Chapter 4

Biorthogonal set of polynomials associated with modified Laguerre polynomials

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Biorthogonal set of polynomials associated with modified Laguerre polynomials

(4.1) INTRODUCTION

In this chapter, we shall study the generalizations of biorthogonal polynomials suggested by modified Laguerre polynomials over the interval \((0, \infty)\) with respect to the distribution function \(x^\alpha \exp(-ax/b)\) and shall obtain associated generating relations for biorthogonal polynomials \(L_{\alpha,b,k,n}(x)\) and \(M_{\alpha,b,k,m}(x)\).

Goyal [6] introduced a modification of Laguerre polynomials defined by generating relation:

\[
(4.1.1) \quad \sum_{n=0}^{\infty} L_{\alpha,b,m,n}(x) w^n = (1-bw)^m \exp[-awx/1-bw] \]

Here when \(m = \alpha + 1, b = a = 1\), we get generalised Laguerre polynomials \(L_{\alpha}(x)\), defined by

\[
(4.1.2) \quad \sum_{n=0}^{\infty} L_{\alpha}^{(\alpha)}(x) t^n = (1+t)^{-\alpha-1} \exp[-x t/(1-t)] \]

Konhauser [7] considered two classes of polynomials \(Y_{\alpha}^{(\alpha)}(x;k)\) in \(x\) and \(Z_{\alpha}^{(\alpha)}(x;k)\) in \(x^k\), \(\alpha > -1, k = 1,2,3,\ldots\)

These polynomials reduce to the Laguerre polynomials \(L_{\alpha}(x)\) in the special cases when \(k=1\), and form a pair of biorthogonal polynomials with biorthogonality relation as:
\[ (4.1.3) \quad \int_0^\infty x^\alpha e^{-x} Y(x;k) Z(x;k) \, dx = \frac{\Gamma(1+\alpha+k_j)}{j!} \delta_{ij}, \]
\[ \forall \ i, j \in \{0, 1, 2, 3, \ldots\}. \]

where \( k \) is a positive integer and \( \delta_{ij} \) denotes the Kronecker delta.

### (4.2) Generalized Modified Laguerre Polynomials

In this section, we propose below a pair of the biorthogonal sets of polynomials associated with \( L_{\alpha,b,k,n}(x) \) as:

\[ (4.2.1) \quad L_{\alpha,b,k,n}(x) = \sum_{p=0}^{\infty} \frac{(a/b)^p (-1)^j}{p!} \left( \frac{(1+\alpha+j)}{k_n} \right) \]

and

\[ (4.2.2) \quad M_{\alpha,b,k,n}(x) = \frac{\Gamma(1+\alpha+km)}{(a/b)^{kn}} \sum_{m=0}^{\infty} \left( -1 \right)^{\frac{km}{m!}} \frac{(s/b)_m}{(s/b)^m} \frac{\Gamma(1+\alpha+km)}{x^m} \]

Interestingly, both these polynomials possesses the following Rodrigue's formula:

\[ (4.2.3) \quad L_{\alpha,b,k,n}(x) = \frac{1}{n!} x^{-\alpha-kn-1} \exp(ax/b) [(x^{k+1}D)^n \{x^{\alpha+1}\exp(-ax/b)\}] \]

and

\[ (4.2.4) \quad M_{\alpha,b,k,n}(x) = \frac{\Gamma(1+\alpha+kn)}{(a/b)^{-\alpha-kn}} \frac{x^{-\alpha}}{n!} \frac{\Delta^n_{\alpha,k}}{\Gamma(1+\alpha)} \]
where $\Delta$ is the difference operator and defined as,

$$\Delta f(\alpha) = f(\alpha+k) - f(\alpha).$$

(4.2.5) \hspace{1cm}

**BIORTHOGONAL RELATION**

The polynomial sets $L_{a,b,k,n}(x)$ and $M_{a,b,k,m}(x)$ are biorthogonal with respect to the weight function $x^\alpha \exp(-ax/b)$ over the interval $(0,\infty)$. The biorthogonal relation is given by

$$\int_0^\infty x^\alpha \exp(-ax/b) \, L_{a,b,k,n}(x) \, M_{a,b,k,m}(x) \, dx$$

$$= \frac{\Gamma(1+\alpha+km)}{(a/b)^{\alpha+km} \, m!} \delta_{m,n}, \forall m,n \in \{0,1,2,\ldots\}$$

where $1+\alpha > 0$ and $k$ is positive integer and $\delta_{m,n}$ is the Kronecker delta.

We shall prove this result later on.

(4.3) **GENERATING FUNCTIONS**

**Hypergeometric form:**

From (4.1.5), we have

$$M_{a,b,k,n}(x) = \frac{\Gamma(1+\alpha+km)}{\sum (-1)^m \, (a/b)^{km} \, x} \sum_{m=0}^{kn} \frac{(a/b)^{km}}{m!}$$

$$n \, m \, n$$

$$\frac{(a/b)^{km}}{m!}$$

$$\Gamma(1+\alpha+km)$$
\[
\begin{align*}
\frac{(1+\alpha)^{kn}}{n!} &= \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \left\{ \frac{(ax/b)^{km}}{(1+\alpha)(1+\alpha+1)\ldots(1+\alpha+km-1)} \right\} \\
\frac{(1+\alpha)^{kn}}{n!} &= \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \left\{ \frac{(ax/b)^{km}}{(1+\alpha)/k(1+\alpha+1)/k\ldots(1+\alpha+k-1)/k} \right\} \\
\frac{(1+\alpha)^{kn}}{n!} &= \frac{\mathbf{1}_F_k [\left[-n; (1+\alpha)/k, (2+\alpha)/k, \ldots, (1+\alpha+k-1)/k; (ax/b)^{km}\right]}{n!} \\
\text{(4.3.1) gives the required hypergeometric form.}
\end{align*}
\]

**GENERATING FUNCTION:**

From equation (4.2.2), we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{1/(1+\alpha)^{kn}}{n!} M_{a, b, k, n}^{(\alpha)}(x) t^n &= \sum_{m=0}^{\infty} \frac{t^n}{(a/b)^{km} n!} \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \left\{ \frac{(ax/b)^{km}}{(1+\alpha)^{km}} \right\} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m n+m}{(a/b)^{km} n!} M_{a, b, k, n}^{(\alpha)}(x) t^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m n+m}{(a/b)^{kn} n!} M_{a, b, k, n}^{(\alpha)}(x) t^n
\end{align*}
\]
\[\sum_{n=0}^{\infty} \frac{\alpha^{n}}{n!} \frac{x^{n}}{k^{n}} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} \frac{\Gamma(\alpha+1/k) \Gamma(\alpha+2/k) \ldots \Gamma(1+\alpha+m/k)}{m^{m}} \frac{(x^{m})^{k}}{m^{m}} \]

\[(4.3.2) = \exp\left[\frac{\alpha}{k}\right] \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+2/k) \ldots \Gamma(1+\alpha+m/k)}{m^{m}} \frac{(x^{m})^{k}}{m^{m}} \frac{(-x/k)^{k} t}{(a/b)^{k}} \]

which gives the required generating function.

\textit{Generating function for } I_{\alpha, b, k, n} (x) \; ;

We write equation (4.2.3) as:

\[\sum_{n=0}^{\infty} I_{\alpha, b, k, n} (x) t = \]

\[= \sum_{n=0}^{\infty} \frac{1}{n!} \alpha t^{n} \exp(ax/b) (x^{1+1}D)^{n} \{x^{\alpha+1} \exp(-ax/b)\} \]

\[= x \exp(ax/b) \left[ \sum_{n=0}^{\infty} \frac{t^{n}}{n!} (x^{1+1}D)^{n} \{x^{\alpha+1} \exp(-ax/b)\} \right] \]

\[= x \exp(ax/b) \left[ \frac{x^{1+\alpha}}{(1-x^{k}kt)^{(1+\alpha)/k}} \exp\left\{\frac{-ax}{b(1-x^{k}kt)^{1/k}}\right\} \right] \]

\[= x \exp(ax/b) \left[ (1-x^{k}kt)^{(1+\alpha)/k} \exp\left\{(ax/b)(1-(1-x^{k}kt)^{-1/k})\right\} \right] \]

\[= x^{1+\alpha}(1-x^{k}kt)^{(1+\alpha)/k} \exp\left\{(ax/b)(1-(1-x^{k}kt)^{-1/k})\right\} \]
(4.4) BIORTHOGONAL RELATION

To prove (4.2.6), Consider the following relation:

\[
\sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \int \frac{x^s}{(1+\alpha)^{km}} \exp(-ax/b) \text{ } L_{a,b,k,n}(x) \text{ } M_{a,b,k,n}(x) \text{ } w^su^m \text{ } dx
\]

\[
= \int x^\alpha \exp(-ax/b) \left\{ \sum_{s=0}^{\infty} L_{a,b,k,n}(x) \text{ } w^s \right\} \sum_{m=0}^{\infty} M_{a,b,k,n}(x) \text{ } u^m \text{ } dx
\]

\[
= \int x^\alpha \exp(-ax/b) \left\{ (1-bw)^{-s} \exp\{-axw/(1-b)\} \right\} \left\{ \exp\{u/(a/b)^k\} \right\}
\]

\[
* \quad _0F_k\left[ \begin{array}{c}
\alpha+1 \\
\alpha+2 \\
\vdots \\
1+\alpha+k-1 \\
\end{array} \right] ; \quad \left[ \frac{(-x/k)^k}{u} \right]
\]

\[
\text{using (4.1.1) and (4.3.2)}
\]

\[
= \left\{ (1-bw)^{-s} \exp\{u/(a/b)^k\} \right\} \sum_{m=0}^{\infty} \int x^\alpha \exp\{-ax/b(1-bw)\} \text{ } (-u)^m
\]

\[
* \quad \frac{(x/k)^{km}}{k} \frac{(x/k)^{km}}{k} \text{ } dx
\]

\[
= \left\{ (1-bw)^{-s} \exp\{u/(a/b)^k\} \right\} \sum_{m=0}^{\infty} \frac{(-u)^m}{b(1-bw)} \frac{\Gamma(\alpha+km+1)}{\Gamma(1+km+1)}
\]

\[
= \frac{\Gamma(1+\alpha)}{(a/b)^{1+\alpha}} \exp\{u/(a/b)^k\} \sum_{m=0}^{\infty} \left\{ \frac{(1-bw)}{(a/b)^k} \right\}^m \frac{(-u)^m}{m!}
\]
\[
\frac{\Gamma(1+\alpha)}{(a/b)^{1+\alpha}} \exp\left[u/(a/b)^k \{1-(1-bw)^k\}\right]
\]

\[
= \frac{\Gamma(1+\alpha)}{(a/b)^{1+\alpha}} \sum_{m=0}^{\infty} \frac{u \{1-(1-bw)^k\}}{(a/b)^k}^m
\]

(4.4.1)

Comparing the coefficient of \(w^n u^m\) on both sides of (4.4.1), we see that coefficient of \(w^n u^m\), when \(n \neq m\), are zero and when \(n = m\) is non-zero and thus;

\[
\int_0^\infty x^\alpha \exp\left(-ax/b\right) L_{a,b,k,n}^{(c)}(x) M_{a,b,k,n}^{(c)}(x) \, dx
\]

\[
= \frac{\Gamma(1+\alpha+kn)}{m! (a/b)^{1+\alpha+km}} \delta_{m,n} \quad \forall m, n \in \{0, 1, 2, \ldots \}
\]

which proves the required biorthogonal relation (4.2.6).

(4.5) INTEGRAL REPRESENTATIONS

(1) Integral Representation for \(L_{a,b,k,n}^{(c)}(x)\)

Osler [8] has given a fractional derivative formula as:

\[
D_{g(z)}^{(\alpha)} \{F(z)\} = D_{h(z)}^{(\alpha)} \left\{ \frac{F(z)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\}_{w=z}
\]

(4.5.1)

where \(D_{g(z)}^{(\alpha)} \{F(z)\}\) denotes the fractional derivative of order \(\alpha\) with respect to \(g(z)\) for \(\alpha = m\) and \(h(z) = z\), \(h(w) = w\), we have

\[
D_{g(z)}^{(\alpha)} \{F(z)\} = D_z^{(\alpha)} \left\{ \frac{F(z)g'(z)}{g(z) - g(w)} \right\}_{w=z}^{m+1}
\]

(4.5.2)
\[ \forall m \in \{0, 1, 2, \ldots\}, \]

For the relatively more familiar derivative of order \( m \).

Now,

\[
(\chi^{k+1}D_\chi)^n [\chi^{1+\alpha} \exp(-ax/b)] = D_\chi \left[ w^{1+\alpha} \exp(-aw/b) w^{-k-1} \right] \]

\[ \quad \left[ \begin{array}{c|c}
\frac{w-x}{x-w} & \frac{n+1}{w=x} \\
\hline
-k & -k
\end{array} \right]
\]

[using (4.5.2)]

\[ (4.5.3) = D_\chi \left[ w^{1+\alpha} \exp(-aw/b) w^{-k-1} \right] \left( \frac{w-x}{x-w} \right)^{n+1} \]

Putting \( x = (1+t)w \) and \( D_\chi = 1/w D_t \) in R.H.S of (4.5.3), equation (4.5.3) becomes:

\[ (4.5.4) \quad (\chi^{k+1}D_\chi)^n [\chi^{1+\alpha} \exp(-ax/b)] = k^{n+1} D_t \left[ \frac{1+\alpha}{1+t} \right]_0^{\infty} \]

Using (4.5.4), equation (4.2.3) yields:

\[ (4.5.5) \quad L_{a,b,k,n}(x) = \frac{1}{n!} \left[ \frac{k^{n+1}}{n} \right] D_t \left[ \frac{1+\alpha}{1+t} \right]_0^{\infty} \]

Cauchy's generalized integral formula about the point \( t = z \) in the Contour \( C \) is given by

\[ (4.5.6) \quad D_z [f(z)] = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} \, dt \]

which for \( t = z + u \) takes the form:
(4.5.7) \[ D_z [f(z)] = \frac{n!}{2\pi i} \int_{C'} \frac{f(z+u)}{(u)^{n+1}} \, du \]

where C' is a contour enclosing the point \( u = 0 \).

Using (4.5.6), we see that (4.5.5) is equivalent to,

(4.5.8) \[ I_{n,b,k,n}^{(\alpha)}(x) = \frac{k^{n+1}}{2\pi i} \int_{C'} \left[ \frac{\exp[-ax/\beta(1/(1+t)-1)]}{(1+t)^{(1+\alpha)}} \right] \frac{1}{(1+t)^{(1+\alpha)}} \, dt \]

which is the required integral representation.

where C is the closed contour enclosing \( t = 0 \), but excluding \( t = -1 \) and roots of the equation \( (1+t)^{k-1} = 0 \).

(ii) INTEGRAL REPRESENTATION FOR \( I_{n,b,k,n}^{(\alpha)}(x) \):

From (4.2.2), we have:

(4.5.9) \[ \int_0^\infty \exp[-(a/b)t] M_{n,b,k,n}(xt) \, t^\theta \, dt \]

\[ = \frac{\Gamma(1+\alpha+kn)}{n! \, (a/b)^n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \, \Gamma(1+\alpha+km)} \int_0^\infty \exp(-agt/b) \, t^{\beta+km} \, dt \]

\[ = \frac{\Gamma(1+\alpha+kn)}{n! \, (a/b)^n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \, \Gamma(1+\beta+km)} \left\{ (x/g)^{\beta+km} \right\}^m \]

\[ = \frac{\Gamma(1+\alpha+kn)}{n! \, (a/b)^n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \, \Gamma(1+\alpha+km)} \left\{ (x/g)^{\beta+km} \right\}^m \]
\[
\frac{\Gamma \left( 1+\alpha+kn \right)}{(a/b)^{1+\alpha+kn}} \sum_{m=0}^{n} \frac{\left( -n \right)^{m} \left( 1+\beta \right)^{kn}}{m! \left( 1+\alpha \right)^{km}} \left( \frac{x}{g} \right)^{k} \\
\]

\[
= \frac{(1+\alpha)^{kn} \Gamma \left( 1+\beta \right)}{(a/b)^{1+\alpha+kn}} \frac{\sum_{m=0}^{n} \left( -n \right)^{m} \left( 1+\beta \right)^{2m+1} ... \left( 1+\beta+1 \right)^{m} / k ;}{\left( x/g \right)^{k}} 
\]

(4.5.10)

In particular, when \( \beta = \alpha \), (5.3.10) reduces to the following form:

\[
(4.5.11) \quad \int_{0}^{\infty} \exp \left( -agt/b \right) M_{\alpha, b, k, n} \left( x, t \right) t^{\alpha} \, dt \\
= \frac{\Gamma \left( 1+\alpha+kn \right)}{(g-x)} \frac{k^{k} n^{n} \left( 1+\alpha+kn \right)^{1+\alpha+kn}}{(a/b)^{1+\alpha+kn} \left( g-x \right)^{1+\alpha+kn} \left( a/b \right)^{1+\alpha+kn}} 
\]

Substituting, \( g = vx \) and \( a t/b = u/x \), in (4.5.11) changes to:

\[
(4.5.12) \quad \int_{0}^{\infty} \exp \left( -uv \right) M_{\alpha, b, k, n} \left( bu/a \right) \, u^{\alpha} \, du \\
= \frac{\Gamma \left( 1+\alpha+kn \right)}{(v-1)} \frac{k^{k} n^{n} \left( 1+\alpha+kn \right)^{1+\alpha+kn}}{(v)^{1+\alpha+kn} \left( a/b \right)^{1+\alpha+kn}} 
\]

or

\[
(4.5.13) \quad L \left[ u^{\alpha} M_{\alpha, b, k, n} \left( bu/a \right) ; v \right] = \frac{\Gamma \left( 1+\alpha+kn \right)}{(v^{k}-1)^{u}} \frac{k^{k} n^{n} \left( 1+\alpha+kn \right)^{1+\alpha+kn}}{(v)^{1+\alpha+kn} \left( a/b \right)^{1+\alpha+kn}} 
\]
where $L \ldots$ denotes the Laplace's transform.

Hence by inverse Laplace's transform;

\[
(4.5.14) \quad M_{a,b,k,n}(bu/a)u^\alpha
\]

\[
= \frac{1}{2\pi i} \int \frac{\exp(uv)\Gamma(1+\alpha+kn)v^{n-1}}{c^n v^{(1+\alpha+kn)}(a/b)} \, dv
\]

where $c$ is a contour enclosing $v = 0$.

Substituting, $bu/a = x$, $v = w$; $(4.5.14)$ changes to;

\[
(4.5.15) \quad M_{a,b,k,n}(x)\frac{\alpha^{\alpha+kn}}{n!x^{(a/b)}}
\]

\[
= \frac{1}{2\pi i} \int \frac{\exp(axw/b)(w-1)^n}{c^{(1+\alpha+kn)}w^{(1+\alpha+kn)}} \, dw
\]

Relation $(4.5.15)$ gives the integral representation for $M_{a,b,k,n}(x)$.

If $\alpha+kn$ is positive integer, then by Cauchy's integral formula;

\[
(4.5.16) \quad (a/b)n!x^\alpha M_{a,b,k,n}(x)
\]

\[
= \frac{1}{\Gamma(1+\alpha+kn)}
\]

\[
= \frac{1}{(\alpha+kn)!}
\]

\[
D_g \left[ (g^{k-1})^n \exp(\alpha x g/b) \right]_{g=0}^{\alpha+kn}
\]
In particular, the above results reduce to corresponding results of Spencer and Fano [9] and Konhauser [7].

(4.6) RECURRANCE RELATIONS

The polynomials $L_{a,b,k,n}(x)$ and $M_{a,b,k,n}(x)$ satisfy the following recurrence relations:

Recurrent relations for $L_{a,b,k,n}(x)$

\[(4.6.1) \quad (bD_x - a) L_{a,b,k,n}(x) = -a L_{a,b,k,n}(x)\]

\[(4.6.2) \quad (bD_x - a)^m L_{a,b,k,n}(x) = (-1)^m a^m L_{a,b,k,n}(x)\]

\[(4.6.3) \quad (-a^{-1}bD_x + 1)^m L_{a,b,k,n}(x) = k L_{a,b,k,n-1}(x) + L_{a,b,k,n}(x)\]

\[(4.6.4) \quad (xD_x - ab^{-1}x - k + \alpha) L_{a,b,k,n}(x) = -(n+1) L_{a,b,k,n+1}(x)\]

\[(4.6.5) \quad (k - \alpha) L_{a,b,k,n}(x) + a b^{-1} L_{a,b,k,n}(x) = (n+1) L_{a,b,k,n+1}(x)\]

\[(4.6.6) \quad [xD_x + (a/b)x + k(n+1) - \alpha + 1] L_{a,b,k,n}(x) = (n+1) L_{a,b,k,n+1}(x)\]

Recurrent relations for $M_{a,b,k,n}(x)$

\[(4.6.7) \quad (x^{-k}D_x) M_{a,b,k,n}(x) = (-k) M_{a,b,k,n-1}(x)\]
\[(4.6.8) \quad (x^{1-k}D_x)^m M_{a,b,k,n}(x) = (-k)^m M_{a,b,k(n-m)}(x)\]

\[(4.6.9) \quad (x D_x - k n) M_{a,b,k,n}(x) = \frac{(\alpha) \Gamma(1+\alpha+kn)}{\Gamma(1+\alpha+k(n-1))} M_{a,b,k(n-1)}(x)\]

\[(4.6.10) \quad n M_{a,b,k,n}(x) + (\alpha) M_{a,b,k,n-1}(x) = \frac{\Gamma(1+\alpha+kn)}{\Gamma(1+\alpha+k(n-1))} M_{a,b,k(n-1)}(x)\]

\[(4.6.11) \quad (xD_x^\alpha) M_{a,b,k,n}(x) = (\alpha+kn) M_{a,b,k,n}(x)\]

\[(4.6.12) \quad (\alpha) M_{a,b,k,n}(x) - (\alpha+kn) M_{a,b,k,n-1}(x) = k x^\alpha M_{a,b,k,n-1}(x)\]

In particular, the above results reduce to the corresponding results of Spencer and Fano [9] and Konhauser [7] and Laguerre [7] polynomials.

**Proofs of the above results**

**Proof of (4.6.1)**

we write equation (4.5.8) as:

\[(4.6.13) \quad L_{a,b,k,a}(x) \exp(-ax/b) = \frac{k^{\alpha+1}}{2\pi i} \left\{ \frac{\exp\left\{ -ax/b(1+i) \right\} (1+i)}{(1+i)^{\alpha+1}} \right\} dt\]

Now, differentiating (4.6.12), with respect to \(x\), we get:
\[(4.6.14) \quad (bD_x-a) L_{a,b,k,n}(x) = (-a) \int \frac{t^{\alpha-1}}{(1+t)^{k+1}} \, dt\]

Using (4.5.8) and after some simplification, we get:

\[(\alpha) \quad (bD_x-a) L_{a,b,k,n}(x) = (-a) L_{a,b,k,n}(x)\]

which proves (4.6.1).

**Proof of (4.6.2)**

Operating both sides of relation (4.6.10) by the operator \((bD_x-a)\), we get:

\[\left( b D_x - a \right)^2 L_{a,b,k,n}(x) = (-a)^2 L_{a,b,k,n}(x)\]

Repeating the above process further \((m-2)\) times, we get:

\[\left( b D_x - a \right)^m L_{a,b,k,n}(x) = (-a)^m L_{a,b,k,n}(x)\]

which proves (4.6.2).

**Proof of (4.6.3)**

we can write equation (4.6.2) in the form:

\[(-a^{-1} b D_x + 1)^m L_{a,b,k,n}(x) = L_{a,b,k,n}(x)\]

\[k^{n+1} \exp\left\{ -a x / b \right\} \left\{ (1/(1+t)-1) \right\} (1+t)^{k-(\alpha+m)} \]

\[= \frac{1}{2\pi i} \int_{C_r} \frac{t^{\alpha-1}}{(1+t)^{k+1}} \, dt\]
\[ k^{n+1} \exp\left\{-ax/b(1+t)\right\} \left[(1+t)^{-1}\right]^{(\alpha+1)+1}(1+t) = \frac{1}{2\pi i} \int_{C} \left[\frac{\exp\left\{-ax/b(1+t)\right\}}{(1+t)^{\alpha+1}} \left\{ \frac{1}{\left(1+t\right)^{\alpha+1}} \right\} \right] dt \]

Using (4.6.13) and after some simplification, we get:

\[ (-a^{-1}b D_{x}^\alpha + 1) L_{a,b,k,n}(x) = k L_{a,b,k,n}(x) + L_{a,b,k,n}(x) \]

which proves (4.6.3):

**PROOF OF (4.6.4)**

Now putting (1+t)x = u in equation (4.5.8), we get:

\[ (4.6.15) \quad L_{a,b,k,n}(x) \exp\left\{-ax/b\right\}^x \]

\[ = \frac{1}{2\pi i} \int_{C} \frac{k^{-1}}{\left(\frac{u}{x}\right)^{\alpha+1}} \exp\left\{-ax^2/ bu\right\} du \]

Now differentiating (4.6.15) both sides with respect to x and after some simplification, we get:

\[ [xD_{x} - a^{-1}b^{-1}x - k^{-1} \alpha] L_{a,b,k,n}(x) = -(n+1) L_{a,b,k,n+1}(x) \]

which proves (4.6.4).

**PROOF OF (4.6.5)**

Eliminating the term \( xD_{x} L_{a,b,k,n}(x) \) from (4.6.1) and (4.6.4), we get:
\[(k-\alpha) L_{a,b,k,n}(x) + (a/b) x L_{a,b,k,n}(x) = (n+1) L_{a,b,k,n+1}(x),\]

which proves (4.6.5):

**PROOF OF (4.6.6)**

Now replacing \(\alpha\) by \(\alpha-kn\) in (4.6.15), we get:

\[(\alpha-kn) L_{a,b,k,n}(x) \exp(-ax/b) x^{k-\alpha-kn+1}\]

\[(4.6.16) \quad \int_{c}^{n+1} \frac{u^{k-\alpha-kn} \exp\{-ax^{2}/bu\}}{2\pi i} du\]

Now differentiating (4.6.16) both sides with respect to \(x\) and after simplification, we get:

\[\left[xD_{x}+(a/b)x+k(n+1-\alpha+1)\right] L_{a,b,k,n}(x) = (n+1) L_{a,b,k,n+1}(x)\]

which proves (4.6.6).

**PROOF OF (4.6.7)**

Now putting \(xw = u\) in (4.5.15); then equation changes to:

\[(4.6.17) \quad M_{a,b,k,n}(x) = \frac{\Gamma(1+\alpha+kn)}{n!2\pi i} \int_{c}^{k/kn} \frac{\exp(au/b)}{u^{1+\alpha+kn}} du\]

Differentiating (4.6.17) both sides with respect to \(x\), and after some simplification, we get;
\begin{equation}
\left(x^{-k}D_x\right) M_{a,b,k,n}^{(\alpha)}(x) = (-k)^{\alpha+k} M_{a,b,k,(n-1)}^{(\alpha+k)}(x)
\end{equation}

which proves (4.6.7).

**PROOF OF (4.6.8)**

Operating both sides of (4.6.7) by the operator \((x^{-k}D_x)\), we get:

\begin{equation}
\left(x^{-k}D_x\right) M_{a,b,k,n}^{(\alpha)}(x) = (-k)^{\alpha+k} M_{a,b,k,(n-1)}^{(\alpha+k)}(x)
\end{equation}

Repeating the above process further \((m-2)\) times, we get:

\begin{equation}
\left(x^{-k}D_x\right)^m M_{a,b,k,n}^{(\alpha)}(x) = (-k)^{\alpha+mk} M_{a,b,k,(n-m)}^{(\alpha+mk)}(x)
\end{equation}

Which proves (4.6.8).

**PROOF OF (4.6.9)**

Now multiplying (4.6.17) both sides by \(x^{-kn}\), we get:

\begin{equation}
M_{a,b,k,n}^{(\alpha)}(x) x^{-kn} = \frac{\Gamma(1+\alpha+kn)}{n!2\pi i} \int_C \frac{x^{-kn} \exp\left(\frac{au}{b}\right) \left(\frac{k}{u-x}\right)^n}{u^{(1+\alpha+kn)}} \, du
\end{equation}

Differentiating (4.6.18) both sides with respect to \(x\) and after simplification, we get:

\begin{equation}
(xD^{-kn}) M_{a,b,k,n}^{(\alpha)}(x) = \frac{\Gamma(1+\alpha+kn)}{(n-1)!2\pi i} \int_C \frac{\exp\left(\frac{au}{b}\right) \left(\frac{k}{u-x}\right)^{n-1}}{u^{1+\alpha+k(n-1)}} \, du
\end{equation}
Now, multiplying \((4.6.19)\) both sides by \(\Gamma\{1+\alpha+k(n-1)\}\), we get:

\[
(xD - k n) M_{\alpha,a,b,k,n}^{(\alpha)}(x) = \frac{(-k) \Gamma(1+\alpha+kn)}{\Gamma(1+\alpha+k(n-1))} M_{\alpha,a,b,k,(n-1)}^{(\alpha)}(x)
\]

Which proves \((4.6.9)\).

**PROOF OF \((4.6.10)\):**

Eliminating the term \(xD_x M_{a,b,k,n}^{(\alpha)}(x)\) from \((4.6.7)\) and \((4.6.9)\) and after simplification, we obtain the following recurrence relation:

\[
M_{a,b,k,n}^{(\alpha)}(x) + k M_{a,b,k,n-1}^{(\alpha+k)}(x) = \frac{(-k) \Gamma(1+\alpha+kn)}{\Gamma(1+\alpha+k(n-1))} M_{a,b,k,(n-1)}^{(\alpha)}(x)
\]

which proves \((4.6.10)\).

**PROOF OF \((4.6.11)\):**

Now, differentiating \((4.5.15)\) both sides with respect to \(x\) and after simplification, we get:

\[
(xD + \alpha) M_{x,a,b,k,n}^{(\alpha)}(x) = (\alpha+kn) M_{a,b,k,n}^{(\alpha-1)}(x)
\]

which proves \((4.6.11)\).

**PROOF OF \((4.6.12)\):**

Eliminating the term \(xD_x M_{a,b,k,n}^{(\alpha)}(x)\), from \((4.6.7)\) and \((4.6.11)\) and after some simplification, we obtain the following recurrence relation:
\[(\alpha^k)_{\binom{a,b,k,n}{x}} = \binom{\alpha+\lambda}{k} \cdot \binom{\alpha}{x} \]

which proves (4.6.12).

**PARTICULAR CASES**

We list below some of the particular cases of (4.2.1) and (4.2.2):

**(I) SPENCER AND FANO POLYNOMIALS**

Taking \(a=1\), \(b=1\), \(k=1\) and \(\alpha = \Omega\), we get:

\[L_{1,1,1,n}^{(\alpha)}(x) = Y_{n}^{(\Omega)}(x)\]

and

\[M_{1,1,1,n}^{(\alpha)}(x) = Z_{1}^{(\Omega)}(x)\]

**(ii) KONHAUSER POLYNOMIALS**

Taking \(a=1\), \(b=1\), we see that

\[L_{1,1,1,n}^{(\alpha)}(x) = Y_{n}^{(\alpha+k)}(x)\]

and

\[M_{1,1,1,n}^{(\alpha)}(x) = Z_{n}^{(\alpha;k)}(x)\]

**(iii) LAGUERRE POLYNOMIALS**

Taking \(a=1\), \(b=1\), \(k=1\), we see that

\[L_{1,1,1,n}^{(\alpha)}(x) = L_{n}(X) = M_{1,1,1,n}^{(\alpha)}(X)\]
(4.7) BILINEAR AND BILATERAL GENERATING RELATION

In the section (4.2), we have derived some generating functions for \( L_{a,b,k,l}(x) \) and \( M_{a,b,k,n}(x) \). Now in this section, we shall adopt group theoretic method to obtain a new class of bilinear and bilateral generating relations associated with \( L_{a,b,k,n}(x) \) and \( M_{a,b,k,n}(x) \). The method has already been discussed in some detail in the section (1.6) of this thesis. All the results derived here appear in the form of some theorems. We prove the following theorems with application.

**THEOREM -1**

If there exists a generating relation of the form:

\[
G(x,w) = \sum_{n=0}^{\infty} a_n L_{a,b,k,n}(x) w^n
\]

then

\[
\exp(wt) G(x, w) = \sum_{n=0}^{\infty} t^n \sigma_n(u),
\]

where

\[
\sigma_n(u) = \sum_{m=0}^{\infty} a_n \left( \frac{1}{m+1} \right) L_{a,b,k,n}(x) u^m
\]

**PROOF**

Consider [6] the linear partial differential operator \( R \) as follows:

\[
R = \left( a \cdot b \left. D_x \right|_{-1} \right) y
\]
such that

\[ (4.7.4) \quad \mathcal{R}_{a,b,k,n}^{(\alpha)} (x, y) = \mathcal{L}_{a,b,k,n}^{(\alpha+1)} (x, y) \]

Hence, clearly \( \mathcal{R} \) forms a raising Lie operator. The multiplier representation of this operator is given by

\[ (4.7.5) \quad \exp(\mathcal{R} \, w) \, f(x, y) = \exp(-yw) \, f(x + bwy/a, y) \]

Let us now consider the generating relation,

\[ (4.7.6) \quad G(x, w) = \sum_{n=0}^{\infty} a_n \mathcal{L}_{a,b,k,n}^{(\alpha)} (x, w) \]

Replacing \( w \) by \( wz \) and then multiplying both sides by \( y \), we get:

\[ (4.7.7) \quad G(x, wz) \, y = \sum_{n=0}^{\infty} a_n \mathcal{L}_{a,b,k,n}^{(\alpha)} (x, wz) \, y \]

Operating both sides of (4.7.7) by \( \exp(wR) \), we get:

\[ (4.7.8) \quad \exp(wR) [G(x, wz) \, y] = \exp(wR) \sum_{n=0}^{\infty} a_n \mathcal{L}_{a,b,k,n}^{(\alpha)} (x, wz) \, y \]

Using (4.7.5), the left hand member of (4.7.8), becomes:

\[ (4.7.9) \quad \exp(-yw) \, G(x + wby/a, wz) \cdot y^{(\alpha)} \]

Also using (4.7.4), the right hand member of (4.5.8), becomes:

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{a_n}{m!} \right) w^m \mathcal{R} \left[ \mathcal{L}_{a,b,k,n}^{(\alpha)} (x, y) \right] \]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{a_n}{m!} \right) w^m z^n \left( (-1)^m L_{a,b,k,n}(x) y \right)
\]

(4.7.10) \( y^{(\alpha)} \sum_{n=0}^{\infty} \left( \frac{a_n}{m!} \right) \sum_{m=0}^{\infty} (-1)^m \left( L_{a,b,k,n-m}(x) \right) \left( \frac{y}{z} \right) \]

Equating (4.7.9) and (4.7.10), we get:

(4.7.11) \( \exp(-yw) G(x+wy/a, wz) \)

\[
= \sum_{n=0}^{\infty} \left( wz \right)^n \sum_{m=0}^{\infty} \left( \frac{a_{n-m}}{m!} \right) \left( L_{a,b,k,n-m}(x) \right) \left( \frac{-y}{z} \right)^m
\]

Finally, putting \( wz = t \) and \( -y/z = u \) in (4.7.11), we get:

\[
\exp(ut) G(x-wby/a, t) = \sum_{n=0}^{\infty} \left( wt \right)^n \sigma(u),
\]

where

\[
\sigma(u) = \sum_{n=0}^{\infty} \left( a_n \right) \left( \frac{1}{m!} \right) L_{a,b,k,n}(x) u^m
\]

This completes the proof of the theorem.

**APPLICATION OF THE THEOREM - 1**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.
Taking, \( a = 1, \ b = 1 \) and \( k = 1 \), (2.7.1) reduces to the following generating function in the form:

\[
(4.7.12) \quad G(x, w) = \sum_{n=0}^{\infty} a_n (\alpha) L_n(x) \ w^n .
\]

Carlitz [1] has given the following generating function for Laguerre polynomial:

\[
(4.7.13) \quad \sum_{n=0}^{\infty} a_n (\alpha) L_n(x) \ t^n = (1-t)^{-\alpha-1} \exp\{x \ t / (t-1)\}
\]

The relation (4.7.12) is type of (4.7.13) with \( a_n = 1 \) and \( w = t \), we get:

\[
(4.7.14) \quad G(x,t) = (1-t)^{-\alpha-1} \exp\{x \ t / (t-1)\}
\]

Applying the theorem, we obtain the following bilateral generating function for Laguerre polynomial:

\[
(4.7.15) \quad (1-t)^{-\alpha-1} \exp[(x-u) t / (t-1)]
\]

\[
= \sum_{n=0}^{\infty} t^n \ \sigma_n(u),
\]

\[
\frac{(1+\alpha+m)_m}{m!} \quad _1F_1\left[ \begin{array}{c} -m; \\ 1+\alpha+m; \end{array} \mid x \right]
\]

where \( \sigma_n(u) = \exp(u) \left[ \frac{\cdots}{m!} \right] _1F_1 \left[ \begin{array}{c} \cdots; \\ 1+\alpha+m; \end{array} \mid x \right] \)

This appears to be a new generating relation.

**THEOREM-2**

If there exists a generating relation of the form:
\[(4.7.16) \quad F(x,w) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_{a,b,k,n}(x) w^n\]

then

\[(4.7.17) \quad (1+ul\kappa) \exp[(ax/b)\{1+(1+ktu)\}] F[(1-\kappa u) x,\{(1-ktu)/u\}, t]\]

\[= \sum_{n=0}^{\infty} t^n \phi_n(u)\]

where

\[(4.7.18) \quad \phi_n(x) = \sum_{n=0}^{\infty} a_{n-m}^{(\alpha-k)} L_{a,b,k,n}(x) u^m\]

**PROOF:**

Consider the linear partial differential operator $\Omega$ as follows:

$$\Omega = y^k z (x \partial/\partial x + y \partial/\partial y - a b^{-1} x - k)$$

such that

\[(4.7.19) \quad \Omega [L_{a,b,k,n}(x) y z^n] = -(n+1) L_{a,b,k,n+1}(x) y z^{n+1}\]

Hence, clearly $\Omega$ forms a raising Lie operator. The multiplier representation of this operator is given by

\[(4.7.20) \quad \exp(w;\Omega) f(x,y,z) = -y^k (zwk - y^k)^{-1}\]

\[\times \exp[ax/b\{1-(zwk-y^k)^{1/k}y^{-1}\}]\]

\[\times f[(zwk+y^k)^{1/k} (x/y), (zwk+y^k)^{1/k}, z]\]

Let us now consider the generating relation:

\[(4.7.21) \quad F(x,w) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_{a,b,k,n}(x) w^n\]
Replacing \( w \) by \( wz \) and then multiplying both side by \( y \), we get:

\[
F(x,w) y = \sum_{n=0}^{\infty} a_n L_{a,b,k,n}(x) (wz)^y
\]

(4.7.22)

Operating both sides of (4.7.22) by \( \exp(w\Omega) \), we get:

\[
\exp(w\Omega) \left[ F(x,wz) y \right] = \exp(w\Omega) \sum_{n=0}^{\infty} a_n L_{a,b,k,n}(x) (wz)^y
\]

(4.7.23)

Using (4.7.20), The left hand member of (4.7.23), becomes:

\[
-y^{k+\alpha} (zwk-y^k)^{-1}
\]

* \( \exp[ax/b\{1-(zwk-y^k)^{1/k}y^{-1}\}] \)

* \( F[(zwk+y^k)^{1/k}(x/y),(zwk+y^k)^{1/k},z] \)

Also using (4.7.19), the right hand member of (4.7.23), becomes

\[
\sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_n/m!}{s+m} w \left[ (-1)^m \left( \begin{array}{c} m \\ s \end{array} \right) \right] L_{a,b,k,n+m}(x)y^z
\]

(4.7.24)

(4.7.25)

\[
y \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (wz) a_{n-m} (-1)^m \left( \begin{array}{c} m \\ s \end{array} \right) L_{a,b,k,n+m}(x)y^z
\]
Equating (4.7.24) and (4.7.25), we get:

\[(4.7.26) \quad -y^k (zwk - y^k)^{-1}\]

\[* \quad \exp\left[ ax / b \left\{ 1 - (zwk - y^k)^{1/k} y^{-1} \right\} \right] \]

\[* \quad F[(zwk+y^k)^{1/k} (x/y), (zwk+y^k)^{1/k}, z] \]

\[= \sum_{n=0}^{\infty} (wz) \sum_{m=0}^{n} a_{n-m} \binom{n}{m} L_{a,b,k,n}(x) (-1/y^k) \]

Finally, putting wz = t, and -1/y^k = u in the (4.7.26), we get:

\[\quad - (1+utk)^{-1} \exp \left[ (ax / b) \left\{ 1 + (1+ktu)^{1/k} \right\} \right] \]

\[* \quad F[(1-tku)^{1/k} x, \{(1-ktu)/u\}^{1/k}, t] \]

\[= \sum_{n=0}^{\infty} t^n \phi_n(u) \]

where

\[\phi_n(x) = \sum_{m=0}^{n} a_{n-m} \binom{n}{m} L_{a,b,k,n}(x) u^m \]

This completes proof of the theorem.

**APPLICATION OF THE THEOREM - 2**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.

Taking a = 1, b = 1 and k = 1, (4.7.16) reduces to the following generating function in the form:
(4.7.27) \[ F(x, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \ w^n \]

Feldhein [4] has given the following generating function for Laguerre polynomial:

\[
(4.7.28) \quad \sum_{n=0}^{\infty} \frac{1}{n!} L_n^{(\alpha)}(x) \ t^n = \frac{\exp(t)}{\Gamma(\alpha+1)} \ {}_0F_1\left[\begin{array}{c}
\alpha+1; \\
\end{array} \right] 
\]

The relation (4.7.27) is type of (4.7.28) with \( a_n = 1 / \Gamma(\alpha+n+1) \) and \( w = t \), we get:

\[
F(x, t) = \frac{\exp(t)}{\Gamma(\alpha+1)} \ {}_0F_1\left[\begin{array}{c}
\alpha+1; \\
-x t
\end{array} \right]
\]

Applying the theorem, we obtain the following bilateral generating function for Laguerre polynomial:

\[
(4.7.29) \quad \left[ \frac{1}{(1+u t)} \right] = \frac{\exp\left[ x + t(x + 1) \right]}{\Gamma(\alpha+1)} \ {}_0F_1\left[\begin{array}{c}
\alpha+1; \\
-(1-t u)x t
\end{array} \right]
\]

\[
= \sum_{n=0}^{\infty} t^n \phi_n(u),
\]

where \( \phi_n(u) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(1+\alpha+n-m)} L_n(x) \ v^m \)

This appears to be new generating relation.
THEOREM-3

If there exists a generating relation of the form:

\[(4.7.30) \quad G(x, w) = \sum_{n=0}^{\infty} \prod_{a,b,k,n}^{(\alpha)} a_n M_a(x) w^n \]

then

\[(4.7.31) \quad w \cdot G\left[ (w+t)x/t, w+t \right] = \sum_{n=0}^{\infty} t^n \sigma(u) \]

where

\[(4.7.32) \quad \sigma(u) = \sum_{p=0}^{\infty} \frac{\Gamma(1+\alpha+kp)}{p! \Gamma(1+\alpha+kp-p)} \prod_{p=0}^{\infty} M_{a,b,k,p}(x) u^p \]

PROOF:

Consider the linear partial differential operator \(\Lambda\) as follows:

\[\Lambda = y^{-1} \left( x \partial/\partial x + y \partial/\partial y \right) \]

such that

\[(4.7.33) \quad \Lambda \left[ M_{a,b,k,n}(x) y \right] = (\alpha+\alpha+n) M_{a,b,k,n}(x) y \]

Hence, clearly \(\Lambda\) forms a raising Lie operator. The multiplier representation of this operator is given by

\[(4.7.34) \quad \exp(\Lambda w) f(x,y) = w f\left[ (w+y)x/y, (w+y) \right] \]

Let us consider the generating relation:

\[(4.7.35) \quad G(x,w) = \sum_{n=0}^{\infty} a_n M_{a,b,k,n}(x) w^n \]

Replacing \(w\) by \(wz\) and then multiplying both sides by \(y\), we get:
(4.7.36) \[ G(x,wz) y = \sum_{n=0}^{\infty} a_n M_{a,b,k,n}(x) (wz) y \]

Operating both sides of (4.7.37) by \( \exp(\Lambda w) \), we get:

\[ \exp(\Lambda w) [G(x, wz) y] = \exp(\Lambda w) \sum_{n=0}^{\infty} a_n M_{a,b,k,n}(x) (wz) y \]

Using (4.7.34), the left hand member of (4.7.37) becomes:

(4.7.38) \[ w G[(w+wz)x/wz, (w+wz)] y \]

Also using (4.7.33), the right side of (4.7.37), becomes:

\[ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{n+p} \Gamma(1+\alpha+kn) \frac{\Gamma(1+\alpha+kn)}{\Gamma(1+\alpha+k(n+p))} M_{a,b,k,n}(x) y \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{n+p} \frac{\Gamma(1+\alpha+k(n+p))}{\Gamma(1+\alpha+k(n+p)-p)} \frac{\Gamma(1+\alpha+k(n+p)-p)}{p!} M_{a,b,k,n}(x) y \]

(4.7.39) \[ = \sum_{n=0}^{\infty} (wz) \sum_{a_{n+p} \frac{\Gamma(1+\alpha+k(n+p))}{\Gamma(1+\alpha+k(n+p)-p)} \frac{\Gamma(1+\alpha+k(n+p)-p)}{p!} M_{a,b,k,n-p}(x) (1/yz) y \]

Now equating (4.7.38) and (4.7.39), we get:

(4.7.40) \[ w G[(w+wz)x/wz, (w+wz)] \]
\[
\begin{align*}
&= \sum_{n=0}^{\infty} \sum_{p=0}^{n} (wz)^n a_n \frac{\Gamma\{1+\alpha+k(n-p)\}}{(n-p)! \Gamma(1+\alpha+k(n-p)-p)} M_{a,b,k,n-p}^{(x)}(1/yz)^p \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{n} a_n \frac{\Gamma\{1+\alpha+k(n-p)\}}{(n-p)! \Gamma(1+\alpha+k(n-p)-p)} M_{a,b,k,n-p}^{(x)}(1/yz)^p \\
&= \sum_{n=0}^{\infty} a_n \frac{\Gamma\{1+\alpha+kp\}}{\Gamma(1+\alpha+kp-p)} M_{a,b,k,p}^{(x)}(u)^p \\
&= \sum_{n=0}^{\infty} \sigma_n (u) \frac{\Gamma\{1+\alpha+kp\}}{\Gamma(1+\alpha+kp-p)} M_{a,b,k,p}^{(x)}(u)^p \\
\end{align*}
\]

Finally, putting \(wz = t\) and \(1/zy = u\) in (4.7.40), we get:

\[
w G [(w+t)x/t, w+t] = \sum_{n=0}^{\infty} t^n \sigma_n (u)
\]

where

\[
\sigma_n (u) = \sum_{p=0}^{n} \frac{a_p \Gamma(1+\alpha+kp)}{\Gamma(1+\alpha+kp-p)} M_{a,b,k,p}^{(x)}(u)^p
\]

This completes proof of the theorem.

**APPLICATION OF THE THEOREM -3**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Konhauser polynomial.

Taking \(a = 1\) and \(b = 1\) in (4.7.30), we get:

\[
(4.7.41) \quad G(x, w) = \sum_{n=0}^{\infty} a_n \frac{(\alpha)}{n} (x;k) w^n
\]

Genin Calveg [5] has given the following generating function for Konhauser polynomial:

\[
(4.7.42) \quad \sum_{n=0}^{\infty} Z_n (x;k) t^n = \exp(t) \frac{\psi(t)}{\varphi(\psi(t+k), \psi(t+k+1), (x/k)^k)}
\]
The relation (4.7.41) is type of (4.7.42) with \( a_n = 1 / (\alpha + 1)_{kn} \) and \( w = t \), we get:

\[
(4.7.43) \quad G(x, w) = \exp(w) \ _0F_k \left[ \begin{array}{c} \vdots \\ \Delta(k; \alpha+1); -(x/k)^k w \end{array} \right]
\]

Applying the theorem, we obtain the following bilateral generating function for Konhauser polynomial:

\[
(4.7.44) \quad w \exp(w + t) \ _0F_k \left[ \begin{array}{c} \vdots \\ \Delta(k; \alpha+1); -((w+t)x / t k) \end{array} \right] (w+t)
\]

\[
= \sum_{n=0}^{\infty} \sum_{n} w^n \sigma(u)
\]

where

\[
\sigma(u) = \sum_{n} \frac{\Gamma(1+\alpha+ kp)}{Z_{p}(x,k)^u} \frac{\alpha - p}{p} \Gamma(1+\alpha+ kp-p)_{kp}
\]

This appears to be new generating relation.

**THEOREM - 4**

If there exists a generating relation of the form:

\[
(4.7.45) \quad G(x, u, w) = \sum_{n=0}^{\infty} a_n w^n L_{a,b,\alpha}(x) L_{c,d,\alpha}(u)
\]

then there exists a generating relation of the form:

\[
(4.7.46) \quad \exp(-2w) G(x + bw/a, x + dw/c; w)
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{\min(a,j)} \sum_{j=0}^{n-j} \frac{w_{a_j} - j}{(j+j)!} \frac{g_{n-j}}{j!}
\]
\[
\begin{align*}
&\begin{pmatrix}
(\alpha_{ij})_{1 \times j} \\
(\beta_{ij})_{2 \\
L_{a,b,k,n-j}(x) \quad L_{c,d,l,n-j}(u)
\end{pmatrix} \\
&\text{PROOF}
\end{align*}
\]

Consider \([6]\) the linear partial differential operator \(R(x,y_i)\) as follows:
\[R(x,y_i) = (a^T b \frac{\partial}{\partial x} - I) y_i, \quad i = 1,2.\]
such that
\[(4.7.47) \quad R(x,y_i) \left[ L_{a,b,k,n}(x) y_i^\alpha \right] = -L_{a,b,k,n}(x) y_i^{\alpha+1}\]

Hence clearly \(R(x,y_i)\) forms a raising Lie operator for the class \((\alpha)\) of the function \(L_{a,b,k,n}(x)\). The extended form of the this operator is given by
\[(4.7.48) \quad \exp(R_1w)f(x,y_i) = \exp(-y_iw)f(x+bwy_i/a ; y_i)\]

Consider the bilinear generating relation:
\[(4.7.49) \quad G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n L_{a,b,k,n}(x) L_{c,d,l,n}(u)\]

If \((4.7.49)\) exists, we substitute \(wy_1y_2g\) in place of \(w\) and then \((\alpha)\) \((\beta)\) multiplying on both sides by \(y_1 y_2\) we get:
\[(4.7.50) \quad y_1 y_2 G(x,u,wy_1y_2g)\]

\[= \sum_{n=0}^{\infty} a_n (wy_1y_2g)^n L_{a,b,k,n}(x) L_{c,d,l,n}(u) (y_1 y_2)\]
Now operating (4.7.50) both sides by \( \exp\{wR(x,y_1)\} \)
\( \exp\{wR(x,y_2)\} \), we get:

\[
\begin{align*}
\text{(4.7.51)} & \quad \exp\{wR(x,y_1)\} \exp\{wR(x,y_2)\} \left[ G(x,u, wy_1 y_2 g) (y_1 y_2) \right] \\
& = \exp\{wR(x,y_1)\}\exp\{wR(x,y_2)\} \sum_{n=0}^{\infty} a_n (wy_1 y_2 g)^n L_{a,b,k,n}(x) L_{c,d,l,n}(u) (y_1 y_2) \\
\end{align*}
\]

Using (4.7.48), the left hand member of (4.7.51) becomes;

\[
\begin{align*}
\text{(4.7.52)} & \quad (y_1 y_2) \exp(-wy_1) \exp(-wy_2) G (x+(bwy_1)/a, u+(dwy_2)/c; wy_1 y_2 g) \\
\end{align*}
\]

Also using (4.7.47), the right hand member of (4.7.51) becomes;

\[
\begin{align*}
\sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{a_n w^{j_1+j_2}}{j_1! j_2!} (gwy_1 y_2)^n R(x,y_1) [ L_{a,b,k,n}(x) y_1 ] \\
& \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{a_n w^{j_1+j_2}}{j_1! j_2!} (gwy_1 y_2)^n (-1)^{j_1+j_2} L_{a,b,k,n}(x) y_1 \\
& \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{a_n w^{j_1+j_2}}{j_1! j_2!} (gwy_1 y_2)^n (-1)^{j_1+j_2} L_{c,d,l,n}(u) y_2 \\
\end{align*}
\]

\[
\begin{align*}
\text{(4.7.53)} & \quad = \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\min(n,j_1)} a_n w^{j_1+j_2} (g)^{n-j_2} \frac{n-j_2}{j_1! j_2!} \frac{(-1)^{j_1+j_2}}{L_{a,b,k,n}(x) y_1} \\
& \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\min(n,j_1)} a_n w^{j_1+j_2} (g)^{n-j_2} \frac{n-j_2}{j_1! j_2!} \frac{(-1)^{j_1+j_2}}{L_{c,d,l,n}(u) y_2} \\
\end{align*}
\]
Equating (4.7.52) and (4.7.53) and then multiplying \( y_1 = y_2 = 1 \), we get:

\[
\exp(-2w) \, G(x^1 \, bw/a, \, x+dw/c; \, wg) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_n^{j+1}}{(j+1)!} \frac{b_n^{j+1}}{(j+1)!} \frac{w^{n+j}}{(n+j)!} \frac{L_{\alpha+j}^1(x)}{\alpha+j} \frac{L_{\beta+j}^2(u)}{\beta+j} \]

This completes proof of the theorem.

**APPLICATION OF THE THEOREM -4**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.

Taking \( a = b = k = 1 \) and \( c = d = 1 = 1 \); (4.7.45) reduces to the following generating function:

\[(4.7.54) \quad G(x,u;w) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n(x) L_n(u) w^n.
\]

Erdelyi et al [3] has given the following bilinear generating function for Laguerre polynomial:

\[(4.7.55) \quad \sum_{n=0}^{\infty} \frac{n! (\lambda)_n}{(\alpha+1)_n (\beta+1)_n} \frac{L_n(x)}{L_n(u)} w^n = (1-t)^x \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n! (\alpha+1)_n (\beta+1)_n} \left\{ \frac{xy}{(1-t)^2} \right\}^n.
\]
The relation (4.7.54) is type of (4.7.55) with $a_n = \frac{n!}{(\alpha+1)_n(\beta+1)_n}$, we get:

$$G(x,y,t) = (1-t)^\lambda \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n! (\alpha+1)_n(\beta+1)_n} \frac{x^y t^n}{(1-t)^2}$$

Applying the theorem, we obtain the following bilinear generating function for Laguerre polynomial:

$$\exp(-2t) (1-tg)^\lambda \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n! (\alpha+1)_n(\beta+1)_n} \frac{(x+t)(u+t)tg^n}{(1-tg)^2}$$

This appears to be new generating relation.
REFERENCES


