CHAPTER VI

DOUBLE SAMPLING FOR STRATIFIED RATIO ESTIMATORS
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Summary
In this chapter, the double sampling separate ratio-type estimator and the combined ratio-type estimator in stratified sampling are proposed. Mean square errors of these estimators are derived.

6.1 Introduction
In most of the surveys, information on an auxiliary variable is available. This information can be suitably used in estimating the mean of the characteristic under study more efficiently. The auxiliary information can be utilised either in selection of the sample or at the estimation stage. The common estimators which utilise the information on an auxiliary variable are ratio and regression estimators.

In sample surveys, the information on an auxiliary variable is utilised many times either in estimation or selection or stratification for increasing the efficiency of the estimator.
If such an information is lacking it is advantageous to take a large preliminary sample for observing the value of the auxiliary variable. Further the information on the characteristic under study is collected on a subsample.

This technique is known as double sampling and was given for the first time by Neyman (1938). This technique is profitable only if the gain in precision is substantial as compared to the increase in the cost due to collection of information on the auxiliary variable for the large sample. In this chapter, we have used this technique for stratified sampling. In stratified sampling, the population is divided into several strata which are homogeneous within themselves and whose means are widely different. The strata weights are used in estimating unbiasedly the mean or the total of the characteristic under study.

When the population is stratified and units are drawn by simple random sampling method from each stratum, there are two ways of obtaining a ratio estimator of the population total \( \gamma \):

i) Separate ratio-type estimator; and

ii) Combined ratio-type estimator.

When auxiliary information is available, while sampling from a finite population, applying the technique of linear combination Prof. D. N. Shah (1987) has
defined two new estimators in stratified sampling. An estimators based on linear combination of ratio and product estimators of stratum means, are defined and then such estimators of stratum means are combined to obtain the estimator of the population mean. Second ratio estimator is obtained by taking linear combination of the combined ratio and product estimators in stratified sampling. For both these estimators he has obtained to the first degree of approximation, bias and variances and the efficiency of these estimators as compared with those of the ratio, product, regression and mean per unit estimators in stratified sampling considering both separate and combined types of estimators.

6.2 Important Notations

We shall assume that the population of size \( N \) is divided into \( L \) strata and that sampling within each stratum is simple random sampling without replacement unless otherwise specified.

Further for \( t^{th} \) stratum \( t=1,2, \ldots, L \) and \( i=1,2, \ldots, N_t \), we denote by

- \( Y_{it} \) the value of the characteristic under study for the \( i^{th} \) unit of the population in the \( t^{th} \) stratum \((i=1,2, \ldots, N_t ; t=1,2, \ldots, L)\)
- \( X_{it} \) the value of the auxiliary characteristic for the \( i^{th} \) unit of the population in \( t^{th} \) stratum
- \( Y \) the total of \( Y \) characteristic of the population
\( X \)  
the total of \( X \) characteristic of the population

\[
R = \frac{Y}{X}
\]
the ratio of the population totals or means of the characteristic \( y \) and \( x \)

\( n_{1t} \)
the sample size of \( X \) observations from \( t^{th} \) stratum (\( t=1,2, \ldots L \))

\( n_{2t} \)
subsample taken from \( n_{1t} \) observations of \( x \) variables in \( t^{th} \) stratum

\( \bar{x}_{1t} \)
sample mean of \( X \) observations based on \( n_{1t} \) observations of \( t^{th} \) stratum

\( \bar{x}_{2t} \)
sample mean of \( X \) observations based on \( n_{2t} \) observations of \( t^{th} \) stratum

\[
W_t = \frac{N_t}{N}, \ (t = 1, 2, \ldots L) \text{ stratum weight of } t^{th} \text{ strata}
\]

\( \bar{y}_t \)
Sample mean of \( y \) variable in \( t^{th} \) stratum, based on \( n_{2t} \) observations

### 6.3 Double Sampling Separate Ratio-Type Estimator

The separate ratio type estimator is of the form

\[
\bar{y}_{sd} = \sum_{t=1}^{L} W_t \hat{y}_{tr}
\]  \hspace{1cm} (6.3.1)

where

\[
\hat{y}_{tr} = \frac{\bar{y}_t}{\bar{x}_{2t}} \bar{x}_{1t}
\]

\[
\hat{y}_{tr} = R_t \bar{x}_{1t}, \quad R_t = \frac{\bar{y}_t}{\bar{x}_{2t}}
\]

Let us consider
\[ \frac{\partial \bar{y}_t}{\partial \bar{y}_t} = \frac{\bar{y}_t - \bar{y}_t}{\bar{y}_t} \Rightarrow \bar{y}_t = \bar{y}_t (1 + \partial \bar{y}_t) \]

\[ \frac{\partial \bar{x}_{tt}}{\partial \bar{x}_t} = \frac{\bar{x}_{tt} - \bar{x}_t}{\bar{x}_t} \Rightarrow \bar{x}_{tt} = \bar{x}_t (1 + \partial \bar{x}_{tt}) \quad \text{and} \]

\[ \frac{\partial \bar{x}_{2t}}{\partial \bar{x}_t} = \frac{\bar{x}_{2t} - \bar{x}_t}{\bar{x}_t} \Rightarrow \bar{x}_{2t} = \bar{x}_t (1 + \partial \bar{x}_{2t}) \]

We have

\[ \frac{\hat{\Delta}}{\hat{y}_{tr}} = \frac{\bar{y}_t - \bar{x}_{tt}}{\bar{x}_{2t}} = \frac{\bar{y}_t (1 + \partial \bar{y}_t) (1 + \partial \bar{x}_{tt})}{(1 + \partial \bar{x}_{2t})} \]

\[ \bar{y}_{tr} = \bar{y}_t (1 + \partial \bar{y}_t) (1 + \partial \bar{x}_{tt}) (1 + \partial \bar{x}_{2t})^{-1} \]

\[ = \bar{y}_t (1 + \partial \bar{x}_{tt} + \partial \bar{y}_t + \partial \bar{y}_t \partial \bar{x}_{tt} + \partial \bar{x}_{2t} + \partial \bar{x}_{2t}^2 ) \]

Considering terms up to the first order of approximation, we get

\[ \frac{\hat{\Delta}}{\hat{y}_{tr}} = \bar{y}_t (1 - \partial \bar{x}_{2t} + \partial \bar{x}_{tt} + \partial \bar{y}_t) \quad (6.3.2) \]

Bias of \( \frac{\hat{\Delta}}{\hat{y}_{tr}} \), is given by

\[ B \left( \frac{\hat{\Delta}}{\hat{y}_{tr}} \right) = E \left( \frac{\hat{\Delta}}{\hat{y}_{tr}} \right) - \bar{y}_t \]

after simplification, we get

\[ B \left( \frac{\hat{\Delta}}{\hat{y}_{tr}} \right) = \sum_{t=1}^{L} \left( \frac{1}{n_{tt}} - \frac{1}{N_t} \right) \frac{S_{by}}{\bar{x}_t} + \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) \frac{S_{by}}{\bar{x}_t} \]

\[ - \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) \frac{S_{by}}{\bar{y}_t} \quad (6.3.3) \]
6.4 Mean Square Error

\[\text{MSE}(\bar{y}_{sd}) = \sum_{t=1}^{L} W_t^2 \cdot \text{MSE} (\hat{y}_{tr}) + \sum_{t=1}^{L} W_t \cdot \text{MSE} (\bar{y}_{tr}, \bar{y}_{tr})\]

\[= \sum_{t=1}^{L} W_t^2 \cdot \mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)^2 + \sum_{t=1}^{L} W_t \cdot \mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)(\bar{y}_{tr} - \bar{Y}_t)\]

To the first order of \(n_t\), we have

\[\sum_{t=1}^{L} W_t \cdot W_t' \cdot \mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)(\bar{y}_{tr} - \bar{Y}_t) = 0\]

\[\text{MSE}(\bar{y}_{sd}) = \sum_{t=1}^{L} W_t^2 \cdot \mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)^2\]  \hspace{1cm} (6.41)

Consider

\[\mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)^2 = \mathbb{E}[ (1 + \partial \tilde{y}_t + \partial \tilde{x}_{tr} - \partial \tilde{x}_{t}) - \bar{Y}_t]^2\]

\[\mathbb{E}(\hat{y}_{tr} - \bar{Y}_t)^2 = \mathbb{E}[ (\partial \tilde{y}_t + \partial \tilde{x}_{tr} - \partial \tilde{x}_{t})^2]\]

\[= \mathbb{E} \tilde{Y}_t^2 \left[ (\partial \tilde{y}_t)^2 + (\partial \tilde{x}_{tr})^2 + (\partial \tilde{x}_{t})^2 + 2(\partial \tilde{y}_t)(\partial \tilde{x}_{t}) \right]
\[= 2(\partial \tilde{y}_t)(\partial \tilde{x}_{tr}) - 2(\partial \tilde{y}_t)(\partial \tilde{x}_{t})\]  \hspace{1cm} (6.42)

We get after simplification

\[\mathbb{E}(\partial \tilde{y}_t)^2 = \sum_{t=1}^{L} \left( \frac{1}{n_t} - \frac{1}{N_t} \right) \frac{S_{\tilde{y}_t}^2}{\tilde{y}_t^2}\]

\[\mathbb{E}(\partial \tilde{x}_{tr})^2 = \sum_{t=1}^{L} \left( \frac{1}{n_t} - \frac{1}{N_t} \right) \frac{S_{\tilde{x}_t}^2}{\tilde{x}_t^2}\]
\[
E(\hat{\delta} \bar{x}_{2t})^2 = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) \frac{S_{tx}^2}{\bar{x}_{t}^2}
\]

\[
E(\hat{\delta} \bar{x}_{1t})(\hat{\delta} \bar{y}_{t}) = \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) \frac{S_{ty}}{\bar{x}_{t} \bar{y}_{t}}
\]

\[
E(\hat{\delta} \bar{x}_{1t})(\hat{\delta} \bar{x}_{2t}) = \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) \frac{S_{tx}^2}{\bar{x}_{t}^2}
\]

using these values in (642), we get

\[
MSE(\bar{y}_{tr} - \bar{y}_{t})^2 = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) \frac{S_{sy}^2}{\bar{y}_{t}^2} + \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) \frac{S_{tx}^2}{\bar{x}_{t}^2} - 2 \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) \frac{S_{t}}{\bar{x}_{t}} \bar{y}_{t} - 2 \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) \frac{S_{tx}^2}{\bar{x}_{t}^2}.
\]

\[
MSE(\bar{y}_{tr} - \bar{y}_{t})^2 = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) S_{sy}^2 + \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) S_{tx}^2 - 2 \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) S_{t} \bar{x}_{t} - 2 \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} - \frac{1}{N_t} \right) S_{tx}^2 \bar{x}_{t}^2.
\]
\[
\begin{align*}
= & \sum_{t=1}^{L} \frac{1}{n_{2t}} \frac{1}{N_t} \bar{S}_{y_t}^2 + \sum_{t=1}^{L} \frac{1}{\bar{X}_t^2} \frac{1}{n_{2t}} \frac{1}{n_{tt}} \bar{S}_{x_t}^2 \\
+ & 2 \sum_{t=1}^{L} \frac{\bar{Y}_t}{n_{tt}} \frac{1}{n_{2t}} \bar{S}_{xy} \left( \frac{1}{n_{tt}} \cdot \frac{1}{n_{2t}} \right)
\end{align*}
\]

We get

\[
\text{MSE}(\bar{y}_{sd}) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \frac{1}{N_t} \right) W_t^2 \bar{S}_{y_t}^2 + \sum_{t=1}^{L} \left( \frac{1}{\bar{X}_t^2} \frac{1}{n_{2t}} \frac{1}{n_{tt}} \right) W_t^2 \bar{S}_{x_t}^2
\]

\[
+ 2 \sum_{t=1}^{L} \frac{1}{n_{tt}} \frac{1}{n_{2t}} \frac{\bar{Y}_t}{\bar{X}_t} W_t^2 \bar{S}_{xy}
\]

\[
\Rightarrow
\]

\[
\text{MSE}(\bar{y}_{sd}) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \frac{1}{N_t} \right) W_t^2 \bar{S}_{y_t}^2 + \sum_{t=1}^{L} \left( \frac{1}{\bar{X}_t^2} \frac{1}{n_{2t}} \frac{1}{n_{tt}} \right) W_t^2 \bar{S}_{x_t}^2
\]

\[
- 2 \sum_{t=1}^{L} \frac{1}{n_{tt}} \frac{1}{n_{2t}} \frac{\bar{Y}_t}{\bar{X}_t} W_t^2 \bar{S}_{xy}
\]

If we take \( R_t = \bar{Y}_t / \bar{X}_t \), then

\[
\text{MSE}(\bar{y}_{sd}) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \frac{1}{N_t} \right) W_t^2 \bar{S}_{y_t}^2 + \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \frac{1}{n_{tt}} \right) R_t^2 W_t^2 \bar{S}_{x_t}^2
\]

\[
- 2 \sum_{t=1}^{L} \left( \frac{1}{n_{tt}} \frac{1}{n_{2t}} \right) R_t W_t^2 \bar{S}_{xy}^2
\]

(643)
Remark: If we take \( n_h = N_h \), then, the result (6.4.3) becomes

\[
\text{MSE} \left( \bar{y}_{sd} \right) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) W_t^2 \, S_{y_t}^2 + \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) R_t^2 \, W_t^2 \, S_{tx}^2
\]

\[
- 2 \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) R_t \, W_t \, S_{xy}^2
\]

\[
\Rightarrow \text{MSE} \left( \bar{y}_{sd} \right) = \sum_{t=1}^{L} W_t^2 \left( \frac{1}{n_{2t}} - \frac{1}{N_t} \right) S_{ytx}^2
\]

(6.4.4)

where \( S_{ytx}^2 = S_{y_t}^2 + R_t^2 \, S_{tx}^2 - 2R_t \, S_{xy} \)

This result is the same as that given on the page number 224 of the book "Sampling Theory of Surveys With Applications" by P.V. Sukhatme and B.V. Sukhatme equation no. 138

6.5 Double Sampling Combined Ratio-Type Estimator

Another way to reduce bias in the estimation of \( \bar{Y} \) when the number of strata is large and the sample sizes \( n_h \)'s are small is to form what is known as combined ratio-type estimator

In case of stratified sampling a combined ratio-type estimator proposed by Hansen, Hurwitz and Gurney (1946), is defined as follows

\[
\bar{y}_c = \frac{\bar{y}_{st}}{\bar{x}_{st}} \bar{X} = \frac{\bar{R}_{st}}{\bar{X}_{st}} \bar{X}
\]

(6.5.1)
where
\[ R_{st} = \frac{\bar{y}_{st}}{\bar{x}_{st}} \]

This is also a biased estimator of \( \bar{Y} \), the bias being given by

\[
\text{Bias}(\bar{y}_c) = E(\hat{R}_{st}) \bar{X} - \bar{Y}
\]

\[
= E(\hat{R}_{st}) E(\bar{x}_{st}) - E(\hat{R}_{st} \bar{x}_{st})
\]

\[
\text{Bias}(\bar{y}_c) = -\text{COV}(\hat{R}_{st}, \bar{x}_{st})
\]

It can be shown that, to the first order of approximation

\[
\text{MSE}_1(\bar{y}_c) = \sum_{t=1}^{L} W_t^2 \left( \frac{1}{n_t} - \frac{1}{N_t} \right) S'_{t(y,x)}^2
\]

\[ (6.5.2) \]

where

\[
S'_{t(y,x)}^2 = S_{ty}^2 + R^2 S_{tx}^2 - 2R S_{txy}
\]

On the lines of Hansen, Hurwitz and Gurney, in this section, a double sampling
combined ratio type estimator in stratified sampling is proposed as follows

\[
\hat{y}_{cd} = \frac{\left( \sum_{t=1}^{L} W_t \cdot \bar{y}_t \right) \left( \sum_{t=1}^{L} W_t \cdot \bar{x}_{it} \right)}{\sum_{t=1}^{L} W_t \bar{x}_{it}}
\]

\[ (6.5.3) \]

We know that
\[ E(\sum_{t=1}^{L} W_t \cdot \bar{y}_t) = \sum_{t=1}^{L} W_t \cdot \bar{Y}_t = \bar{Y} \]

Consider

\[ \Delta \bar{y} = \sum_{t=1}^{L} W_t \cdot \bar{y}_t - \bar{Y} \]

\[ \Rightarrow \sum_{t=1}^{L} W_t \cdot \bar{y}_t = \bar{Y} + \Delta \bar{y} = \bar{Y}(1 + \frac{\Delta \bar{y}}{\bar{Y}}) \]

Similarly, we get

\[ \Delta \bar{x}_{it} = \sum_{t=1}^{L} W_t \cdot \bar{x}_{it} - \bar{X} \]

\[ \Rightarrow \sum_{t=1}^{L} W_t \cdot \bar{x}_{it} = \bar{X} \left(1 + \frac{\Delta \bar{x}_{it}}{\bar{X}}\right) \quad \text{and} \]

\[ \sum_{t=1}^{L} W_t \cdot \bar{x}_{2t} = \bar{X} \left(1 + \frac{\Delta \bar{x}_{2t}}{\bar{X}}\right) \]

... \(\bar{y}_{cd}\) becomes

\[
\hat{\bar{y}}_{cd} = \frac{\Delta \bar{y} \left(1 + \frac{\Delta \bar{x}_{it}}{\bar{Y}}\right) \left(1 + \frac{\Delta \bar{x}_{it}}{\bar{X}}\right)}{\Delta \bar{x}_{2t} \left(1 + \frac{\Delta \bar{x}_{2t}}{\bar{X}}\right)} \quad (6.54)
\]
\[ y_{cd} = \bar{Y}(1+ \frac{\Delta \bar{y}}{\bar{Y}})(1+ \frac{\Delta \bar{x}_{1t}}{\bar{X}})(1+ \frac{\Delta \bar{x}_{2t}}{\bar{X}}) \]

\[ y_{cd} = \bar{Y}(1 - \frac{\Delta \bar{x}_{2t}}{\bar{X}} + \frac{\Delta \bar{x}_{1t}}{\bar{X}} + \frac{\Delta \bar{y}_t}{\bar{Y}}) \]

Consider the first order of approximation, we get

\[ y_{cd} = \bar{Y}(1 + \frac{\Delta \bar{y}}{\bar{Y}} + \frac{\Delta \bar{x}_{1t}}{\bar{X}} - \frac{\Delta \bar{x}_{2t}}{\bar{X}}) \]

Taking expectation of \( \hat{y}_{cd} \), we get to the first order of approximation

\[
E(\hat{y}_{cd}) = \bar{Y} \\
E(\Delta \bar{y}) = E(\Delta \bar{x}_{1t}) = E(\Delta \bar{x}_{2t}) = 0
\]

### 6.6 Mean Square Error (\( \hat{Y}_{cd} \))

\[
MSE(\hat{y}_{cd}) = E(\hat{y}_{cd} - \bar{Y})^2
\]

\[
= E \left[ \frac{\Delta \bar{y}}{\bar{Y}} + \frac{\Delta \bar{x}_{1t}}{\bar{X}} + \frac{\Delta \bar{x}_{2t}}{\bar{X}} \right] - \bar{Y}^2
\]

\[
= E \left[ \frac{\Delta \bar{y}}{\bar{Y}} + \frac{\Delta \bar{x}_{1t}}{\bar{X}} + \frac{\Delta \bar{x}_{2t}}{\bar{X}} \right]^2
\]

\[
= E(\Delta \bar{y})^2 + \frac{\bar{Y}^2}{\bar{X}^2} E(\Delta \bar{x}_{1t})^2 + \frac{\bar{Y}^2}{\bar{X}^2} E(\Delta \bar{x}_{2t})^2 + \frac{2 \bar{Y}}{\bar{X}} E(\Delta \bar{y})(\Delta \bar{x}_{1t})
\]
\[
\tilde{Y} - 2 \frac{E(\Delta \tilde{y})(\Delta \tilde{x}_t)}{\tilde{X}} - 2 \frac{\tilde{Y}^2}{\tilde{X}^2} E(\Delta \tilde{x}_{tt})(\Delta \tilde{x}_t)
\]

Now consider

\[
E(\Delta \tilde{y})^2 = \sum_{t=1}^{L} W_t (\tilde{y}_t - \tilde{Y}_t)^2
\]

\[
= \sum_{t=1}^{L} W_t^2 E(\tilde{y}_t - \tilde{Y}_t)^2 + \sum_{t=1}^{L} W_t W_t' E(\tilde{y}_t - \tilde{Y}_t)(\tilde{y}_t' - \tilde{Y}_t')
\]

\[
E(\Delta \tilde{y})^2 = \sum_{t=1}^{L} W_t^2 E(\tilde{y}_t - \tilde{Y}_t)^2
\]

\[
\sum_{t=1}^{L} W_t W_t' E(\tilde{y}_t - \tilde{Y}_t)(\tilde{y}_t' - \tilde{Y}_t') = 0
\]

\[
E(\Delta \tilde{y})^2 = \sum_{t=1}^{L} \frac{1}{n_{2t} N_t} W_t^2 \cdot S_y^2
\]

Similarly, we get

\[
E(\Delta \tilde{x}_{tt})^2 = \sum_{t=1}^{L} \frac{1}{n_{tt} N_t} W_t^2 \cdot S_x^2
\]

\[
E(\Delta \tilde{x}_t)^2 = \sum_{t=1}^{L} \frac{1}{n_{2t} N_t} W_t^2 \cdot S_x^2
\]

\[
E(\Delta \tilde{x}_{tt})(\Delta \tilde{x}_t) = \sum_{t=1}^{L} \frac{1}{n_{tt} N_t} W_t^2 \cdot S_{xy}
\]
\[ E(\Delta \bar{y})(\Delta \bar{x}_t) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \cdot S_{tx} \]

\[ E(\Delta \bar{x}_t)(\Delta \bar{x}_2t) = \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} W_t^2 \cdot S_{tx}^2 \]

Thus, we get \( \text{MSE}(\bar{y}_{cd}) \) as follows

\[
\text{MSE}(\bar{y}_{cd}) = \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} \bar{y}_t^2 + \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} \bar{x}_t^2 - \frac{\bar{y}_t^2}{\bar{x}_t^2}
\]

\[
+ \frac{\bar{y}_t^2}{\bar{x}_t^2} \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} S_{tx}^2 + 2 \frac{\bar{y}_t^2}{\bar{x}_t^2} \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} S_{tx}^2
\]

\[
- 2 \frac{\bar{y}_t^2}{\bar{x}_t^2} \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} S_{txy} - 2 \frac{\bar{y}_t^2}{\bar{x}_t^2} \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} W_t^2 \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} S_{txy}^2
\]

\[
= \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \cdot S_{txy}^2 + \sum_{t=1}^{L} \left( \frac{1}{n_{1t}} \right) \frac{1}{N_t} W_t^2 \cdot S_{tx}^2
\]

\[
+ 2 \sum_{t=1}^{L} \left( \frac{1}{n_{2t}} \right) \frac{1}{N_t} W_t^2 \cdot S_{txy}
\]
\[
\begin{align*}
\bar{Y} & = \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{ty}^2 + \sum_{t=1}^{L} \frac{1}{n_{2t} n_{1t}} \left[ - \frac{1}{X_t} \right] W_t^2 S_{tx}^2 \\
\bar{X} & = \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{xy}
\end{align*}
\]

\[
\Rightarrow \text{MSE}(\hat{\gamma}_{cd}) = \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{ty}^2 + R^2 \sum_{t=1}^{L} \frac{1}{n_{2t} n_{1t}} \left[ - \frac{1}{X_t} \right] W_t^2 S_{tx}^2
\]

\[
- 2R \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{xy}^2
\]

(6.6.2)

where \( R = \frac{\bar{Y}}{\bar{X}} \). If \( n_{1t} = N_t \), then equation (6.6.2), reduces as follows

\[
\text{MSE}(\hat{\gamma}_{cd}) = \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{ty}^2 + R^2 \sum_{t=1}^{L} \frac{1}{n_{2t} n_{1t}} \left[ - \frac{1}{X_t} \right] W_t^2 S_{tx}^2
\]

\[
- 2R \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] W_t^2 S_{xy}^2
\]

\[
= \sum_{t=1}^{L} \frac{1}{n_{2t}} \left[ - \frac{1}{N_t} \right] \left[ S_{ty}^2 + R^2 S_{tx}^2 - 2R S_{xy} \right]
\]

(6.6.3)
where

\[ S'_{t(y,x)}^2 = S_{ty}^2 + R^2 S_{tx}^2 - 2R S_{txy} \]

This result is the same as that given in the equation (6.5.2) as \( n_{x} = n_{t} \) for all \( t \).