CHAPTER-2

Generalization of Konhauser polynomials and associated generating relations

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CHAPTER 2

Generalization of Konhauser polynomials and associated generating relations

(2.1) INTRODUCTION

In this chapter, we shall study the generalizations of biorthogonal polynomials suggested by Konhauser polynomials over the interval \((0, \infty)\) with respect to the distribution function \(w(x) = x^{(\alpha)} \exp(-px^r) \, dx\)

and also obtain associated generating relations for \(Y_m(x, r, p, k)\)

and \(Z_m(x, r, p, k)\).

We recall the polynomials \(G_n(x, r, p, k)\), which are introduced by Srivastava and Singhal [8], in attempt to provide a elegant unification of various known generalizations of classical Hermite and Laguerre polynomials. These polynomials are defined by the generalized Rodrigues' formula (1.3.10),

\[
(2.2.1) \quad G_n(x, r, p, k) = x^{4n-\alpha} \exp(px^r)(1/n!) \left( x^{k+1} D_x \right)^n \{ x^{\alpha} \exp(-px^r) \},
\]

where \(D_x = d/dx\), and parameters \(\alpha, k, p\) and \(r\) are unrestricted in general. The explicit expansion for (1.3.10) is given as (1.3.15).

\[
(2.1.2) \quad G_n(x, r, p, k) = \frac{k^n}{n!} \sum_{i=0}^{n} \prod_{j=0}^{i-1} (-1)^j \frac{j!}{i!} \quad \left( \frac{\alpha}{1+i} \right)^{\frac{(i+1)+\alpha}{i}}
\]

It is worth mentioning here that other than Srivastava and Singhal [8], Chaudh [1] and Srivastava, P.N. [10] also considered the polynomials defined (1.3.10) simultaneously.
Comparing (1.3.15) with (1.4.21), we get:

\[(2.1.3)\]
\[
Y_n(x;k) = k^n G_n(x, 1, 1, k)
\]

Thus, we observe that (1.3.10) provides a generalization of one member of the pair of Konhauser biorthogonal polynomials. This leads us to consider a pair of biorthogonal polynomials, one of which is connected with (1.3.10).

(2.2) GENERALIZED KONHAUSER POLYNOMIALS

We propose below a pair of biorthogonal sets of polynomials \(Z_n(x, r, p; k)\) and \(Y_n(x, r, p; k)\), where \(Z_n(x, r, p; k)\) is a polynomial of degree \(n\) in \(x^r\) (\(k\) is fixed) while \(Y_n(x, r, p; k)\) is a polynomial of degree \(n\) in \(x^r\) (\(r\) is fixed integer).

\[(2.2.1)\]
\[
Z_n(x, r, p; k) = \frac{\Gamma((\alpha+1+kn)/r)}{\Gamma((\alpha+1+km)/r)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{x}{p} \right)^m
\]

and

\[(2.2.2)\]
\[
Y_n(x, r, p; k) = \frac{1}{n!} \sum_{i=0}^{\infty} \frac{x^i}{i!} \sum_{j=0}^{\alpha+1+r} (-1)^j \binom{\alpha+1+r}{j} (\frac{x^i}{k})^j
\]

where \((\alpha+1)/r > 0\) and \(k/r\) is a positive integer.

BIORTHOGONAL RELATION

The polynomials sets \(Z_n(x, r, p; k)\) and \(Y_n(x, r, p; k)\) are biorthogonal with respect to the distribution function \(w(x) = x \exp(-px^r)\) over the interval \((0, \infty)\).
The biorthogonal relation is given by

\[
(2.2.3) \int_0^\infty x^a \exp(-px^b) Y_n(x,r,p;k) Z_n(x,r,p;k) \, dx = \frac{\Gamma\{\alpha+1+kn\}/r!}{r \, m! \, p^{\alpha+1+kn}/r}
\]

where \(\delta_{mn}\) is Kronecker delta and \(k/r\) is a positive integer. We shall prove this relation later on.

\textbf{(2.3) GENERATING FUNCTION FOR } Z_n(x,r,p;k) \textbf{ AND } Y_n(x,r,p;k)

From (2.2.1), we have

\[
Z_n(x,r,p;k) = \frac{\Gamma\{(\alpha+1+kn)/r\}}{p^{\alpha+1+kn}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{km/r}{p} \frac{km}{x}
\]

\[
Y_n(x,r,p;k) = \frac{\Gamma\{(\alpha+1)/r\}}{p^{\alpha+1/km}} \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \frac{km/r}{p} \frac{km}{x}
\]

\[
= \frac{\Gamma\{(\alpha+1)/r\}}{p^{\alpha+1/km}} \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \frac{km/r}{p} \frac{km}{x} \prod_{n=1}^{\infty} \left[\frac{\Gamma\{(\alpha+1)/r+\tau-1/q\}}{q^{\tau}}\right]
\]

\[
(2.3.1) \quad \frac{\Gamma\{(\alpha+1)/r\}}{p^{\alpha+1/km}} \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \frac{km/r}{p} \frac{km}{x} \prod_{n=1}^{\infty} \left[\frac{\Gamma\{(\alpha+1)/r+\tau-1/q\}}{q^{\tau}}\right]
\]

Thus, \(Z_n(x,r,p;k)\) is in the hypergeometric form.
where \(k/n = q\), a positive integer.
Now,
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{m!} \frac{p^{nq}}{n!} \frac{x^{km}}{(q+1/n)^{km}} \quad (\alpha=1/r, \beta=n/q, \gamma=m/\beta)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \frac{x^{km}}{(q+1/n)^{km}} \quad \sum_{m=0}^{\infty} \frac{p^{nq}}{(q+1/n)^{km}} \quad \Pi_{s=1}^{n+1} \left[ \frac{(\alpha+1/r + s - 1)/q}{q} \right]
\]

\[
= \frac{(-t)^n}{p^{ks/r}} \Pi_{s=1}^{n+1} \left[ \frac{(\alpha+1/r + s - 1)/q}{q} \right]
\]

which is the generating function for \( Z_n(x,r,p;k) \).

From (2.1.1) and (2.2.2), we observe that

\[
(2.3.3) \quad \gamma_n(x,r,p;k) = k^n \cdot G_n(x,r,p;k)
\]

Hence

\[
(2.3.4) \quad \gamma_n(x,r,p;k) = \frac{x^{-\alpha-kn-1} \exp(px^r)}{k^n n!}
\]

Now using the analogous results (1.3.14), we get a generating function for \( \gamma_n(x,r,p;k) \) as:

\[
(2.3.5) \quad \sum_{n=0}^{\infty} \gamma_n(x,r,p;k) t^n = \frac{(1-t)^{-(\alpha+1)/k}}{\exp[px^r(1-(1-t)^{1/r})^{1/k}]}
\]
(2.4) BIORTHOGONAL RELATION

We have to prove relation (2.2.3);

Consider,

\[
\sum_{\alpha} \sum_{\beta} \int x^\alpha \exp(-px^\beta) Y_n(x,r,p;k) Z_m(x,r,p;k) \left[ \frac{1}{\Gamma(\{1+\alpha\}/r)} \right] u^{\beta} \, dx
\]

\[
= \int x^\alpha \exp(-px^\beta) \sum_{\alpha} Y_n(x,r,p;k) t^\beta \left[ \frac{1}{\Gamma(\{1+\alpha\}/r)} \right] u^m \, dx
\]

\[
= \int_0^\infty \frac{x^\alpha \exp(-px^\beta)}{(1-t)^{(\alpha+1)/k}} \exp[-p\left(1-(1-t)^{m}\right)] \exp(-u/p)^{\beta/r}
\]

\[\cdot \frac{\Gamma(\{\alpha+1\}/r)}{(1-t)^{(\alpha+1)/k}}\]

\[\left[ \text{using } (2.3.2) \text{ and } (2.3.5) \right]\]

\[
= (1-t)^{-\alpha-1} \frac{u^{\beta}}{\Gamma(\{1+\alpha\}/r)} \sum_{\alpha=0}^\infty \frac{x^\alpha u^{m}}{\Gamma(\{1+\alpha\}/r)} \exp(-p\left(1-(1-t)^{m}\right) \, dx
\]

After some simplification, we get:

\[
(2.4.1) \int x^\alpha \exp(-px^\beta) \sum_{\alpha} Y_n(x,r,p;k) t^\beta \left[ \frac{1}{\Gamma(\{1+\alpha\}/r)} \right] u^m \, dx
\]

\[
= \frac{\Gamma(\{\alpha+1\}/r)}{\Gamma(\{\alpha+1\}/r)} \exp(ut/p^{\beta})
\]

\[\cdot \frac{p(\alpha+1)}{p^{\{\alpha+1\}/r}}\]
\[
\Gamma \left( \frac{(\alpha+1)}{\rho} \right) = \sum_{m=0}^{\infty} \frac{(1/m!)}{(ut^m/p^m)}
\]

Comparing the coefficient of \(u^m t^n\) on both sides of (2.4.1). We see that coefficient of \(u^m t^n\) when \(m \neq n\), the right hand member of (2.4.1) is zero and when \(n = m\), then the right hand member is non zero. Thus, proves the relation (2.2.3).

(2.5) INTEGRAL REPRESENTATIONS

(i) Integral representation for \(Y_n(x, r, p; k)\)

Osler [5] has given a fractional derivative formula as:

\[
(2.5.1) \quad D_{\frac{d}{dz}}^{\alpha} \{f(z)\} = D_{\frac{d}{dz}}^{\alpha} \left[ \frac{f(z)}{g(z)} \frac{g'(z)}{h(z)} \right]_{w=t}^{z}
\]

\(\alpha\)

where \(D_{\frac{d}{dz}}^{\alpha} \{f(z)\}\) denotes the fractional derivation of order \(\alpha\) with respect to \(g(z)\).

For \(\alpha = m\) and \(h(z) = z\), we have

\[
(2.5.2) \quad D_{\frac{d}{dz}}^{\alpha} \{f(z)\} = D^{m} \left[ f(z), g'(z) \left( \frac{z-w}{g(z)-g(w)} \right)^{m+1} \right]_{w=z}
\]

\(\forall \quad m \in \{0, 1, 2, \ldots\}\).

For the relatively more familiar derivative of order \(m\), Now from (2.5.2); we have

\[
(2.5.3) \quad Y_n(x, r, p; k) = \frac{x^{-\alpha-1+k} \exp(px^r)(x^{k+1} D)}{k^n n!}
\]
\[ k \, x^{b-\alpha-1} = \frac{\exp(-px^r)}{n!} \left[ x^{\alpha+\ln} \exp(-px^r) \right] \frac{x - y}{x^k - y^k} y = x \]

From (2.5.3), we get integral representation as:

\[ (\alpha) \quad Y_n(x, r, p; k) = \frac{k \, x^{\alpha+\ln}}{2\pi i} \int_C \left[ \frac{\exp\{p(u^r-x^r)\}}{u^{\alpha+\ln}} \frac{1}{(u^k-x^k)^{n+1}} \right] du \]

Taking \( y = (1+t) \) and after simple manipulation, we get:

\[ (\alpha) \quad Y_n(x, r, p; k) = \left. \frac{k}{\pi i} \int_0^1 \frac{(1+t)^{\alpha+\ln} \exp\{-px^r(1+t)^r\} \, t^{n+1}}{((1+t)^k-1)^{n+1}} \, dt \right|_{t=0}^{t=1} \]

From (2.5.4), we easily write the integral representation for

\[ (\alpha) \quad Y_n(x, r, p; k) \]

as:

\[ (\alpha) \quad Y_n(x, r, p; k) = \left. \frac{k}{2\pi i} \int_C \frac{(1+t)^{\alpha+\ln} \exp\{-px^r(1+t)^r\}}{((1+t)^k-1)^{n+1}} \, dt \right|_{t=0}^{t=1} \]

where \( C \) is a closed contour enclosing \( t = 0 \), but excluding \( t = 1 \) and the roots of the equation \((t+1)^k - 1 = 0\).

(ii) **Integral representation for** \( Z_n(x, r, p; k) \)

We consider,

\[ \int_0^\infty \exp(-ps^r x^r) \, t^\beta \, Z_n(x, r, p; k) \, dt \]

\[ = \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} \frac{1}{\Gamma\left\{(\alpha+1+\ln)/r\right\}} \int_0^\infty t^{\beta+\ln} \exp(-ps^r x^r) dt \]
\[
\Gamma\{(\alpha+1+kn)/r\} \frac{n}{\Gamma\{(k\alpha+\beta+1)/r\}} \frac{\Sigma_{m=0} \Gamma\{(z+1+km)/r\}}{m!} \Gamma\{(\alpha+1+km)/r\}
\]

\[
(2.5.7)
\]

\[
\Gamma\{(\alpha+1+kn)/r\} \frac{n}{\Gamma\{(k\alpha+\beta+1)/r\}} \frac{\Sigma_{m=0} \Gamma\{(z+1+km)/r\}}{m!} \Gamma\{(\alpha+1+km)/r\}
\]

\[
* \ _{q+1}F_q\left[ \begin{array}{c}
-\beta, \beta+1/rq, \ldots, \beta+1+r(q-1)/rq \\
(\alpha+1)/rq, (\alpha+1+r)/rq, \ldots, (\alpha+1+r(q-1)/rq)
\end{array} ; (x/s)^k \right]
\]

In particular for \( \alpha = \beta \), \(2.5.7\) reduces to the following form:

\[
(2.5.8)
\int_0^\infty \exp(-psx) \ t^\alpha Z_n(x,r,p;k) \ dt
\]

\[
= \frac{\Gamma\{(\alpha+1+kn)/r\} (s^k-x^k)^n}{p^{(kn+\alpha+1)/r} \Gamma\{(1+\alpha+kn)/r\} r . n! \ s^{1+\alpha+kn}}
\]

Now applying inverse Laplace transform technique to \(2.5.8\), we get the integral

representation for \( Z_n(x,r,p;k) \):

\[
(2.5.9)
\frac{\Gamma\{(1+\alpha+kn)/r\} (\alpha)}{\Gamma\{(1+\alpha+kn)/r\}} Z_n \left( u^{1/r}, r, p; k \right)
\]

\[
= \frac{1}{\Gamma\{(1+\alpha+kn)/r\}} \int_{c-i\infty}^{c+i\infty} \frac{\exp(ut) \ [ (t/p)^{k/r} - 1]^n}{\Gamma\{(1+\alpha+kn)/r\}} \ dt
\]

Putting \( u = x^r \) and \( t = ps^r \) in \(2.5.9\), we get:
\[
\frac{n! \, p^{(1+\alpha+r+k,n)/r}}{\Gamma((1+\alpha+k)/r)} \, x^{\alpha+r+1} \, Z_n(x, r, p; k) \quad (\alpha)
\]

\[
= \frac{1}{2\pi i} \int_C \frac{\exp(ps^*x^r) \left[ s^k - 1 \right]^n}{s^{(2+\alpha-r+kn)}} \, ds
\]

where \( C \) is a contour enclosing \( s = 0 \), when \( \alpha, r, k \) and \( n \) are integers.

We also have differential formula for \( Z_n(x, r, p; k) \) as,

\[
\frac{p^{(1+\alpha+r+k,n)/r}}{\Gamma((1+\alpha+k)/r)} \, x^{\alpha+r+1} \, Z_n(x, r, p; k) \quad (\alpha)
\]

\[
= \frac{1}{(1+\alpha+kn)!} \, D_s \left[ \frac{\exp(ps^*x^r) \left[ s^k - 1 \right]^n}{s^{(2+\alpha+kn)}} \right]_{s = 0}
\]

In particular, the above results reduce to the corresponding results Spencer and Feno [7] and Konhauser [3].

(2.6) RECURRANCE RELATIONS

The polynomials \( Y_n(x, r, p; k) \) and \( Z_n(x, r, p; k) \) satisfy the recurrence relations:

(i) The recurrence relations for \( Y_n(x, r, p; k) \)

\[
(D_x - p \, r \, x^{r-1}) \, Y_n(x, r, p; k) = (-p \, r \, x^{r-1}) \, Y_n(x, r, p; k) \quad (\alpha+r)
\]

or

\[
(p^{-1}r^{-1}x^{1-r} \, D_x - 1) \, Y_n(x, r, p; k) = - Y_n(x, r, p; k) \quad (\alpha+r)
\]
\( (2.6.3) \quad (p^{-1}x^{1-r}D_x -1)^n(y_n(x,r,p;k) = (-1)^m y_n(x,r,p;k) \)

\( (2.6.4) \quad (-p^{-1}x^{1-r}D_x + 1)^q y_n(x,r,p;k) = y_n(x,r,p;k) \)

where \( k/r = q \), is a positive integer.

\( (2.6.5) \quad [(-p^{-1}x^{1-r}D_x + 1)^q - 1] y_n(x,r,p;k) = y_{n-1}(x,r,p;k) \)

\( (2.6.6) \quad y_n(x,r,p;k) - y_{n-1}(x,r,p;k) = y_{n-1}(x,r,p;k) \)

\( (2.6.7) \quad (xD_x + \alpha + kn - pr) y_n(x,r,p;k) = k(n+1) y_n(x,r,p;k) \)

\( (2.6.8) \quad (xD_x + \alpha + 1 - pr) y_n(x,r,p;k) = k(n+1) y_{n+1}(x,r,p;k) \)

\( (2.6.9) \quad (\alpha + 1 - k) y_n(x,r,p;k) = pr x^\alpha y_n(x,r,p;k) + k(n+1) y_{n+1}(x,r,p;k) \)

\( (2.6.10) \quad k(n+1)y_{n+1}(x,r,p;k) = (\alpha + kn + 1) y_n(x,r,p;k) - pr x^\alpha y_{n+1}(x,r,p;k) \)

(ii) The recurrence relations for \( Z_n(x,r,p;k) \)

\( (2.6.11) \quad D_x Z_n(x,r,p;k) = -k x^{k-1} Z_{n-1}(x,r,p;k) \)

\( (2.6.12) \quad (x^{1-k}D_x^n Z_n(x,r,p;k) = (-k)^n Z_{n-m}(x,r,p;k) \)
(2.6.13) \[ (xD_x - kn) Z_n(x,r,p;\lambda) \]
\[ \frac{k \Gamma\{(1+\alpha+kn)/r\}}{r \Gamma\{(1+\alpha+k(n-1))/r\}} \]
\[ \times \left( \frac{\Gamma\{(1+\alpha+kn)/r\}}{Z_{n-1}(x,r,p;\lambda)} \right) \]

(2.6.14) \[ (xD_x+\alpha-r+1)) Z_n(x,r,p;\lambda) = (1+\alpha-r+kn) Z_n(x,r,p;\lambda) \]

(2.6.15) \[ (1+\alpha-r+kn) Z_n(x,r,p;\lambda) - (1+\alpha-r+kn) Z_{n-1}(x,r,p;\lambda) \]

\[ \frac{k \Gamma\{(1+\alpha+kn)/r\}}{r \Gamma\{(1+\alpha+k(n-1))/r\}} \]

(2.6.16) \[ [(p^t r^{-1} x^{1+\alpha} \Gamma(x))^{-1} \right] x^{1+\alpha-r} Z_n(x,r,p;\lambda) \]
\[ = (n+1) x^{1+\alpha-r} k(\alpha-k) Z_{n+1}(x,r,p;\lambda) \]

where \( q = k/r \).

**Proofs from (2.6.1) to (2.6.4)**

Rewrite equation (2.5.6) in the following form:

\[ \exp(-px^t) \frac{(1+t)^{\alpha+kn}}{\Gamma\{(1+\alpha+k(n-1))/r\}} \exp\{-px^t(1+t)^r\} \]
\[ \frac{\Gamma\{(1+\alpha+kn)/r\}}{2\pi i} \int_C \]

Differentiating both sides with respect to \( x \), we get:
\[(D_x - p r x^{r_1}) Y_n (x, r, p; k) = (-p r x^{r_1}) Y_n (x, r, p; k)\]

which proves (2.6.1).

Now, multiplying (2.6.1) both sides by \(p^{-1} r^{-1} x^{r_1}\) and after simplification we get result (2.6.2).

Result (2.6.3) and (2.6.4) are obvious iterations of (2.6.2).

**PROOF OF (2.6.5)**

Subtracting (2.6.4) both sides by \(Y_n (x, r, p; k)\) and after some simplification, we get:

\[
[(-p^{-1} r^{-1} x^{r_1} D_x + 1)^a - 1] Y_n (x, r, p; k) = Y_{n-1} (x, r, p; k)
\]

which proves (2.6.5).

**PROOF OF (2.6.6)**

From equation (2.6.4) and (2.6.5), we get:

\[
Y_n (x, r, p; k) - Y_n (x, r, p; k) = Y_{n-1} (x, r, p; k)
\]

which proves (2.6.6).

**PROOF OF (2.6.7)**

Rewrite equation (2.5.4), in the following form:

\[
\exp(-px^c) Y_n (x, r, p; k) = \frac{k}{2\pi i} \left[ \frac{\exp(-pu^c) u^{\alpha+1}}{(u/x)^{k-1} - 1} \right] du
\]

where \(\alpha > 0\) and \(\beta > 0\).
Differentiating it on both sides with respect to $x$ and after simplification, we get:

$$(\alpha) \quad (xD_x + \alpha + k \lambda - p \lambda x^2) \Psi_n(x, r, p; k) = k (n+1) \Psi_n(x, r, p; k)$$

which proves (2.6.7).

**PROOF OF (2.6.8)**

Rewrite the equation (2.5.4) in the following form:

$$x \exp(-px^2) \Psi_n(x, r, p; k) = \frac{k}{2\pi i} \int \left[ \frac{\exp(-pu^2)(u/x)^\alpha}{(u/x)^k - 1} \right] du$$

Differentiating both sides with respect to $x$, and rearranging terms, we get:

$$(\alpha) \quad (xD_x + \alpha + 1 - k - p \lambda x^2) \Psi_n(x, r, p; k) = k (n+1) \Psi_{n+1}(x, r, p; k)$$

which proves (2.6.8).

**PROOF OF (2.6.9)**

Eliminating the term $xD_x \Psi_n(x, r, p; k)$ between (2.6.1) and (2.6.8), we get:

$$(\alpha) \quad (\alpha + 1 - k) \Psi_n(x, r, p; k) - rpx^2 \Psi_n(x, r, p; k) = k (n+1) \Psi_{n+1}(x, r, p; k)$$

On transposition, we get:

$$(\alpha + 1 - k) \Psi_n(x, r, p; k) = rpx^2 \Psi_n(x, r, p; k) = k (n+1) \Psi_{n+1}(x, r, p; k)$$

which proves (2.6.9).
PROOF OF (2.6.10)

Eliminating the term \( xD_x \gamma_n(x,r,p;k) \) between (2.6.1) and (2.6.7), we get:

\[
(\alpha+1+k n) \gamma_n(x,r,p;k) - rpx^\alpha \gamma_n(x,r,p;k) = k(n+1) \gamma_{n+1}(x,r,p;k)
\]

which proves (2.6.10).

PROOF OF (2.6.11)

Now putting \( u = sx \) in relation (2.5.9), we get:

\[
\frac{n! \, p^{(1+\alpha-r+k n)/r}}{\Gamma((1+\alpha+k n)/r)} \frac{(\alpha)}{Z_n(x,r,p;k)}
\]

\[
= \frac{\exp(pu^r) \left[ u^k - x^k \right]^n}{\int_{2\pi i} \frac{\exp(pu^r) \left[ u^k - x^k \right]^n}{u \{ (2+\alpha-r+k n)/r \}} du}
\]

Differentiating it on both sides with respect to \( x \), we get:

\[
\frac{n! \, p^{(1+\alpha-r+k n)/r}}{\Gamma((1+\alpha+k n)/r)} \frac{(\alpha)}{D_x \left[ Z_n(x,r,p;k) \right]}
\]

\[
= \frac{r \, n \, k \, x^{k-1} \, \exp(pu^r) \left[ u^k - x^k \right]^{n-1}}{\int_{2\pi i} \frac{\exp(pu^r) \left[ u^k - x^k \right]^{n-1}}{u \{ 2+\alpha-r+k (n-1)/r \}} du}
\]

After some simplification, we get:

\[
D_x \left[ Z_n(x,r,p;k) \right] = -k \, x^{k-1} \left[ Z_{n+1}(x,r,p;k) \right]
\]

which proves (2.6.11).
**PROOF OF (2.6.12)**

From equation (2.6.11), we have

\[
x^{k-1} D_x Z_n(x,r,p;:k) = -k \frac{(\alpha)}{(\alpha+k)} Z_{n-1}(x,r,p;k)
\]

which on iteration, m times further gives the recurrence relation:

\[
(x^{k-1} D_x)^m Z_n(x,r,p;:k) = (-k)^m \frac{(\alpha+k)}{(\alpha+k+n)} Z_{n-m}(x,r,p;k)
\]

which proves (2.6.11).

**PROOF OF (2.6.13)**

Now, putting \( u = sx \) in relation (2.5.9) and rewrite the equation in the following form:

\[
\frac{n! \, p^{(1+\alpha-tkn)/r}}{\Gamma((1+\alpha+kn)/r)} \frac{x^{kn}}{Z_n(x,r,p;k)} = \frac{\exp(pu^r)}{2\pi c} \int \frac{[(u/x)^{k-1}]^n}{u^{(2+\alpha-tkn)/r}} \, du
\]

Differentiating it on both sides with respect to \( x \), we get:

\[
\frac{n! \, p^{(1+\alpha-tkn)/r}}{\Gamma((1+\alpha+kn)/r)} \frac{x^{kn}}{Z_n(x,r,p;k)} = \frac{\exp(pu^r)}{2\pi c} \int \frac{[(u/x)^{k-1}]^{n-1}}{u^{(2+\alpha-t(k+1)n)/r}} \, du
\]

or
\[(xD_x - kn) \quad Z_n(x,r,p;k) \]
\[= -k \quad p^k \Gamma\{(1+\alpha+kn)/r\} \quad Z_{n-1}(x,r,p;k)\]
\[\Gamma \left\{\{1+\alpha+k(n-1)\}/r\right\}\]

which proves (2.6.13).

**PROOF OF (2.6.14)**

Differentiating (2.5.9), both sides with respect to \(x\), we get:

\[(xD_x + \alpha - r + 1) \quad Z_n(x,r,p;k)\]
\[= \Gamma \left\{(1+\alpha+kn)/r\right\} \quad Z_{n-1}(x,r,p;k)\]
\[\Gamma \left\{(1+\alpha-r+kn)/r\right\}\]

or

\[(xD_x + \alpha - r + 1)) \quad Z_n(x,r,p;k) = (1+\alpha-r+kn) \quad Z_n(x,r,p;k)\]

which proves (2.6.14).

**PROOF OF (2.6.15)**

Eliminating the common term \(xD_x \quad Z_n(x,r,p;k)\) from (2.6.13) and (2.6.14), we obtain the following recurrence relation:

\[(2.6.15) \quad (1+\alpha - r + kn) \quad Z_n(x,r,p;k) - (1+\alpha - r + kn) \quad Z_n(x,r,p;k)\]
\[= k \quad p^k \Gamma\{(1+\alpha+kn)/r\} \quad Z_{n-1}(x,r,p;k)\]
\[\Gamma \left\{\{1+\alpha+k(n-1)\}/r\right\}\]
which proves (2.6.15).

**PROOF OF (2.6.16)**

Rewrite the equation (2.5.10) in the following form:

\[
\frac{n! \ p^{\left(1+\alpha-r+k n\right)/r} \ x^{\alpha+r+1}}{\Gamma\left(1+\alpha-r+k n\right)/r\right)} Z_n(x, r, p; k)
\]

\[
= \frac{r \ \exp(ps^x) \ p^s s^k [s^k-1]^n}{2\pi \ c \ \frac{\Gamma\left(2+\alpha-r+k n\right)}{s\left(2+\alpha-r+k n\right)}} \ ds
\]

where \( k/r = q \),

or

\[
= \frac{r \ p^q \ \exp(ps^x) [s^k-1]^n}{2\pi \ c \ \frac{\Gamma\left(2+\alpha-r+k n\right)}{s\left(2+\alpha-r+k n\right)}} \ ds
\]

\[
+ \frac{r \ p^q \ \exp(ps^x) [s^k-1]^{n+1}}{2\pi \ c \ \frac{\Gamma\left(2+\alpha-r+k n+1\right)}{s\left(2+\alpha-r+k n+1\right)}} \ ds
\]

Differentiating both sides with respect to \( x^r \), \( q \)-times, we get:

\[
\left[(p^{-1} r^{-1} x^{1-r} D_{x})^q -1\right] x^{1+\alpha-r} Z_n(x, r, p; k)
\]

\[
= (n+1) \ x^{1+\alpha-r+k(\alpha-k)} Z_{n+1}(x, r, p; k)
\]

which proves (2.6.16).
\textbf{(2.7) DIFFERENTIAL EQUATIONS}

The differential equation for $Y_n(x, r, p; k)$ is given by

\[(2.7.1) \quad \{(1-p^{-1}r^{-1}x^{1-r}D_x)^q - 1\} \{xD_x + \alpha + 1 - k - prx^{r-1}\} - k(n+1)\] $Y_n(x, r, p; k)$

\[= 0\]

\[\text{[using (2.6.5) and (2.6.8)]}\]

which is equivalent to

\[(2.7.2) \quad \{(1-p^{-1}r^{-1}x^{1-r}D_x)^q - 1\} \{xD_x + \alpha + 1 - prx^{r-1}\} - kn\] $Y_n(x, r, p; k) = 0$

The differential equation for $Z_{m}(x, r, p; k)$ is given by

\[(2.7.3) \quad (x^{1-k}D) [x^{1-\alpha+r}r (p^{-1}r^{-1}x^{1-r}D_x)^{q - 1}] x^{1+\alpha-x} Z_{m}(x, r, p; k)$

\[+ (n+1) Z_{m+1}(x, r, p; k) = 0.\]

\[\text{[using (2.6.11) and (2.6.16)]}\]

which is equivalent to,

\[(2.7.4) \quad (p^{-1}r^{-1}x^{1-r}D_x)^q (x^{2+\alpha-x}D_x) Z_{m}(x, r, p; k)$

\[= x^{2+\alpha-x}D_x Z_{m}(x, r, p; k) - kn x^{1+\alpha-x} Z_{m}(x, r, p; k)\]
SPECIAL CASES:

The following known special cases of (2.2.1) and (2.2.2) are

(i) **Spencer and Fano polynomials**

Taking, \( k = 2, r = l, p = l \); we get:

\[
Z_1(x) = \gamma_1(x,1,1,2) \quad \text{and} \quad \gamma_n(x) = Z_n(x,1,1,2)
\]

(ii) **Konhauser polynomials**

Taking, \( r = l, p = l \); we get:

\[
\gamma_n(x; k) = \gamma_n(x,l,l,k) \quad \text{and} \quad Z_n(x,k) = Z_n(x,l,l,k).
\]

(iii) **Laguerre polynomials**

Taking, \( k = 1, r = 1, p = 1 \); we get:

\[
L_n(x) = \gamma_n(x,1,1,1) = Z_n(x,1,1,1)
\]

(iv) **Bessel polynomials**

Other than above (2.2.1) and (2.2.2) also give rise to biorthogonal polynomials sets associated with Bessel polynomials given below for \( k = -1 \) and \( r = -1 \), we get:

\[
\gamma_n(x, -1, \beta, -1) = [(—1)/n!] (\beta/n) \quad \gamma_n(x, \alpha + \beta - 2n, \beta)
\]
where $Y_n(x, \alpha, \beta)$ is generalized Bessel polynomials of Krall and Frink [2], defined by

$$Y_n(x, -1, \beta, -1) = \beta^{-\alpha + 2} x^{-\alpha} \exp(\beta/x) D^n \left[ x^{\alpha - 2 + 2n} \exp(-\beta/x) \right]$$

Clearly the above polynomials satisfy the biorthogonal property,

$$\int_{-\infty}^{\infty} x^\alpha e^{-\beta/x} Y_n(x, \alpha + \beta - 2n, \beta) q_m(x, \alpha + \beta, \beta) \, dx$$

$$= \frac{(-1)^{n-1} n! \Gamma(m-\alpha+1)}{m! \beta^{m-\alpha-1}} \delta_{mn}.$$
(2.3) BILINEAR AND BILATERAL GENERATING RELATIONS

In the section (2.3), we have derived some generating function

\[ G_n(x, r, p; k) \] and \[ Z_n(x, r, p; k) \].

Now in this section, we shall adopt group theoretic method to obtain a new class of bilinear and bilateral generating relations

\[ \sum_{n=0}^{\infty} \binom{\alpha}{n} Y_n(x, r, p; k) w^n \]

associated with \( Y_n(x, r, p; k) \) and \( Z_n(x, r, p; k) \). The method has already been discussed in some detail in the section (1.6) of this thesis. All the results derived here appear in the form of some theorems. We prove the following theorems with application.

**THEOREM - 1**

If there exists a generating relation of the form:

(2.8.1) \[ G(x, w) = \sum_{n=0}^{\infty} \binom{\alpha}{n} Y_n(x, r, p; k) w^n \]

(2.8.2) \[ \exp\left[ p x^n \{ 1 - (1-t)^\alpha / k \} \right] (1-t)^{-(n+1)/k} G \left[ x(1-t)^{-1/k}, tv(1-t) \right] \]

\[ \sum_{n=0}^{\infty} \binom{\alpha}{n} Y_n(x, r, p; k) t^n \sigma_n(v) \]

where

(2.8.3) \[ \sigma_m(v) = \sum_{m=0}^{n} m! a_m \binom{n}{m} v^m \]

**PROOF:**

Consider the linear partial differential operator \( \Omega \) as follows:

\[ \Omega = xy \partial / \partial x + ky^2 \partial / \partial x + (\alpha + 1 - r) px^r y \]
such that

\[(2.8.4) \quad \Omega = \left[ \gamma_n(x,r,p;k) \right] n! y^n = k(n+1) y^{n+1} \gamma_{n+1}(x,r,p;k)\]

Hence, clearly \( \Omega \) forms a raising Lie-operator for the class of function \( \gamma_n(x,r,p;k) \). The multiplier representation of this operator is given by

\[(2.8.5) \quad \exp(w\Omega) f(x,y) = \exp[p x^{\tau} \{ 1-(1-kwy)^{-r/k} \} ] \ (1-kwy)^{(x+1)/k} f \ [x(1-kwy)^{-1/k}, y(1-kwy)^{-1}]\]

Let us now consider the following generating relation:

\[(2.8.6) \quad G(x,w) = \sum_{n=0}^{\infty} a_n n! \ \gamma_n(x,r,p;k) \ w^n\]

Replacing \( w \) by \( wyz \) in \( (2.8.6) \), we get:

\[(2.8.7) \quad G(x,wyz) = \sum_{n=0}^{\infty} a_n n! \ \gamma_n(x,r,p;k) \ (wyz)^n\]

Operating both sides of \( (2.8.7) \) by \( \exp(w\Omega) \), we get:

\[(2.8.8) \quad \exp(w\Omega) G(x,wyz) = \exp(w\Omega) \sum_{n=0}^{\infty} a_n n! \ \gamma_n(x,r,p;k) \ (wyz)^n\]

Now, using \( (2.8.5) \), the left hand member of \( (2.8.8) \) becomes,

\[(2.8.9) \quad \exp[p x^{\tau} \{ 1-(1-kwy)^{-r/k} \} ] * (1-kwy)^{(x+1)/k} G(x(1-kwy)^{-1/k}, y(1-kwy)^{-1}]\]

Also using \( (2.8.4) \), the right hand member of \( (2.8.8) \) becomes,

\[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m w^{n+m} z^n \ (1/ml) \ \Omega \left[ \gamma_n(x,r,p;k) \right] y^n n!\]
\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^n a_n (1/m!) (n+m)! \ w^n \ z^n \ y^{n+m} \ \left[ \gamma_{n+m} (x,r,p;k) \right] \]

\[ (2.8.10) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (1/m!) \ n! \ (kwy)^m (yzw)^{n-m} \ \left[ \gamma_{n} (x,r,p;k) \right] \]

Equating (2.8.9) and (2.8.10) and then putting \( kwy = t \), \( zt / k = v \), we get the following relation:

\[ \exp[ p x^{\alpha} \{1 - (1-t)^{-r/k}\} ] (1-t)^{-(\alpha+1)/k} \ G[ x(1-t)^{-r/k}, tv(1-t)] \]

\[ = \sum_{n=0}^{\infty} \gamma_{n} (x,r,p;k) \ t^n \sigma_n(v) \]

where

\[ \sigma_n(v) = \sum_{m=0}^{n} m! \ a_m \left( \begin{array}{c} n \\ m \end{array} \right) v^m \]

This completes proof of the theorem.

**PARTicular cases**

(i) **Laguerre polynomials:**

Putting \( r=1, p=1, k=1 \) in (2.2.2), we get:

\[ \gamma_{n} (x,1,1,1) = L_n(x) \]

(ii) **Bilateral generating relation**

Putting \( r=1, p=1, k=1 \) in (2.8.2), we get:

\[ (1-t)^{-(\alpha+1)} \exp[x\{1-(1-t)^{-1}\}] \ G[ x(1-t)^{-1}, tv(1-t)] \]

\[ = \sum_{n=0}^{\infty} L_n(x) \ t^n \sigma_n(v) \]
(iii) **Hypergeometric function**

Putting \( a_n = (1/m!) \) in (2.8.3), we get:

\[
\sigma_n(v) = \sum_{n=0}^{\infty} \binom{n}{m} v^m
\]

\[
= (1-v)^n
\]

\[
= _1F_0 \left[ \begin{array}{c} n; \quad \ldots \ldots; \quad v \end{array} \right]
\]

**APPLICATION OF THE THEOREM -1**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.

Now, taking \( p = r = k = 1 \); (2.8.1) reduces to the following form:

(2.8.12)

\[
G(x,w) = \sum_{n=0}^{\infty} a_n \frac{(\alpha)}{n!} L_n(x) \ w^n
\]

We consider the Laguerre polynomials satisfying the following generating relation:

(2.8.13)

\[
\sum_{n=0}^{\infty} \left\{ \frac{(\lambda)_n}{(\alpha+1)_n} \right\} \frac{(\alpha)}{\lambda} L_n(x) \ w^n
\]

\[
= (1-t)^{-\lambda} \ _1F_1 \left[ \begin{array}{c} \lambda \ ; \quad x \ t \ \ldots \ldots \ \alpha+1 \ ; \quad t-1 \end{array} \right]
\]

The relation (2.8.12) is type of (2.8.13) with \( a_n n! = \{(\lambda)_n / (\alpha+1)_n\} \) and \( w = t \), we get:

(2.8.14)

\[
G(x,t) = (1-t)^{-\lambda} \ _1F_1 \left[ \begin{array}{c} \lambda \ ; \quad x \ t \ \ldots \ldots \ \alpha+1 \ ; \quad t-1 \end{array} \right]
\]
Applying the theorem, we obtain the following relation on bilateral generating relation involving Laguerre polynomials:

\[
(2.8.15) \quad \exp\{x(1-(1-t)^{-1})\} (1-t)^{(\alpha+1)} [1-tv(1-t)^{-\lambda}]_t F_1 \left[ \begin{array}{c}
\lambda, \\
xtv
\end{array} \right]_{\alpha+1; vt\{1-t\}-1}^{
\alpha+1; vt\{1-t\}-1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} \lambda^n x^n \left(2F_1 \left[ \begin{array}{c}
-n, \\
\lambda
\end{array} \right]_{\alpha+1; vt\{1-t\}-1}^{
\alpha+1; vt\{1-t\}-1}
\right)
\]

This appears to be a new generating relation.

**Theorem 2**

If there exists a bilinear generating relation of the form:

\[
(2.8.16) \quad G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n (n!)^2 \chi_n(x; t) \chi_n(u; s)
\]

then there exists a generating relation of the form:

\[
(2.8.17) \quad (1-tw)^{(\alpha+1)/2} (1-sw)^{(\beta+1)/2} \exp[p\{x(1-(1-wt)^{-1/2}) + u\{1-(1-sw)^{-1/2}\}]^2]
\]

\[
G\left[ x(1-tw)^{-1/2}, u(1-sw)^{-1/2}; wg(1-wt)^{-1}(1-ws)^{-1}\right]
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (wg)^n n! f_n(w, g, x; u, s) \chi_n(u; s)
\]

where

\[
(2.8.18) \quad f_n(w, g, x) = \sum_{j=0}^{\min(n, j)} a_{n-j} (wt)^{j-j/2} g^{j-j/2} \frac{(j-j)!}{(j/2)! (j/2)!}
\]

\[
* (n+j-2 j/2)! \chi_n^{(\alpha)}(x, t)
\]
Consider the linear partial differential operator $\Omega_1$ as follows:

$$\Omega_1 = y_1 \left( x \partial / \partial x + k y_1 \partial / \partial y_1 + \alpha + 1 - r p x^r \right); \quad i = 1, 2.$$ 

such that

$$(2.8.19) \quad \Omega_1 \left[ Y_0^\alpha (x,t) n! y_1 \right] = k (n+1)! y_1 \quad Y_{n+1}^\alpha (x,t)$$

Hence, clearly $\Omega_1$ forms a raising Lie-operator for the class of

$$(\alpha)$$

function $Y_0^\alpha (x,t,p;k)$. The multiplier representation of this operator is given by

$$(2.8.20) \quad \exp(w_1 \Omega_1) f(x,y_1) = (1-kwy_1)^{(\alpha+1)/k} \exp[px^r \{1-(1-k wy_1)^{r/k}\}] f[\{x(1-kwy_1)^{1/k}, y_1(1-kwy_1)^{-1}\}]$$

Assuming that (2.8.16) exists, we substitute $wy_1 y_2 g$ in the place of $w$ and operating both sides by $\exp(w_1 \Omega_1) \exp(w_2 \Omega_2)$, we get:

$$(2.8.21) \quad \exp(w_1 \Omega_1) \exp(w_2 \Omega_2) G(x,u;wy_1 y_2 g)$$

$$= \exp(w_2 \Omega_2) \sum_{n=0}^{\infty} a_n (wy_1 y_2 g)^n (n!)^2 Y_0^\alpha (x,t) \quad Y_n^\beta (u;s)$$

Now, using (2.8.20), the left hand member of (2.8.21) becomes:

$$(2.8.22) \quad \exp(w_1 \Omega_1) \left[ (1-sw_2 y_2)^{(\beta+1)/s} \exp[pu^r \{1-(1-sw_2 y_2)^{r/s}\}] \right]$$

$$G[\{u(1-sw_2 y_2)^{-1/s}, y_2(1-sw_2 y_2)^{-1}\}]$$

$$= (1-twy_1)^{(\alpha+1)/t}(1-sw_2 y_2)^{(\beta+1)/s}$$

$$\exp[px^r \{1-(1-twy_1)^{r/t}\} + pu^r \{1-(1-sw_2 y_2)^{r/s}\}]$$

$$G[x(1-twy_1)^{-1/k}, u(1-sw_2 y_2)^{-1/s}, wy_1 y_2 g(1-twy_1)^{-1}(1-sw_2 y_2)^{-1}]$$
Also using (2.8.20), we see that the right member side of (2.8.21) becomes,

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} a_n \left( wy_1 y_2 g \right) \frac{(n!)^2}{j! j! j!'!} \frac{\Omega_1^1}{\Omega_2^2} \frac{w^j}{\Omega_1^1} \frac{\Omega_1^1}{\Omega_2^2} \frac{w^{j'}}{\Omega_1^1} \frac{\Omega_1^1}{\Omega_2^2} \frac{Y_n(x,t)}{Y_n(u,s)}
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} a_n \frac{w^{\frac{1}{2} j}}{\Omega_1^1} g^n \left\{ \frac{\Omega_1^1}{\Omega_2^2} \left( Y_n(x,t) \right) \left( n! \right) \right\} \left\{ \frac{\Omega_1^1}{\Omega_2^2} \left( Y_n(u,s) \right) \left( n! \right) \right\}
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} a_n \frac{w^{\frac{1}{2} j}}{\Omega_1^1} g^n \left\{ t^1 Y^1 \left( x, t \right) \left( n+j \right) \left( n+j \right) \right\} \left\{ s Y^2 \left( u, s \right) \left( n+j \right) \left( n+j \right) \right\}
\]

\[
(2.8.23) = \sum_{w=0}^{\infty} \sum_{j=0}^{\infty} a_n \frac{w^{\frac{1}{2} j}}{\Omega_1^1} g^n \left\{ \frac{\Omega_1^1}{\Omega_2^2} \left( Y^1 \left( x, t \right) \right) \left( n+j \right) \left( n+j \right) \right\} \left\{ \frac{\Omega_1^1}{\Omega_2^2} \left( Y^2 \left( u, s \right) \right) \left( n+j \right) \left( n+j \right) \right\}
\]

\[
= \sum_{n=0}^{\infty} a_n \frac{w^{\frac{1}{2} j}}{\Omega_1^1} \frac{\Omega_1^1}{\Omega_2^2} \frac{w^{j'}}{\Omega_1^1} \frac{\Omega_1^1}{\Omega_2^2} \frac{\Omega_1^1}{\Omega_2^2} \frac{Y_n(x,t)}{Y_n(u,s)} \left( n! \right)^2 \left( n! \right)^2 \left( n! \right)^2 \frac{\Omega_1^1}{\Omega_2^2} \frac{\Omega_1^1}{\Omega_2^2} \frac{\Omega_1^1}{\Omega_2^2} \frac{\Omega_1^1}{\Omega_2^2}
\]

[ by series manipulation]

Equating (2.8.22) and (2.8.23) and then putting \( y_1 = y_2 = 1 \), we obtain the required relation (2.8.17).

**APPLICATION OF THE THEOREM -2**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.

Taking \( p = r = s = k = 1 \) and \( t = 1 \); (2.8.16) reduces to the following form:

\[
(2.8.24) \quad G(x,u,w) = \sum_{n=0}^{\infty} a_n \frac{w^n}{(n!)^2} L_n(x) L_n(u)
\]
Manocha [4] has given the following bilinear generating relation (1.5.62) for Laguerre polynomial:

\[
(\mu)_n
\]

The relation (2.8.24) is type of (1.5.62) with \( n! a_m = \frac{x^m y^n}{(\alpha+1)_n (\beta+1)_n} \), \( w = t \) and \( u = y \), then we get:

\[
(2.8.25) \quad G(x,y;\lambda) = (1-t)^\lambda \sum_{n=0}^{\infty} \frac{\lambda_n x^m y^n}{(\alpha+1)_n (\beta+1)_n (1-t)^n} \\
\psi_2 \left[ \mu+n; (\alpha+n+1, (\beta+n+1); x/t, y/t; (1-t), 1) \right], |t| < 1.
\]

where

\[
\psi_2 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} x^m y^n}{(\gamma) (\gamma') m! n!}.
\]

\(|x| < 1, |y| < \infty.
\]

Applying the theorem, we obtain the following relation on bilinear generating function involving Laguerre polynomials:

\[
(2.8.26) \quad (1-w)^{-2(\alpha+1)} \exp\left[ -w(x+u)/(1-w) \right] \left\{ 1-w(1+g)/(1-w) \right\}^\lambda
\]

\[
\psi_2 \left[ \mu+n; (\alpha+n+1, (\beta+n+1); xwg(1-w)/(w(g+1)-1), ywg(1-w)/(w(g+1)-1) \right]
\]

\[
= \sum_{n=0}^{\infty} \sum_{j_1=0}^{(\beta)} (wg)^n n! f_{n,j_1} (w,g,x) L_n(u)
\]
where
\[
f_n (w, g; x) = \sum_{j=0}^{\min(n, j)} \frac{a_{n-j} \binom{w}{j} \binom{j-1}{j/2} \binom{j}{2}}{j! (j-2)! (j-1)!} g^2\]

\[
\overset{(\alpha)}{=}(n+j-2)_{j/2}! L_{n+j-2}_{j/2} (x)
\]

and
\[
\psi_2 = \sum_{m,n=0}^{\infty} \binom{\alpha}{m+n} x^m y^n
\]

This appears to be a new generating function.

**THEOREM-3**

If there exists a generating relation of the form:

\[
(2.8.27) \quad G(x, w) = \sum_{n=0}^{\infty} \binom{\alpha}{n} Z_n (x,r,p;k) \ w^n
\]

then there exists a generating relation of the form:

\[
(2.8.28) \quad \sum_{n=0}^{\infty} t^n \sigma_n (x, v)
\]

where

\[
(2.8.29) \quad \sigma_n (x, v) = \sum_{m=0}^{n} \binom{a_m}{m!} \binom{1+\alpha+km}{r} Z_m (x,r,p;k) v^m
\]
PROOF

Consider the linear partial differential operator $\Delta$ as follows:

$$\Delta = y^x (\partial/\partial x + y \partial/\partial y - r + 1)$$

such that

$$(\alpha) \Delta [ Z_\alpha (x,r,p,k) y^\alpha ] = (1 + \alpha - r + k) Z_\alpha (x,r,p,k)$$

(2.8.30)

Hence, clearly $\Delta$ forms a raising Lie-operator for the class of function $Z_\alpha (x,r,p,k)$. The multiplier representation of this operator is given by

$$(\alpha) \exp(w\Delta) f(x,y) = (rw+y^r)^{1/r} \ y^f \ \{ x((wr+y^r)^{1/r}) y_s((wr+y^r)^{1/r}) \}$$

(2.8.31)

Now consider the following generating relation:

$$(\alpha) G(x,w) = \sum_{n=0}^{\infty} a_n Z_\alpha (x,r,p,k) w^n$$

(2.8.32)

Replacing $w$ by $wz$ and then multiplying both sides of (2.8.32) by $y^\alpha$, we get:

$$G(x,wz) y^\alpha = y^\alpha \sum_{n=0}^{\infty} a_n Z_\alpha (x,r,p,k) (wz)^n$$

(2.8.33)

Operating both sides of (2.8.33) by $\exp(w\Delta)$, we get:

$$\exp(w\Delta) [G(x,wz) y^\alpha] = \exp(w\Delta)[y^\alpha \sum_{n=0}^{\infty} a_n Z_\alpha (x,r,p,k) (wz)^n]$$

(2.8.34)

Now, using (2.8.31), the left hand member of (2.8.34), becomes:

$$(rw+y^r)^{1/r} \ y^r G \{ x((wr+y^r)^{1/r}) y_s((wr+y^r)^{1/r}) \}$$

(2.8.35)

Also using (2.8.30), we see that the right hand member of (2.8.34) becomes:
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n w^{n+m} Z_n^{(a)} \Delta^m \left[ Z_m(x,r,p;k) \right] y^\alpha
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \frac{1}{m!} w^{n+m} Z_n^{(a-m)} \left( -r \right)^m \left( 1 - \{1+\alpha+kn\}/r \right)_m Z_m(x,r,p;k) y^{\alpha-mn}
\]

(2.8.36) \[
= y^\alpha \sum_{n=0}^{\infty} (wz)^n \sum_{m=0}^{n} a_{n-m} \left( \frac{1}{m!} \right) \left( 1 - \{1+\alpha+k(n-m)\}/r \right)_m Z_{n-m}(x,r,p;k)
\]

\[
* \quad (-r/zy^f)^m
\]

Equating (2.8.35) and (2.8.36), we get:

(2.8.37) \[
(rw+y^f)^{1/r} y^f G \left[ \left\{ x(wr+y^f)^{1/r} \right\}/y, (w^-+y^f)^{1/r} \right]
\]

\[
= \sum_{n=0}^{\infty} (wz)^n \sum_{m=0}^{n} a_{n-m} \left( \frac{1}{m!} \right) \left( 1 - \{1+\alpha+k(n-m)\}/r \right)_m Z_{n-m}(x,r,p;k)
\]

\[
* \quad (-r/zy^f)^m
\]

Finally putting \( wz = t \) and \(-r/zy^f = v\) in (2.8.37), we get:

\[
[rw+(-v t/wr)]^{1/r} (-vt/wr) G \left[ x(r^2 w^2 - vt)^{1/r}, (r^2 w^2 - vt)^{1/r} \right]
\]

\[
= \sum_{n=0}^{\infty} t^n \sigma_n(x,v)
\]

where

\[
\sigma_n(x,v) = \sum_{m=0}^{n} \left( \frac{a_m}{m!} \right) \left( 1 - \{1+\alpha+km\}/r \right)_m Z_m(x,r,p;k) v^m
\]

This completes proof of the theorem.
APPLICATION OF THE THEOREM -3

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Konhauser polynomial.

Taking \( \mu = 1 \) and \( \tau = 1 \); (2.8.27) reduces to the following form:

\[(2.8.38)\quad G(x, w) = \sum_{n=0}^{\infty} a_n \left( z_n(x; k) w^n \right)\]

Srivastava, H.M. [9] has given the following generating function for Konhauser polynomial:

\[(2.8.39)\quad \sum_{n=0}^{\infty} \left[ \frac{1}{(\alpha+1)_k} \right] t^n\]

\[= \exp(t) \quad _0F_K \left[ \begin{array}{c} \left( -\frac{x}{k} \right) t \\ \Delta(K; \alpha+1) \end{array} \right] \]

The relation (2.8.38) is type of (2.8.39) with \( a_n = \left[ \frac{1}{(\alpha+1)_k} \right] \)

and \( w = 1 \), then we get:

\[(2.8.40)\quad G(x, 1) = \exp(t) \quad _0F_K \left[ \begin{array}{c} \left( -\frac{x}{k} \right) t \\ \Delta(K; \alpha+1) \end{array} \right] \]

Applying the theorem, we obtain the following function as a bilateral generating function for Konhauser polynomials:

\[(2.8.41)\quad (-v t /w) \exp[(w^2- vt) /w] \exp(t) \quad _0F_K \left[ \begin{array}{c} \left( -\frac{x(vi-w)}{vt} \right) \left\{ (w^2- vt) /w \right\} \\ \Delta(K; \alpha+1) \end{array} \right] \]
\[
= \sum_{n=0}^{\infty} t^n \sigma_n(x,v)
\]

where

\[
(2.8.42) \quad \sigma_n(x,v) = \sum_{m=0}^{n} (-\alpha-km)_{m} \left[ 1/m! (\alpha+1)_{km} \right] Z_{m}(x, k) v^m
\]

This appears to be new generating function.

**THEOREM 4**

If there exists a generating relation of the form:

\[
(2.8.43) \quad F(x,w) = \sum_{n=0}^{\infty} a_n \gamma_n(x,r,p;k) w^n
\]

then there exists a generating relation of the form:

\[
(2.8.44) \quad \exp(u) \left[ \left\{ (-u t/p)+x^t \right\}^{1/t} , t \right] = \sum_{n=0}^{\infty} t^n \sigma_n(x,u)
\]

where

\[
(2.8.45) \quad \sigma_n(x,u) = \sum_{j=0}^{n} \alpha_n \left( 1/j! \right) \gamma_n(x,r,p,k) u^j
\]

**PROOF**

Consider the linear partial differential operator \( \Phi \) as follows,

\[
\Phi = y^\nu \left( x^{\nu+1} \partial / \partial x - p r \right)
\]

such that

\[
(2.8.46) \quad \Phi \left[ \gamma_n(x,r,p;k) y^\nu \right] = -p r \gamma_n(x,r,p;k) y^{\nu+r}
\]

The multiplier representation of this operator is given by

\[
(2.8.47) \quad \exp(w\Phi) f(x,y) = \exp(-p r y f(w)) f \left[ (r y f + x^r)^{1/r} , y \right]
\]
Let us now consider the following generating relation:

\[(2.8.48) \quad F(x,w) = \sum_{n=0}^{\infty} a_n \Psi_n^r(x,r,p;k) w^n\]

Replacing \(w\) by \(wz\) and then multiplying both sides of \((2.8.48)\) by \(y^\alpha\), we get:

\[(2.8.49) \quad y^\alpha F(x,wz) = y^\alpha \sum_{n=0}^{\infty} a_n \Psi_n^r(x,r,p;k) (wz)^n\]

Operating both sides of \((2.8.49)\) by \(\exp(w\Phi)\), we get:

\[(2.8.50) \quad \exp(w\Phi) [y^\alpha F(x,wz)] = \exp(w\Phi) [y^\alpha \sum_{n=0}^{\infty} a_n \Psi_n^r(x,r,p;k) (wz)^n]\]

Now, using \((2.8.47)\), we see that the left hand member of \((2.8.50)\) becomes:

\[(2.8.51) \quad \exp(-pr y^r w) y^\alpha F [(r y^r w + x^r)^{1/r}, wz]\]

Also using \((2.8.46)\), the right hand member of \((2.8.50)\) becomes:

\[= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \left( \sum_{i=0}^{n} a_{n-i} \Psi_i^r(x,r,p;k) (wz)^j \right) y^\alpha\]

\[= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \left( \sum_{i=0}^{n} a_{n-i} \Psi_i^r(x,r,p;k) (wz)^j \right) y^\alpha\]

\[= \sum_{n=0}^{\infty} (wz)^n \sum_{j=0}^{n} a_{n-j} \Psi_j^r(x,r,p;k) (wz)^j y^\alpha\]

Equating \((2.8.51)\) and \((2.8.52)\), we get:

\[(2.8.53) \quad \exp(-pr y^r w) y^\alpha F [(r y^r w + x^r)^{1/r}, wz]\]

\[= \sum_{n=0}^{\infty} (wz)^n \sum_{j=0}^{n} a_{n-j} \Psi_j^r(x,r,p;k) (wz)^j y^\alpha\]
Finally, putting \( wz = t \) and \((-rpy^z/z) = u\) in (2.8.53), we get:

\[
\exp(ut) \ F\{\{ut/p\} + x^r\}^{1/r}, t\} = \sum_{n=0}^{\infty} t^n \ \sigma_n(x,u)
\]

where,

\[
\sigma_n(x,u) = \sum_{j=0}^{n} a_n (1/j!) \ Y_{(\alpha+j)}^{(\alpha)}(x,r,p;k) \ u^j
\]

This completes proof of the theorem.

**APPLICATION OF THE THEOREM -4**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomial.

Taking \( r = 1 \), \( p = 1 \) and \( k = 1 \), (2.8.43) reduces to the following form:

(2.8.54) \[
F(x,w) = \sum_{n=0}^{\infty} a_n L_n(x,r,p;k) \ w^n
\]

Consider the generating function for Laguerre polynomial as follows:

(2.8.55) \[
\sum_{n=0}^{\infty} \left[ 1/ (\alpha+1)_n \right] L_n(x) \ t^n = \exp(t) \ _0F_1 \left[ -\alpha+1; -xt \right]
\]

The relation (2.8.54) is type of (2.8.55), with \( a_n = \left[ 1/ (\alpha+1)_n \right] \) and \( w = t \), we get:

(2.8.56) \[
F(x,w) = \exp(w) \ _0F_1 \left[ -\alpha+1; -wx \right]
\]

Applying the theorem, we obtain the following result on bilateral generating function for Laguerre polynomials:

(2.8.57) \[
\exp{(w+1)u} \ _0F_1 \left[ -\alpha+1; -w(x-wu) \right]
\]
\[ \sum_{n=0}^{\infty} w^n \sigma_n(x,u) \]

where,

(2.8.58) \[ \sigma_n(x,u) = \sum_{j=0}^{n} (a_{n/j}) \ \ell_n(x) \ u \]

This appears to be new generating function.

**THEOREM - S**

If there exists a bilinear generating function of the form:

(2.8.59) \[ G(x,u;w) = \sum_{n=0}^{\infty} a_n \ Y_n(x,r,p,k) \ Y_n(u,s,q,l) \ w^n \]

then there exists a generating relation of the form:

(2.8.60) \[ \exp[-w(pr+qs)] \ G[ (rw+x_r) \ (tw+u_s) ; \ wg ] \]

\[ = \sum_{n=0}^{\infty} \sum_{j_1=0}^{\min(n,j_1)} a_{n-j_2} \ \ell_{j_1}^{w_1} \ (pr)^{j_1} \ (tw)^{j_2} \]

\[ \times \ Y_{n-j_2}(x,r,p,k) \ Y_{n-j_2}(u,s,q,l) \]

**PROOF**

Consider the linear partial differential operator \( \phi_i \) as follows:

\[ \phi_i = y^i_j \ (x^i \ \partial \partial x - pr) ; \ j = 1, 2 \]

such that
(2.8.61) \[ \phi_1 \left[ \gamma_n (x, r, p, k) y_i \right] = -pr \gamma_n (x, r, p, k) y_i \]

Hence, clearly \( \phi_1 \) forms a raising Lie-operator for the class of function \( \gamma_n (x, r, p, k) \). The multiplier representation of this operator is given by:

(2.8.62) \[ \exp (w \phi_1) f (x, y_i) = \exp (-pr y_i w) f [ry_i w + x r]^{1/r} y_i \]

Assuming that (2.8.59) exists, we substitute \( wy_1 y_2 \) in the place of \( w \) and then multiplying \( y_1 y_2 \) both sides of (2.8.59), we get:

(2.8.63) \[ y_1 y_2 G(x, u; wy_1 y_2) = \sum_{n=0}^{\infty} \gamma_n (x, r, p, k) \gamma_n (u, s, q, l) (wy_1 y_2)^n \]

\[ * (y_1 y_2) \]

Now, operating both sides by \( \exp (\phi_1 w) \exp (\phi_2 w) \), we get:

(2.8.64) \[ \exp (\phi_1 w) \exp (\phi_2 w) \left[ y_1 y_2 G (x, u ; wy_1 y_2) \right] \]

\[ = \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a_n (\phi_1 w) (\phi_2 w) (1/j_1! j_2! ) \gamma_n (x, r, p, k) \gamma_n (u, s, q, l) \]

\[ * (wy_1 y_2)^n \]

Using (2.8.62), the left hand member of (2.8.64) becomes:

(2.8.65) \[ \exp [-w pry_1^r + qsy_2^s)] \]

\[ * G [(ry_1^r w + x^r)]^{1/r}, (sy_2^s w + u^s)^{1/s}, wy_1 y_2 \]
Also using (2.8.61), the right hand member of (2.8.64) becomes:

\[
\sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} a_n \left( w \right)^n \left( g y_1 y_2 \right)^n \left( \phi_1 \right)^n \left[ Y_n(x;r,p,k) y_1 \right] 
\]

\[
\phi_2 \left[ Y_n(u;s,q,l) y_2 \right] 
\]

\[
\sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a_{n-j_1} \left( -1 \right)^{n-j_2} \left( g \right)^{n-j_2} \left( pr \right)^{j_1} \left( q s \right)^{j_2} 
\]

\[
\left( 2.8.66 \right) = \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a_{n-j_1} \left( w \right)^n \left( -1 \right)^{n-j_2} \left( g \right)^{n-j_2} \left( pr \right)^{j_1} \left( q s \right)^{j_2} 
\]

\[
\exp[-w(pr+qs)] G[\left( rw+ux \right)^{1/2}, \left( sw+ux \right)^{1/2}; wg ] 
\]

\[
= \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} a_{n-j_1} \left( w \right)^n \left( -1 \right)^{n-j_2} \left( g \right)^{n-j_2} \left( pr \right)^{j_1} \left( q s \right)^{j_2} 
\]

\[
\left( \alpha+r(j_1-j_2) \right) \left( \beta+s(j_2) \right) \cdot Y_n-j_2(x;r,p,k) Y_n-j_2(u;s,q,l) 
\]

Equating (2.8.65) and (2.8.66) and then putting \( y_1 = y_2 \), we get:

This completes the proof of the theorem.

**APPLICATION OF THE THEOREM-5**

Although the above theorem can be applied to many well known classical polynomials, we give below one special case for Laguerre polynomials.
Taking \( r = p = k = 1, s = q = 1 = 1 \) and \( \beta = \alpha \), the relation (2.8.59) reduces to the following form:

\[
(2.8.67) \quad G(x,u;w) = \sum_{n=0}^{\infty} a_n \frac{(x)}{n!} L_n(x) L_n(u) w^n
\]

Rangrajan [6] has given the following formula for Laguerre polynomials:

\[
(2.8.68) \quad \sum_{n=0}^{\infty} \frac{[n!/((\alpha+1))] (x)}{n!} L_n(x) L_n(y) t^n
\]

\[
= (1-t)^{-\alpha-1} \exp\left[-(x+y)t/(1-t)\right] \quad {}_0F_1\left[\frac{\cdots}{\cdots}; \alpha+1; \frac{xy}{(1-t)^2}\right]
\]

The relation (2.8.67) is type of (2.8.68) with \( a_n = [n!/((\alpha+1))] \), \( u = y \) and \( w = t \), we get:

\[
(2.8.69) \quad G(x,y,t) = (1-t)^{-\alpha-1} \exp\left[-(x+y)t/(1-t)\right] \quad {}_0F_1\left[\frac{\cdots}{\cdots}; \alpha+1; \frac{xy}{(1-t)^2}\right]
\]

Applying the theorem, we obtain the following bilinear generating relation for Laguerre polynomials:

\[
(2.8.70) \quad \exp(-2t) (1-tg)^{-\alpha-1} \exp[-(2t+x+y)tg/(1-tg)]
\]

\[
\quad {}_0F_1\left[\frac{(x+y)tg}{(1-tg)^2}; \frac{\cdots}{\cdots}; \alpha+1; \frac{xy}{(1-tg)^2}\right]
\]
\[
\sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \min(n, j_1) \quad a_{n-j_2} \quad t^{n+j_2} \quad \frac{j_1! \cdot (j_1-1)!}{j_2! \cdot (j_2-1)!} \\
\]

\[= \frac{(\alpha+j_1-j_2)}{L_{n+j_1}(x)} \quad \frac{\alpha+j_2}{L_{n+j_2}(u)} \]

where

(2.8.71) \quad a_{n-j_2} = (n-j_2)! / (\alpha+1)_n

This appears to be a generating relation.
REFERENCES


