2 e^{-\mu_2 t}\Gamma_t(\mu_1 - \mu_2, k) \tag{6.8.3}$$

$$\begin{aligned}
g_{1,u_1}(t) &= h(t)\Gamma_t(\lambda, a-m-1) \\
g_{2,u_2}(t) &= \sum_{q=0}^{a-n-2}\int_0^t H(s)f(\lambda, q; s)f(\lambda, c-m-q-1; (t-s)ds \\
&= \sum_{q=0}^{a-n-2}\int_0^t \left[\Gamma_t(\mu_1, k) - (\frac{\mu_1}{\mu_1-\mu_2})^k e^{-\mu_2 t}\Gamma_t(\mu_1-\mu_2; k)\right]. \\
&\qquad \times \left[\frac{\lambda^{c-m-1}e^{-\lambda t}}{\Gamma q\Gamma(c-m-q-1)}s^{q-1}(t-s)^{c-m-q-2}\right] ds
\end{aligned}$$

$$\begin{aligned}
&= (a-m-1)f(\lambda, c-m-1; t) \\
&- \sum_{q=0}^{a-m-2}(\frac{\mu_1}{\mu_1-\mu_2})^k\sum_{l=0}^{\infty}(\frac{-\mu_2}{\lambda})^l\frac{f(\lambda, c-m+l-1; t)}{lB(q,l)} \\
&- \sum_{q=0}^{a-m-2}\sum_{j=0}^{k-1}\sum_{l=0}^{\infty}\left[(\frac{\mu_1}{\lambda})^j - (\frac{\mu_1}{\mu_1-\mu_2})^k(\frac{\mu_1-\mu_2}{\lambda})^j\right] \\
&\times \left[(\frac{-\mu_1}{\lambda})^l\frac{\Gamma(l+j+q)}{l!j!\Gamma q}f(\lambda, c-m+l-1; t)\right]
\end{aligned}$$

Therefore the p.d.f of U is given by

$$
\begin{aligned}
g_u(t) &= h(t)\Gamma_t(\lambda, a-m-1) + (a-m-1)f(\lambda, c-m-1;t) \\
&- \sum_{q=0}^{a-m-2}(\frac{\mu_1}{\mu_1-\mu_2})^k \sum_{l=0}^{\infty}(\frac{-\mu_2}{\lambda})^l \frac{f(\lambda, c-m+l-1;t)}{lB(q,l)} \\
&- \sum_{q=0}^{a-m-2}\sum_{j=0}^{k-1}\sum_{l=0}^{\infty}\left[(\frac{\mu_1}{\lambda})^j - (\frac{\mu_1}{\mu_1-\mu_2})^k(\frac{\mu_1-\mu_2}{\lambda})^j\right] \\
&\times \left[(\frac{-\mu_1}{\lambda})^l \frac{\Gamma(l+j+q)}{l!j!\Gamma q} f(\lambda, c-m+l-1;t)\right]
\end{aligned}
$$

In case (vii) the arriving unit has to wait for the service completion of (k+1) batches, which has the pdf

$$
h(t) = (\frac{\mu_1}{\mu_1-\mu_2})^k \mu_2 e^{-\mu_2 t}\Gamma_t(\mu_1-\mu_2, k)
$$

Therefore the waiting time distribution is given by

$$
\begin{aligned}
w(t) &= \sum_{n=0}^{a-1} P(0,n)f(\lambda, c-n-1;t) + \sum_{n=a}^{c-2} P(0,n)f(\lambda, c-n-1;t) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{a-2} P(1, dk+m)f_z(t) + \sum_{k=0}^{\infty}\sum_{m=a-1}^{d-1} P(1, dk+m)f(\mu_1, k+1;t) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{a-2} P(2, c-1+dk+m)g_u(t) + \sum_{k=0}^{\infty}\sum_{m=a-1}^{d-1} P(2, c-1+dk+m)h(t)
\end{aligned}
\tag{6.8.4}
$$

$$
\begin{aligned}
dW(t) \;=\; & P(1,0)\{[T_2\lambda + \frac{\mu_1 r}{1-r} + \frac{T_1}{1-\theta_1}(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K_1(\theta_1)})\theta_1]A_1(1,c-1,c-a-1) \\
& -\frac{\mu_1 r^{c-1}}{1-r}A_1(r,c-1,c-a-1) + T_3\lambda E(c-a-1,\lambda t) \\
& -\frac{T_1}{1-\theta_1}(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K_1(\theta_1)})\theta_1^{c-1}A_1(\theta_1,c-1,c-a-1) \\
& +(a-1)f(\lambda,c-1;t) + \mu_1 e^{-\mu_1 t}\{[\frac{r^{a-1}-r^d}{1-r} + A(r,a-1)]e^{\mu_1 r^d t} \\
& -\frac{T_1\theta_1^d\theta_6}{K_1(\theta_1)}\theta_1^m[E(a-1,\lambda t) - 1 + (\frac{\theta_1^{a-1}-\theta_1^d}{1-\theta_1} + A(\theta_1,a-1))e^{\mu_1\theta_1^d t}]\} \\
& -\sum_{q=0}^{a-2}\sum_{l=0}^{\infty}(\frac{-\mu_1}{\lambda})^l\frac{f(\lambda,c+l-1;t)}{lB(q,l)} + \sum_{m=1}^{a-2}(r^m - \frac{T_1\theta_1^d\theta_6}{K_1(\theta_1)}\theta_1^m) \\
& \times[(a-m-1)f(\lambda,c-m-1;t) - \sum_{q=0}^{a-m-2}\sum_{l=0}^{\infty}(\frac{-\mu_1}{\lambda})^l\frac{f(\lambda,c+l-1;t)}{lB(q,l)}] \\
& +\sum_{k=1}^{\infty}\sum_{m=0}^{a-2}[r^{dk+m} - \frac{T_1\theta_1^d\theta_6}{K_1(\theta_1)}\theta_1^{dk+m}] \times [(a-m-1)f(\lambda,c-m-1;t) \\
& -\sum_{q=0}^{a-m-2}\sum_{j=0}^{k}\sum_{l=0}^{\infty}\frac{\Gamma(l+j+q)}{l!j!\Gamma q}(-1)^l(\frac{\mu_1}{\lambda})^{l+j}f(\lambda,c-m+l+j-1;t)] \\
& +T_1\mu_1\mu_2[\frac{\theta_1^{a-1}-\theta_1^d}{1-\theta_1} + A(\theta_1,a-1)] \times [\frac{e^{-\mu_2 t} - e^{-\mu_1(1-\theta_1^d)t}}{\mu_2(1-\theta_1^d)-\mu_2}] \\
& +\frac{T_1}{1-\theta_1^d}\sum_{m=0}^{a-2}\theta_1^m(a-m-1)f(\lambda,c-m-1;t) \\
& -T_1\sum_{k=0}^{\infty}\sum_{m=0}^{a-2}\theta_1^m\sum_{q=0}^{a-m-2}\sum_{l=0}^{\infty}[\frac{(\mu_1-\mu_2)}{\mu_1(1-\theta 1^d)-\mu_2}(\frac{\mu_1}{\mu_1-\mu_2})^k(\frac{-\mu_2}{\lambda})^l \\
& \frac{f(\lambda,c-m+l-1;t)}{lB(q,l)} - \sum_{j=0}^{k-1}[(\frac{\mu_1}{\lambda})^j - (\frac{\mu_1}{\mu_1-\mu_2})^k(\frac{\mu_1-\mu_2}{\lambda})^j] \\
& \times(\frac{-\mu_1}{\lambda})^l\frac{\Gamma(l+j+q)}{l!j!\Gamma q}f(\lambda,c-m+l+j-1;t)]]\}, \quad t \geq 0 \qquad (6.8.5)
\end{aligned}
$$

Expected waiting time $E(w_t) = \int_0^\infty w(t)dt$

$$
\begin{aligned}
E(W(t)) = & P(1,0)\{[T_2\lambda + \frac{\mu_1 r}{1-r} + \frac{T_1}{1-\theta_1}(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K_1(\theta_1)})\theta_1]B_1(1,c-1,c-a-1) \\
& -\frac{\mu_1 r^{c-1}}{1-r}B_1(r,c-1,c-a-1) + T_3\frac{(c-a-1)(c-a)}{\lambda} \\
& -\frac{T_1}{1-\theta_1}(\mu_2 - \frac{\mu_1\theta_1^d\theta_6}{K_1(\theta_1)})\theta_1^{c-1}B_1(\theta_1,c-1,c-a-1) \\
& +B_1(1,c-1,c-a-1) + (1+T_2T_3\theta_4)\frac{(c-a)(c-a-1)}{2\lambda} \\
& +\mu_1\{[\frac{r^{a-1}-r^d}{(1-r)\mu_1^2(1-r^d)^2} + B(r,a-1,\theta_{\mu_1(1-r^d)})] - \frac{T_1\theta_1^d\theta_6}{K_1(\theta_1)}[\frac{a(a-1)}{\lambda^2} - \frac{1}{\mu_1^2} \\
& +(\frac{\theta_1^{a-1}-\theta_1^d}{(1-\theta_1)\mu_1^2(1-\theta_1^d)^2} + B(\theta_1,a-1),\theta_{\mu_1(1-\theta_1^d)})]\} \\
& +\frac{(a-1)(c-1)}{\lambda}[\frac{T_1}{1-\theta_1}(1+\frac{\theta_1^{2d}\theta_6}{K_1(\theta_1)}) + \frac{1}{1-r^d}] \\
& -\frac{(c-1)(1-\theta_4^{a-1})}{\lambda(1-\theta_4)} + \frac{\theta_4\mu_1}{\lambda}K(a-1,\theta_4) \\
& -\frac{T_1(\mu_1-\mu_2)}{\mu_1(1-\theta_1^d)-\mu_2}[\frac{(c-1)(1-\theta_1^{a-1})}{\lambda(1-\theta_1)} - \frac{\theta_1\mu_1}{\lambda}K(a-1,\theta_1)] \\
& +\frac{1}{\lambda}[\frac{L_2(a-1,r)}{1-r^d} - L_3(a-1,r,\theta_{\mu_1})] - \frac{\theta_1^d\theta_6}{\lambda K_1(\theta_1)}[\frac{L_2(a-1,\theta_1)}{1-\theta_1^d} \\
& -L_3(a-1,\theta_1,\theta_{\mu_1})] - \sum_{k=1}^{\infty}\sum_{m=0}^{a-2}[r^{dk+m} - \frac{T_1\theta_1^d\theta_6}{K_1(\theta_1)}\theta_1^{dk+m}]\sum_{j=0}^{k}(\frac{\mu_1}{\lambda})^j L(q,j) \\
& +\frac{T_1\mu_1\mu_2}{\mu_1(1-\theta_1^d)-\mu_2}[B(\theta_1,a-1,\theta_{\mu_2}) + \frac{\theta_1^{a-1}-\theta_1^d}{1-\theta_1}\frac{\mu_1^2(1-\theta_1^d)^2-\mu_2}{\mu_1^2\mu_2^2(1-\theta_1^d)^2} \\
& -B(\theta_1,a-1,\theta_{\mu_1(1-\theta_1^d)})] + T_1[\frac{L_2(a-1,\theta_1)}{\lambda(1-\theta_1)} - \frac{(\mu_1-\mu_2)L_3(a-1,\theta_1,\theta_{\mu_2})}{\lambda(\mu_1(1-\theta_1^d)-\mu_2)}] \\
& -T_1\sum_{k=0}^{\infty}\sum_{m=0}^{a-2}\theta_1^{dk+m}\sum_{j=0}^{k-1}[(\frac{\mu_1}{\lambda})^j - (\frac{\mu_1}{\mu_1-\mu_2})^k(\frac{\mu_1-\mu_2}{\lambda})^j]L(q,j)]\}
\end{aligned}
$$

where

$$
\begin{aligned}
L(q,j) &= \sum_{q=0}^{a-m-2}\frac{\theta_4^{q+j}}{jB(q,j)}[(c-m+j-1)-(q+j)\theta_4], \\
\theta_d &= \frac{\lambda}{\lambda+\mu_1(1-\theta_1^d)} \quad \text{and} \quad \theta_r = \frac{\lambda}{\lambda+\mu_1(1-r^d)}
\end{aligned}
$$

6.9 Optimal Control Limits

In this section, a cost function is defined by incorporating the expected queue length and expected busy periods of server in type-I and type-II services to find an optimal values for the control limits '**a**' and '**c**'. In this model, the busy period of the server may include the busy periods of the server in type-I or both type-I and type-II depending on the queue size at the service completion of batches. So the cost function defined in terms of the expected queue length and the busy periods of the server in type-I and type-II services gives an optimal value for the control limits. Here, assume that costs are charged for unit time of the server in type-I and type-II services and for holding the customers in the queue. Also, assume that an over head cost is charged to initiate the batch service.

Let C_0 be the over head cost to initiate a batch service, C_h be the holding cost per unit time for a customer in the queue, C_1 denote cost per unit time for the server in type-I service and C_2 be the cost per unit time for the server in the type-II service. Then the expected cost function can be defined by

$$E_{a,c}cost = C_0 + C_h L_q + C_1 E_{b_1} + C_2 E_{b_2} \tag{6.9.1}$$

where

L_q = expected queue length,

E_{b_1}= expected busy period of the server in type-I service and

E_{b_2}= expected busy period of server in type-II service

6.10 Numerical Illustration

A numerical analysis of the model is discussed in this section, we have computed steady state probabilities and performance measures of the model for an arbitrary **set:** λ **= 0.9**, μ_1 **=0.5**, μ_2 **=0.8** and for different values of **a** and **c**. Also, we have obtained an optimal solution for the control limit **a** and **c**.

The steady state probabilities queue size can be computed by using the equations (6.3.9) to (6.3.14). For a=7, c=16 and d=20 the steady state probabilities are given by

Table 6.10.1

Steady state probabilities

Lq (for λ=0.9, μ_1=0.5, μ_2=0.8, a=7, c=16)

n	P(0,n)	P(1,n)	P(2,n)	n	P(0,n)	P(1,n)	P(2,n)
0	0.0041	0.0038	0	9	0.0002	0.0026	0.0004
1	0.0350	0.0028	0	10	0.0698	0.0001	0.0003
2	0.0517	0.0020	0	11	0.0698	9.547e-005	0.0002
3	0.0658	0.0014	0	12	0.0698	6.822e-005	0.0002
4	0.0607	0.0010	0	13	0.0698	4.875e-005	0.0001
5	0.0656	0.0007	0	14	0.0698	3.484e-005	0.00007
6	0.0683	0.0005	0	15	0.0698	2.489e-005	0.0331
7	0.0698	0.0004	0.0005	16	0	1.779e-005	0.0175
8	0.0698	0.0003	0.0010	17	0	1.271e-005	0.0093

The expected queue length Lq computed by using the equation (6.4.1) for **d=20** and for different values of **a** and **c** are given by

Table-6.10.2

Expected queue length (for λ=0.9, μ_1=0.5, μ_2=0.8,d=20)

c\a	0	1	2	3	4	5	6
8	3.7247	3.9650	4.1895	4.3732	4.5127	4.6144	4.6879
9	4.1988	4.4558	4.6929	4.8854	5.0311	5.1375	5.2147
10	4.6851	4.9589	5.2082	5.4082	5.5580	5.6664	5.7445
11	5.1799	5.4703	5.7311	5.9374	6.0899	6.1989	6.2763
12	5.6806	5.9870	6.2584	6.4702	6.6244	6.7328	6.8086
13	6.1849	6.5063	6.7875	7.0039	7.1591	7.2665	7.3402
14	6.6909	7.0264	7.3164	7.5367	7.6925	7.7986	7.8701
15	7.1975	7.5458	7.8436	8.0674	8.2235	8.3283	8.3975

Remarks: Here it can be seen that the expected queue length increases as **a** and **c** increases.

The expected busy period of the server in type-I service(E_{b_1}) calculated by using the equation (6.6.12) for d=20 and for different values of **a** and **c** are given by

Table-6.10.3

Expected busy period in type-I service (for λ=0.9, μ_1=0.5, μ_2=0.8,d=20)

c\a	0	1	2	3	4	5	6
8	2.1459	2.1459	2.1459	2.1459	2.1459	2.1459	2.1459
9	2.1021	2.1021	2.1021	2.1021	2.1021	2.1021	2.1021
10	2.0719	2.0719	2.0719	2.0719	2.0719	2.0719	2.0719
11	2.0509	2.0509	2.0509	2.0509	2.0509	2.0509	2.0509
12	2.0361	2.0361	2.0361	2.0361	2.0361	2.0361	2.0361
13	2.0257	2.0257	2.0257	2.0257	2.0257	2.0257	2.0257
14	2.0183	2.0183	2.0183	2.0183	2.0183	2.0183	2.0183
15	2.0130	2.0130	2.0130	2.0130	2.0130	2.0130	2.0130

Remarks: Here we can observe that the expected busy period of the server in type-I service decreases for increasing values of '**c**' but '**a**' has no influence on E_{b_1}.

The expected busy period of the server in type-II service(E_{b_2}) computed by using the equation (6.7.17) for d=20 and for different values of **a** and **c** are given by

Table-6.10.4

Expected busy period in type-II service (for λ=0.9, μ_1=0.5, μ_2=0.8, d=20)

c\a	0	1	2	3	4	5	6
8	3.5310	2.3682	1.9366	1.7444	1.6513	1.6041	1.5798
9	3.5391	2.3742	1.9419	1.7493	1.6560	1.6088	1.5844
10	3.5433	2.3774	1.9446	1.7519	1.6585	1.6112	1.5868
11	3.5456	2.3791	1.9461	1.7533	1.6598	1.6125	1.5881
12	3.5468	2.3800	1.9469	1.7540	1.6605	1.6132	1.5888
13	3.5474	2.3805	1.9473	1.7544	1.6609	1.6136	1.5891
14	3.5478	2.3807	1.9475	1.7546	1.6611	1.6138	1.5893
15	3.5479	2.3809	1.9476	1.7547	1.6612	1.6139	1.5894

Remarks: Here it is clear that the expected busy period of the server in type-II service increases as '$\underline{c}$' increases and decreases for increasing values of '$\underline{a}$'. This shows that the secondary limit '$\underline{a}$', introduced in the general bulk service rule makes the model more realistic.

The expected cost ($E_{a,c}cost$) can be obtained by using the equation (3.10.1). The table (3.12.5) below gives ($E_{a,c}cost$) for the chosen arbitrary set when d=20, C_0=200,C_h=25, C_1=100 and C_2=30

Table-6.10.5

Expected Cost (for λ=0.9, μ_1=0.5, μ_2=0.8, d=20, C_0=200,C_h=25, C_1=100, C_2=30)

c\a	0	1	2	3	4	5	6	7	8
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1	740.27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	657.21	629.54	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	623.58	594.84	586.51	0.00	0.00	0.00	0.00	0.00	0.00
4	609.94	580.72	572.61	570.45	0.00	0.00	0.00	0.00	0.00
5	606.25	576.94	569.06	**567.33**	567.44	0.00	0.00	0.00	0.00
6	608.20	579.04	571.42	570.00	570.47	571.38	0.00	0.00	0.00
7	613.64	584.76	577.43	576.25	576.95	578.07	579.18	0.00	0.00
8	621.36	592.84	585.79	584.83	585.67	586.91	588.11	589.16	0.00
9	630.62	602.49	595.74	594.96	595.90	597.19	598.41	599.47	600.37

Remarks: Here we can note that the expected cost is minimum when **a** is less than **c** and it attains the minimum at (a=3, c=6). The optimum cost for λ=0.9, μ_1=0.5, μ_2=0.8 when C_0=100, C_h=20, C_1=80, C_2= 30 is **567.33**. So the optimal control limits are **a=3**, **c=7**.