CHAPTER - 4
Multi-Objective Fuzzy Inventory Model Of Deteriorating Items With Fuzzy Lead-Time
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4.1 Introduction

In inventory system shortages may occur due to different causes, viz. delayed supply or production, transportation problem, sudden increase in demand, artificial crisis, etc. Though shortages bring loss of goodwill, still allowing shortages is one of the common managerial decisions for many reasons such as to optimize the objective(s), so the occurrence of the shortages may appear in different ways, especially in the case of finite time horizon problems.

In shortage cases, lead-time issue becomes more important. In the existing literature, inventory models are generally developed under the assumption of constant or stochastic lead-time. A number of research papers have already been published in this direction by Foote et al, [15] and others. Recently, Kalpakam and Sapan [22-23], Mohebbi and Posner [35] studied a perishable inventory model with stochastic lead-time. Lead-time stochastically can be represented only if past data regarding the supply of the item is available. For new multinational companies being established recently in the market, no past data is available for them. Moreover, for innovative products, it is not possible to have any past record. Hence, now a day, it is more practical to consider lead-time as imprecise and flexible in nature. So it is
represented by a fuzzy or interval valued number as their estimation are
done by experts opinion.

In this Chapter, an inventory model is developed for a
deteriorating item with constant demand and fuzzy lead-time. Shortages
are allowed and fully backlogged. Here, the objective is to maximize the
profitability function, which is the ratio of profit and invested cost. The
imprecise parameter is transformed into an interval number and then
following the interval mathematics, the fractional problem has been
transformed into an equivalent multi-objective deterministic inventory
problem defined by the left limit and center value of an interval and
solved using Fast and Elitist multi-objective Genetic Algorithm (FEMOGA)
(1.8.2). The model is illustrated numerically with an example.

**Notation:**

D = Demand rate

θ = Rate of deterioration for deteriorating items at time t

T = Duration of each business cycle length for infinite time
    horizon Models,

  t = time,

Q = initial / maximum inventory level,

P = selling price per unit,

Q_r = reorder level of an item,

S_d = deteriorated units per cycle,

S_c = Total shortage cost per cycle,

C_s = Shortage cost per unit per unit time,

C_p = Purchase / production cost per unit,
\[ c_0 = \text{setup / ordering cost for each business cycle / time period}, \]
\[ c_h = \text{inventory holding cost per unit quantity per unit time}, \]
\[ C_H = \text{Total holding cost per cycle}, \]
\[ q(t) = \text{inventory level at time } t \]
\[ F(Q, Q_r) = \text{Total profit}, \]
\[ G(Q, Q_r) = \text{Invested cost}. \]

**Assumptions**

An inventory model of deteriorating item is developed by considering fractional objectives with fuzzy lead-time. This model is solved with the help of FEMOGA (1.8.2) to maximize the objective function. The following are the assumption for developing the inventory model:

i) Limited shortages are allowed and fully backlogged.

ii) The rate of replenishment is infinite.

iii) Inventory system involves only one item and one stocking point.

iv) \( t_1 \) is the length of time when new order is placed.

v) \( t_2 \) is the time length when inventory reaches zero.

vi) Lead-time is fuzzy in nature and it is assumed that \( \tilde{L} = (a_1, a_2, a_3) \) a TFN/PFN (1.9) and its nearest interval is \([L_1, L_2]\).

**4.2 Mathematical Formulation**

In this case the inventory level \( q(t) \) at time \( t \ (0 \leq t \leq T) \) satisfies the following differential equations

\[
\frac{dq(t)}{dt} = -(q(t) + D) \quad 0 \leq t \leq t_2
\]

\[ = -D, \quad t_2 \leq t \leq T \quad (4.1)\]
with the boundary conditions,

\[
q(t) = \begin{cases} 
Q, & \text{at } t = 0 \\
0, & \text{at } t = t_2 \\
-Q_s, & \text{at } t = T 
\end{cases} 
\tag{4.2}
\]

and \(q(t) = Q_r\) at \(t = t_1\).

![Graph showing inventory level over fuzzy lead-time for model - 4.1](image)

**Fig.4.1:** inventory level over fuzzy lead-time for model - 4.1

The solutions of the differential equations (4.1) using the boundary conditions (4.2) are

\[
q(t) = \begin{cases} 
\frac{D}{\theta} \left[ e^{\theta(t_2 - t)} - 1 \right], & 0 \leq t \leq t_2 \\
D(t_2 - t), & t_2 \leq t \leq T 
\end{cases} 
\tag{4.3}
\]
\[ Q = \frac{D}{\theta} \left( e^{\theta t_2} - 1 \right), \quad Q_r = \frac{D}{\theta} \left( e^{\theta (t_2 - t_1)} - 1 \right), \]
\[ Q_s = D(T - t_2) = D(t_1 + L - t_2) \quad (4.4) \]

The total holding cost \( (C_H) \) and deteriorated units \( (S_d) \) per cycle are \( c_n I_1 \) and \( \theta I_1 \) respectively where

\[ I_1 = \int_0^{t_2} q(t) dt = \frac{D}{\theta} \left[ \frac{Q}{D} - \frac{1}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right] \quad (4.5) \]

The total shortage cost \( (S_c) \) per cycle is given by

\[ S_c = c_s \int_{t_2}^{T} q(t) dt = \frac{c_s D}{2} \left[ \frac{1}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) - L \right]^2 \quad (4.6) \]

The total profit \( F(Q, Q_r) \) of the system is given by

\[ F(Q, Q_r) = (P - c_p)(Q + Q_s) - C_H - S_C - P S_d - c_0 \quad (4.7) \]

The invested cost \( G(Q, Q_r) \) of the system is given by

\[ G(Q, Q_r) = c_p(\text{Q} + Q_s) \quad (4.8) \]

Now the profit function is given by

\[ \text{PRF}(Q, Q_r) = \frac{F(Q, Q_r)}{G(Q, Q_r)} = \frac{[F_1(Q, Q_r), F_3(Q, Q_r)]}{[G_1(Q, Q_r), G_3(Q, Q_r)]} = [\text{PRF}_1, \text{PRF}_3] \quad (4.9) \]

where

\[ \text{PRF}_1 = \frac{F_1(Q, Q_r)}{G_3(Q, Q_r)}, \quad \text{PRF}_3 = \frac{F_3(Q, Q_r)}{G_1(Q, Q_r)} \]

\[ G_1(Q, Q_r) = c_p \left\{ Q + DL_1 - \frac{D}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} \]

\[ G_3(Q, Q_r) = c_p \left\{ Q + DL_2 - \frac{D}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} \]

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\[ F_1(Q, Q_r) = \left( P - c_p \right) Q + \left\{ \left( P - c_p \right) + \frac{c_3}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} DL_1 \]
\[-\frac{D}{\theta} \left\{ \left( P - c_p \right) + \frac{c_3}{2\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} \log \left( 1 + \frac{Q \theta}{D} \right) \]
\[-\frac{D(c_h + \theta P)}{\theta^2} \left\{ \frac{Q}{D} - \frac{1}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} - \frac{c_4 DL_1^2}{2} - c_0 \]

\[ F_3(Q, Q_r) = \left( P - c_p \right) Q + \left\{ \left( P - c_p \right) + \frac{c_4}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} DL_2 \]
\[-\frac{D}{\theta} \left\{ \left( P - c_p \right) + \frac{c_4}{2\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} \log \left( 1 + \frac{Q \theta}{D} \right) \]
\[-\frac{D(c_h + \theta P)}{\theta^2} \left\{ \frac{Q}{D} - \frac{1}{\theta} \log \left( 1 + \frac{Q \theta}{D} \right) \right\} - \frac{c_4 DL_1^2}{2} - c_0 \]

The multi-objective non-linear fractional programming is defined as

\[
\text{Maximize} \left\{ \text{PRF}_1(Q, Q_r), \text{PRF}_2(Q, Q_r) \right\}
\]  \hspace{1cm} (4.10)

where \( \text{PRF}_2 = \frac{1}{2} \left[ \frac{F_1(Q, Q_r)}{G_3(Q, Q_r)} + \frac{F_3(Q, Q_r)}{G_1(Q, Q_r)} \right] \)

\[
= \frac{F_1(Q, Q_r) G_1(Q, Q_r) + F_3(Q, Q_r) G_3(Q, Q_r)}{2 G_1(Q, Q_r) G_3(Q, Q_r)}
\]  \hspace{1cm} (4.11)

Like using the transformation \( y_1 = Q, y_2 = Q, t \),
\( t = 1/G_3(Q, Q_r) \) and \( t = 1/2G_1(Q, Q_r) G_3(Q, Q_r) \) the problem (4.10) can be replaced by the following two problems

\[
\text{Maximize} \ t \left\{ F_1 \left( \frac{y_1}{t}, \frac{y_2}{t} \right) G_1 \left( \frac{y_1}{t}, \frac{y_2}{t} \right) + F_3 \left( \frac{y_1}{t}, \frac{y_2}{t} \right) G_3 \left( \frac{y_1}{t}, \frac{y_2}{t} \right) \right\}
\]  \hspace{1cm} (4.12)

\[
\text{Maximize} \ t F_1 \left( \frac{y_1}{t}, \frac{y_2}{t} \right)
\]
Subject to \( t \ G_3 (Q, Qr) = 1 \) and \( 2 \ t \ G_1 G_3 = 1 \)
\[ t > 0, \ y_1, y_2 > 0 \]

Following the equality constraints in (4.12) can be replaced as
\[ t \ G_3(Q, Q_3) \leq 1 \text{ and } 2 \ t \ G_1 G_3 \leq 1 \]  \hspace{1cm} (4.13)

The only difference of the problems (4.12) and (4.13) is that the equality constraints of (4.12) are replaced by the inequality constraints. The reason for such replenishment is that the feasible set with these equality constraints will not be convex, where as the feasible set with the inequality constraints will some time be convex.

### 4.3 Solution methodology

Using the interval arithmetic, the above fractional programming problem is transformed to multi-objective optimization problem and then FEMOGA is used to solve the transformed problem.

### 4.4 Numerical methodology

To illustrate the proposed inventory models, following input data are considered.
\[ c_h = $0.60, \ c_s = $1.2, \ c_0 = $80, \ c_p = $8, \ P = $15, \ D = 445, \]
\[ a_1 = 0.35, \ a_2 = 0.5, \ a_3 = 0.6, \ \theta = 0.08 \]

**Case-I:** Considering the fuzzy parameter \( \tilde{L} \) as TFN, the nearest interval approximations are
\[ \tilde{L} = (a_1, a_2, a_3) = [0.425, 0.55], \text{ then the results are obtained via MOGA and are presented in table- 4.1.} \]

**Case-II:** Considering the fuzzy parameter \( \tilde{L} \) as PFN, the nearest interval approximations is
\( \tilde{L} = ((a_1, a_2, a_3)) = [0.400, 0.5667] \). Then the earlier mentioned input data, results are obtained via MOGA and are presented in table- 4.2.

Table 4.1
Optimal results when \( \tilde{L} \) is TFN for model - 4.1

<table>
<thead>
<tr>
<th>( t )</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( Q )</th>
<th>( Q_r )</th>
<th>([PRF_1, PRF_3])</th>
<th>([F_1, F_3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.537 ( \times 10^{-7} )</td>
<td>0.44908</td>
<td>0.45058</td>
<td>204.17</td>
<td>0.668</td>
<td>[0.457, 0.541]</td>
<td>[1435.98, 1602.45]</td>
</tr>
<tr>
<td>0.533 ( \times 10^{-7} )</td>
<td>0.45046</td>
<td>0.45058</td>
<td>204.17</td>
<td>0.05</td>
<td>[0.458, 0.541]</td>
<td>[1440.01, 1606.44]</td>
</tr>
<tr>
<td>0.534 ( \times 10^{-7} )</td>
<td>0.45019</td>
<td>0.45058</td>
<td>204.17</td>
<td>0.169</td>
<td>[0.458, 0.541]</td>
<td>[1439.25, 1605.69]</td>
</tr>
<tr>
<td>0.524 ( \times 10^{-7} )</td>
<td>0.44744</td>
<td>0.45142</td>
<td>204.55</td>
<td>2.973</td>
<td>[0.454, 0.538]</td>
<td>[1419.72, 1586.33]</td>
</tr>
<tr>
<td>0.542 ( \times 10^{-7} )</td>
<td>0.44268</td>
<td>0.45142</td>
<td>204.55</td>
<td>3.187</td>
<td>[0.454, 0.538]</td>
<td>[1418.32, 1584.94]</td>
</tr>
<tr>
<td>0.537 ( \times 10^{-7} )</td>
<td>0.44908</td>
<td>0.45129</td>
<td>204.49</td>
<td>0.988</td>
<td>[0.456, 0.540]</td>
<td>[1432.89, 1599.39]</td>
</tr>
</tbody>
</table>

Table 4.2
Optimal results when \( \tilde{L} \) is PFN for model - 4.2

<table>
<thead>
<tr>
<th>( t )</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( Q )</th>
<th>( Q_r )</th>
<th>([PRF_1, PRF_3])</th>
<th>([F_1, F_3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.537 ( \times 10^{-7} )</td>
<td>0.44976</td>
<td>0.45030</td>
<td>204.04</td>
<td>2.466</td>
<td>[0.443, 0.555]</td>
<td>[1396.69, 1618.79]</td>
</tr>
<tr>
<td>0.520 ( \times 10^{-7} )</td>
<td>0.45046</td>
<td>0.45058</td>
<td>204.17</td>
<td>0.169</td>
<td>[0.445, 0.556]</td>
<td>[1411.38, 1633.29]</td>
</tr>
<tr>
<td>0.534 ( \times 10^{-7} )</td>
<td>0.45046</td>
<td>0.45058</td>
<td>204.17</td>
<td>0.053</td>
<td>[0.445, 0.556]</td>
<td>[1412.14, 1634.05]</td>
</tr>
<tr>
<td>0.542 ( \times 10^{-7} )</td>
<td>0.44744</td>
<td>0.45326</td>
<td>205.40</td>
<td>3.793</td>
<td>[0.439, 0.550]</td>
<td>[1383.90, 1606.11]</td>
</tr>
<tr>
<td>0.542 ( \times 10^{-7} )</td>
<td>0.44269</td>
<td>0.45029</td>
<td>204.04</td>
<td>3.383</td>
<td>[0.442, 0.554]</td>
<td>[1390.69, 1612.84]</td>
</tr>
<tr>
<td>0.537 ( \times 10^{-7} )</td>
<td>0.44744</td>
<td>0.45029</td>
<td>204.04</td>
<td>2.475</td>
<td>[0.443, 0.555]</td>
<td>[1396.63, 1618.73]</td>
</tr>
</tbody>
</table>

Solving (4.12) and (4.13) via FEMOGA method, we obtain a set of results given in table 4.1[following the input data] when the fuzzy parameter \( \tilde{L} \) is taken as TFN. Among these results, Decision Maker (DM) may choose the one suited to him/her.
If \( \bar{L} \) is PFN, the set of results are given in Table-4.2. As such the results in table-4.1 and - 4.2 cannot be compared. Still if the DM wanted to judge the effect of different lead-time representations, he/she may compare the center values of the best profit intervals [\( F_1, F_3 \)]. In that case it is observed that the result from table-4.1 is better that in table- 4.2.

### 4.5 Practical Implications

Due to introduction of open market, multi-nationals market their products in different countries. There is a keen competition of these companies to capture the market in different ways. As a result, they change their product specifications very frequently to attract the customers (e.g., stationary goods, cars, motor cycles, etc.). Before launching new products, these companies normally create an artificial crisis of these products. In this circumstances, the retailers of these multi-national products face a problem of lead-time for replenishment. As these companies are new, the past data about the lead-time of their product replenishment is not available. As a result retailers estimate an imprecise value of lead-time by expert opinions and depending upon that they estimate their strategies. This model can be applied in these real-life situations.
4.6 Conclusion

In this chapter an inventory model is developed for a deteriorating item with constant demand and fuzzy lead-time. Shortages are allowed and fully backlogged. Here, the objective is to maximize the profitability function, which is the ratio of profit and invested cost. The imprecise parameter is transformed to an interval number and then following the interval mathematics, the fractional objective function has been transformed into an equivalent deterministic multi-objective inventory functions defined by the left limit and center of an interval and solved using Fast and Elitist multi-objective genetic Algorithm (FEMOGA). The model is illustrated and results are presented for some numerical data. The model can also be applied in stochastic, fuzzy-stochastic environments.