Chapter IV

SOME DERIVED MODELS

4.1. INTRODUCTION

There are several attempts in literature to construct bivariate distributions which have specified forms for its marginal and conditional distributions of which the systems with specified marginals are reviewed in Johnson and Kotz (1972). Seshadri and Patil (1964) studied the problem of determining the joint distribution of $X_1$ and $X_2$ given the marginal distribution of $X_i$ and conditional distribution of $X_j$ given $X_i = x_i$, $i, j = 1, 2$, $i \neq j$. They showed that a sufficient condition for the uniqueness of the joint density function of $X_1$ and $X_2$ is that the conditional distribution of $X_i$ given $X_j$ is of the exponential form. The question of determining the joint distribution using the conditional distributions has received considerable attention in the recent times, on the ground that information about the conditional densities are available in many real life phenomenon. Some recent papers in this area are of Castillo...
and Galambos (1987) Arnold (1987), Arnold and Strauss (1988), etc. Arnold and Press (1989) determined a necessary and sufficient condition for the existence of joint density given the conditional densities. Gourieroux and Monfort (1979). Most of the attempts in these papers were to obtain joint distributions which have a specified form for their conditionals, such as bivariate distributions whose conditionals are normal, Weibull, Pareto etc. In the following section, we provide a uniform framework in which a class of bivariate distributions can be generated. This class contains models whose conditionals are exponential, Weibull, Pareto I, Pareto II and finite range distributions.

4.2. DERIVATION OF THE FAMILY.

The lack of memory property defined by Nair and Nair (1991) given in equation (1.36) is generalised here as follows.

\[ P(X_i > G(t_i, s_i) | X_i \geq s_i, X_j = x_j) = P(X_i \geq t_i | X_j = x_j) \]  

(4.1)

Thus the problem of finding the bivariate distributions...
for all $s_i, t_i, x_j$ in $(u, c)$ holds, $i, j = 1, 2$ and $i \neq j$ where $G(\ldots), u, c$ etc; are all as explained in the beginning of Chapter II.

Writing the conditional survival function of $X_i$ given $X_j = x_j$ as,

\[
P(X_i > x_i | X_j = x_j) = S(x_i, x_j)
\]

equation (4.1) becomes,

\[
P(X_i > \eta(t_i, s_i) | X_j = x_j) = P(X_i > t_i | X_j = x_j).
\]

Using (4.2) and (4.3) we have

\[
S(G(t_i, s_i), x_j) = S(t_i, x_j) \cdot S(s_i, x_j).
\]

For a fixed, but otherwise arbitrary $x_j$ (4.4) has the solution, following the arguments in Muliere and Scarsini (1987)

\[
S(x_i, x_j) = \exp[-\lambda_i(x_j)g(x_i)], \lambda_i(x_j) > 0.
\]

Thus the problem of finding the bivariate distributions
characterized by (4.1) reduces to find the joint distribution of \((X_1, X_2)\), where the conditional distribution of \(X_i\) given \(X_j = x_j\) forms,

\[
\mathbb{P}(X_1 \geq x_1 | X_2 = x_2) = \exp[-\lambda_1(x_2)g(x_1)] \tag{4.6}
\]

and

\[
\mathbb{P}(X_2 \geq x_2 | X_1 = x_1) = \exp[-\lambda_2(x_1)g(x_2)] \tag{4.7}
\]

The probability density function corresponding to (4.6) and (4.7) are then

\[
f(x_1 | x_2) = \frac{\lambda_1(x_2) e^{-\lambda_1(x_2)g(x_1)} g'(x_1)}{f_2(x_2)} \tag{4.8}
\]

and

\[
f(x_2 | x_1) = \frac{\lambda_2(x_1) e^{-\lambda_2(x_1)g(x_2)} g'(x_2)}{f_1(x_1)} \tag{4.9}
\]

respectively.

Representing the marginal densities of \(X_1\) and \(X_2\) by \(f_1(x_1)\) and \(f_2(x_2)\) we arrive at the identity,

\[
\frac{\lambda_1(x_2) e^{-\lambda_1(x_2)g(x_1)} g'(x_1)}{f_2(x_2)} \frac{\lambda_2(x_1) e^{-\lambda_2(x_1)g(x_2)} g'(x_2)}{f_1(x_1)} = \lambda_2(x_1) e^{-\lambda_2(x_1)g(x_2)} g'(x_2) f_1(x_1) \tag{4.10}
\]
or equivalently for all $x_1, x_2$ in $(u, c)$.

\[
\log \lambda_1(x_2) - \lambda_1(x_2) g(x_1) + \log g'(x_1) + \log f_2(x_2)
\]

\[
= \log \lambda_2(x_1) - \lambda_2(x_1) g(x_2) + \log g'(x_2) + \log f_1(x_1)
\]

(4.11)

with primes indicating differentiation. Differentiating (4.11) with respect to $x_2$,

\[
\frac{\partial}{\partial x_2} \log \lambda_1(x_2) - \frac{\partial}{\partial x_2} \lambda_1(x_2) g(x_1) + \frac{\partial}{\partial x_2} f_2(x_2) = -\frac{\partial}{\partial x_2} \lambda_2(x_1) \frac{\partial g(x_2)}{\partial x_2} + \frac{\partial}{\partial x_2} g'(x_2)
\]

(4.12)

Now differentiating (4.12) with respect to $x_1$, we have

\[
\frac{\partial \lambda_1(x_2)}{\partial x_2} \cdot \frac{\partial g(x_1)}{\partial x_1} = \frac{\partial \lambda_2(x_1)}{\partial x_1} \cdot \frac{\partial g(x_2)}{\partial x_2}
\]

or

\[
\frac{\partial \lambda_1(x_2)}{\partial x_2} \cdot \frac{\partial g(x_1)}{\partial x_2} = \frac{\partial \lambda_2(x_1)}{\partial x_1} \cdot \frac{\partial g(x_2)}{\partial x_1}
\]

(4.13)
For equation (4.11) to be true for all $x_1, x_2$ it must be true that

$$\frac{\partial \lambda_i(x_j)}{\partial x_j} = \theta, \quad i, j = 1, 2 \quad i \neq j$$

(4.14)

where $\theta$ is a constant independent of both $X_1$ and $X_2$. Since this solution is unique, the value of $\lambda_i(x_j)$ that satisfy (4.10) is

$$\lambda_i(x_j) = (\alpha_i + \theta g(x_j)).$$

(4.15)

Introducing this value of $\lambda_i(x_j)$ in (4.10) and simplifying

$$[g'(x_2)]^{-1} f_2(x_2) (\alpha_1 + \theta g(x_2)) \exp(\alpha_2 g(x_2))$$

$$= [g'(x_1)]^{-1} f_1(x_1) (\alpha_2 + \theta g(x_1)) \exp(\alpha_1 g(x_1))$$

for all $x_1, x_2$. This however means that for some constant $c > 0$,

$$f_1(x_1) = C g'(x_1) (\alpha_j + \theta g(x_1))^{-1} \exp(\alpha_1 g(x_1)).$$

(4.16)

From (4.10), (4.14) and (4.16) the joint density of $(x_1, x_2)$ is
\[ f(x_1, x_2) = C \cdot g'(x_1)g'(x_2) \exp(-\alpha_1 g(x_1) - \alpha_2 g(x_2) - \Theta g(x_1)g(x_2)) \]  

(4.17) 

\( x_1, x_2 \) belonging to \((u, c)\), \( \alpha_i > 0, \Theta > 0, \ i = 1, 2. \)

In particular when \( \Theta = 0 \), we have the case of independence of \( x_1, x_2 \). The constant \( C \) can be obtained as follows.

We have

\[
\int_0^\infty \int_0^\infty f(x_1, x_2) dx_1 dx_2 = 1 \\
\int_u^\infty \frac{-\alpha_1 g(x_1)}{C} \int_u^\infty g'(x_1) (\alpha_2 + \Theta g(x_1))^{-1} dx_1 = 1.
\]

That is,

\[
C \theta^{-1} \exp(-\alpha_1 \alpha_2 \theta^{-1}) E_1(\alpha_1 \alpha_2 \theta^{-1}) = 1
\]

or

\[
C = \theta \exp(-\alpha_1 \alpha_2 \theta^{-1}) \left[ E_1(\alpha_1 \alpha_2 \theta^{-1}) \right]^{-1}.
\]

(4.18)

Corresponding survival function is obtained as

\[
R(x_1, x_2) = \int_\infty^\infty \int_\infty^\infty f(x_1, x_2) dx_1 dx_2.
\]

\[
= C \int_\infty^\infty \int_\infty^\infty g'(x_1) g'(x_2) e^{-\alpha_1 g(x_1) - \alpha_2 g(x_2) - \Theta g(x_1)g(x_2)} dx_2 dx_1
\]
\[
\int_{x_1}^{\infty} e^{-\alpha_2 g(x_1) - \theta g(x_1) g(x_2)} e^{-\alpha_1 g(x_1) - \theta g(x_1) g(x_2)} g'(x_1) dx_1
\]

and gives

\[
\int_{x_1}^{\infty} e^{-\alpha_2 g(x_1) - \alpha_1 g(x_1) + \alpha_1 g(x_1) + \alpha_2 g(x_2) + \theta g(x_1) g(x_2)}
\]

(4.19)

4.3. PARTICULAR CASES.

1. Taking \( G(x_1, x_2) = x_1 + x_2 \)

\[
f(x_1, x_2) = C \exp(-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2)
\]

\( x_1, x_2 > 0 \), which is the bivariate exponential distribution obtained in Arnold and Strauss (1988) and Abrahams and Thomas (1984).

2. \( G(x_1, x_2) = x_1 x_2 \)

implies

This is the bivariate distribution with Pareto I model as

a bivariate distribution with Weibull conditionals results.


\[ g(x_i) = a \log x_i, \quad i = 1, 2 \]

and gives

\[ f(x_1, x_2) = c x_1^{-(\alpha_1 + 1)} x_2^{-(\alpha_2 + 1)} e^{-\theta \log x_2}, \]

\[ \alpha_1, \alpha_2 > 0, \quad \theta \geq 0, \quad x_1, x_2 > 1. \]

This is the bivariate distribution with Pareto I model as conditionals.

3. \( g(x_1, x_2) = (x_1^{\beta} + x_2^{\beta})^{1/\beta} \)

gives

\[ f(x_1, x_2) = C \beta \beta^{-1} x_1^{\beta-1} x_2^{\beta-1} \exp(-\alpha_1 x_1^\beta - \alpha_2 x_2^\beta - \theta x_1^\beta x_2^\beta), \]

\[ \beta > 0, \quad \alpha_1 > 0, \quad \theta \geq 0, \quad i = 1, 2, \quad x_1, x_2 > 0. \]

A bivariate distribution with Weibull conditionals results.

4. \( g(x_1, x_2) = x_1 + x_2 + ax_1 x_2 \)

implies

\[ g(x_1 + x_2 + ax_1 x_2) = g(x_1) + g(x_2) \]

and

\[ g(x_i) = \log(1 + ax_i), \quad i = 1, 2. \]

The joint density is
\[ f(x_1', x_2) = C (1+ax_1)^{-\alpha_1} (1+ax_2)^{-\alpha_2} \theta \log(1+ax_2) \]
\[
\alpha_1, \alpha_2, a > 0, \theta \geq 0.
\]

5. \[ G(x_1, x_2) = x_1 + x_2 / \left[ 1 + \frac{x_1 x_2}{p^2} \right] \]

implies

\[ f(x_1, x_2) = \frac{4Cp^2}{(p^2-x_1^2)(p^2-x_2^2)} \left( \frac{p-x_1}{p+x_1} \right)^{-\alpha_1} \left( \frac{p-x_2}{p+x_2} \right)^{-\alpha_2} \theta \log \left( \frac{p-x_2}{p+x_2} \right) \left( \frac{p-x_1}{p+x_1} \right) \]

\[ 0 < x_1 < p, 0 < x_2 < p, \alpha_1, \alpha_2 > 0, \theta \geq 0, p > 0. \]

In all these cases \( C \) is as in equation (4.18). The solution of the functional equations in the examples are available in Aczel (1966).

Remarks

The unique bivariate distribution with Pareto II conditionals obtained in example 4 of above differs from a similar model derived in Arnold (1987). Arnold chooses the
scale parameters to depend on the conditioned variable and the shape parameter is fixed. In the model described here, the scale parameter remains unaltered, while the shape parameter changes with the values of the conditioned variable.

4.4. RANDOM ENVIRONMENTAL MODELS.

A working system is often affected by the changes in its surroundings. The environment in which the system is working need not be the same as the laboratory environment, under which the system was designed and the prospective reliability was determined. The working environment comprises of a number of observable and unobservable factors whose intensities change over time in a random manner. For example, the system might have been built on the premise that the components are structurally independent so that when they work in a common environment, the expectation is that they fail independently. However the common working condition may induce certain kind of relationships among the
components that makes the assumption independent failure times untenable. Thus the reliability of the system is often affected sometimes adversely and sometimes favourably, when the system operates in places different from the initial test site. It is important to assess the manner and extent by which the reliability is affected due to a change in environment and therefore extensive studies have been earned out by various researchers on models that can explain this fact.

Lindley and Singpurwalla (1986) have studied systems sharing common environment and Currit and Singpurwalla (1988) analysed the reliability function of Lindley and Singpurwalla model, in the parallel and series systems and have obtained a formula for making Bayesian inferences for the reliability function. Nayak (1987), Cinlar and Özelkici (1987), Roy (1989), Bandyopadhyay and Basu (1990), Gupta and Gupta (1990) Lee and Gross (1991), Sankaran and Nair (1993), Singpurwalla and Youngren (1993) etc; have considered environmental models in detail. It is
customary in modelling problems to assume that the failure rate of the system working in the new environment is given by $\eta h(x_1, x_2)$ where $h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$ is the vector failure rate, when the system has worked in the test environments. In this representation, $\eta$ stands for the effect on the failure rate due to the change in environment. Thus when the environment factor $\eta > 1$ ($\eta < 1$, $\eta = 1$) the new working conditions are assumed to be harsher (milder, same as) than the original work site. Since the influence of the changed environment is seldom known exactly, it is reasonable to take $\eta$ as a random variable and to assume a suitable probability density function for it. One such choice for the distribution of $\eta$ is the gamma density

$$f(\eta|m,p) = \frac{m^p}{\Gamma(p)} e^{-m\eta} \eta^{p-1}, \ m > 0, \ p > 0. \quad (4.20)$$

Consider a two component system, with life lengths $X_1$ and $X_2$. Originally, the system is assumed to have a distribution function specified by (2.8). The vector valued failure rate of the system is given in equation (2.58). While working in an environment with environment factor $\eta$, 

$$\eta h(x_1, x_2) = (\eta h_1(x_1, x_2), \eta h_2(x_1, x_2))$$
its failure rate vector get changed to $\eta \ h(x_1, x_2)$, as

$$\eta \ h(x_1, x_2) = (\eta (\alpha_1 + \theta g(x_2))g'(x_1), \eta (\alpha_2 + \theta g(x_1))g'(x_2)) \quad (4.21)$$

which gives the new survival function of $(X_1, X_2)$ as

$$R(x_1, x_2) = \exp(-\eta \alpha_1 g(x_1) - \eta \alpha_2 g(x_2) - \eta \theta g(x_1)g(x_2)). \quad (4.22)$$

Accounting the uncertainty of $\eta$, by averaging this over the distribution of $\eta$, given by (3.20)

$$R_\eta(x_1, x_2) = \int_0^\infty \frac{m}{\Gamma(p)} \frac{e^{-m\eta}}{\eta^{p-1}} e^{-\eta(\alpha_1 g(x_1) + \alpha_2 g(x_2) + \theta g(x_1)g(x_2))} d\eta$$

$$= [1 + a_1 g(x_1) + a_2 g(x_2) + b g(x_1)g(x_2)]^{-p} \quad (4.23)$$

where $a_i = \frac{\alpha_i}{m}$, and $b = \theta / m$, $i = 1, 2$.

The corresponding density function is given by

$$f(x_1, x_2) = p[p(a_1 + bg(x_2)(a_2 + bg(x_1)) + a_1 a_2 - b)] g'(x_1)g'(x_2) \quad (p+2)$$

$$[1 + a_1 g(x_1) + a_2 g(x_2) + b g(x_1)g(x_2)]^{-p-2} \quad (4.24)$$

and the marginal density functions are
The conditions on the parameters of the model derive from

\[ R(x_1, u) \geq R(x_1, x_2) \]

or

\[ (1 + a_1 g(x_1))^p \geq (1 + a_1 g(x_1) + a_2 g(x_2) + bg(x_1)g(x_2))^p \]

\[ 1 + a_1 g(x_1) \leq (1 + a_1 g(x_1) + a_2 g(x_2) + bg(x_1)g(x_2)) \]

\[ 0 \leq (a_2 + bg(x_1))g(x_2). \]

Since \( g(.) \) is monotonic increasing and \( g(u) = 0 \), the above inequality holds good for all \( x_1, x_2 \) if and only if \( a_2 > 0 \) and \( b > 0 \). Similarly we get

\[ 0 \leq (a_1 + bg(x_2))g(x_1) \]

which gives \( a_1 > 0, b > 0 \). From the assumption of gamma density one gets \( p > 0 \).

Also \( f(u, u) \geq 0 \) leaves the condition,

\[ p(p(a_1 a_2) + a_1 a_2 - b) > 0 \]

or

\[ (p+1)a_1 a_2 \geq b. \]
Thus \( 0 < b \leq (p+1)a_1a_2 \).

Thus the conditions on the parameters are

\[
a_i > 0, \ i = 1, 2, \ p > 0, \ 0 \leq b \leq (p+1)a_1a_2.
\]

The family of distributions obtained under aforementioned framework includes a large class of distributions, like Pareto distributions of Hutchinson (1979), Lindley and Singpurwalla (1986), Burr distributions of Takahasi (1965), Durling et al (1970). These distributions are considered in the forthcoming section.

4.5. PARTICULAR CASES.

1. When \( g(x_i) = x_i, \ i = 1, 2, \ u = 0 \), the form of original distribution is Gumbel's bivariate exponential distribution specified by (2.22) corresponding environmental model takes the form,

\[
R(x_1, x_2) = [1+a_1x_1+a_2x_2+bx_1x_2]^{-p}; \ x_1, x_2 > 0 \quad (4.26)
\]

which is bivariate Pareto of Hutchinson (1979). By taking \( b=0 \) in equation (3.26), bivariate Pareto
distribution of Lindley and Singpurwalla (1986) is obtained.

2. \[ g(x_i) = \log x_i \quad i=1,2 \] original distribution is bivariate Pareto Type I, specified by equation (2.23) and accordingly, equation (4.23) changes to
\[ R(\eta | x_1, x_2) = [1 + a_1 \log x_1 + a_2 \log x_2 + b \log x_1 \log x_2]^{-\eta x_1, x_2 > 1}. \quad (4.27) \]

3. \[ g(x_i) = x_i^{\beta} \] the parent distribution becomes bivariate Weibull given by (2.27), and the environmental model is
\[ R(\eta | x_1, x_2) = [1 + a_1 x_1^{\beta} + a_2 x_2^{\beta} + b x_1^{\beta} x_2^{\beta}]^{-\eta x_1, x_2 > 0}. \quad (4.28) \]
which is bivariate Burr distribution of Durling et al (1970). When \( b = 0 \), bivariate Burr distribution of Takahasi (1965) results.

4. When \[ g(x_i) = \log(1 + ax_i) \] the parent distribution becomes bivariate Pareto Type II, and the environmental model arising from (4.23) is
\[ R(\eta | x_1, x_2) = [1 + a_1 \log(1 + cx_1) + a_2 \log(1 + cx_2) + b \log(1 + cx_1) \log(1 + cx_2)]^{-\eta}. \quad (4.29) \]
5. When \( g(x_i) = \log \frac{c+x_i}{c-x_i} \), with \( 0 < x_i < c \), \( i=1,2 \) the form of original distribution is bivariate finite range distribution specified by (2.32). Corresponding environmental model is

\[
R_\eta(x_1, x_2) = \left[ 1 + a_1 \log \frac{c+x_1}{c-x_1} + a_2 \log \frac{c+x_2}{c-x_2} + b \log \frac{c+x_1}{c-x_1} \log \frac{c+x_2}{c-x_2} \right]^{-p}. \tag{4.30}
\]

To be able to analyse the reliability of this system in a changed environment, we note that, the vector valued failure rate of the system is

\[
h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))
\]

with

\[
h_i(x_1, x_2) = \frac{p(a_i + bg(x_j)) g'(x_i)}{[1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2)]}. \tag{4.31}
\]

Thus for a quantitative assessment of the effect of the new environment on the system the objects of comparison are the failure rates in (4.31) and (2.58). Using the superscripts \'e' and \'o' to differentiate the failure rates of changed
and original environments the relative measures that facilitate comparison are

\[
\frac{h_i^e(x_1, x_2) - h_i^o(x_1, x_2)}{h_i^o(x_1, x_2)} = \frac{h_i^e(x_1, x_2)}{h_i^o(x_1, x_2)} - 1. \tag{4.32}
\]

Thus when actually the system is operated in a different set of conditions, the error that would be committed through the measurement of the failure rate would be positive or negative according as

\[
\frac{h_i^e(x_1, x_2)}{h_i^o(x_1, x_2)} > 1.
\]

With respect to our model, this happens when

\[
p(a_1 + bg(x_j)) > \frac{(1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))(a_1+\theta g(x_j))}{(1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))} < 1.
\]

For \( i = 1 \), the first condition reduces to

\[
P(a_1 + bg(x_2)) > (1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2))(a_1+\theta g(x_2)) = m\left(1+a_1 g(x_1)+a_2 g(x_2)+bg(x_1)g(x_2)\right)(a_1+\theta g(x_2))
\]

or when

\[
\int_{\lambda_1}^{\lambda_2} \lambda_1 g(x_2) \, dx_2 = \frac{1}{\lambda_1} - \int_{\lambda_2}^{\lambda_1} \lambda_1 g(x_2) \, dx_2
\]
Since $E(\eta) = p/m$, we conclude that whenever

$$E(\eta) > (\leq) (1 + a_1 g(x_1) + a_2 g(x_2) + b g(x_1) g(x_2))$$

the changed environment would cause failures more (less) frequently than in the test condition, when

$$E(\eta) = [1 + a_1 g(x_1) + a_2 g(x_2) + b g(x_1) g(x_2)],$$

the two environments are identical.

4.6. CHARACTERIZATIONS

As discussed in Section 3.2, measures similar to AFR, GFR and HFR can be obtained if the concept of failure rate in them is replaced by mean residual life in the bivariate case. Accordingly, the arithmetic mean mean residual life (AM MRLF) is defined as the vector,

$$K(x_1, x_2) = (K_1(x_1, x_2), K_2(x_1, x_2))$$

where

$$K_i(x_1, x_2) = \frac{1}{x_i} \int_{x_1}^{x_i} r_1(x_1, x_2) \, dx_1 \quad (4.33)$$

and

$$\frac{1}{x_i} \int_{x_1}^{x_i} \log r_1(x_1, x_2) \, dx_1 \quad (4.34)$$
and

\[ r_i(x_1, x_2) = \frac{1}{R(x_1, x_2)} \int_{x_1}^{\infty} R(x_1, x_2) \, dx_1, \quad i=1,2 \]

is the mean residual life of \( i \)th component.

Likewise, the bivariate geometric mean residual life (GM MRLF) is defined as,

\[ L(x_1, x_2) = (L_1(x_1, x_2), L_2(x_1, x_2)) \]

where

\[ L_i(x_1, x_2) = \exp \left\{ \frac{1}{x_1} \int_{0}^{x_1} \log r_i(x_1, x_2) \, dx_1 \right\} \quad (4.34) \]

and the bivariate harmonic mean residual life (HM MRLF) is defined as

\[ M(x_1, x_2) = (M_1(x_1, x_2), M_2(x_1, x_2)) \]

with

\[ M_i(x_1, x_2) = \left[ \frac{1}{x_1} \int_{0}^{x_1} \frac{1}{r_i(x_1, x_2)} \, dx_1 \right]^{-1} \quad (4.35) \]

Using the concepts of AM MRLF, GM MRLF and HM MRLF together with AFR, GFR and HFR we can characterize some of the models already considered in the sequel.
Theorem 4.1

A random vector \( X = (x_1, x_2) \) in \( \mathbb{R}^2 \) with absolutely continuous distribution satisfies the property

\[
K(x_1, x_2) = r(x_1, x_2). \tag{4.36}
\]

For every \( x_1, x_2 > 0 \) if and only if the distribution of \( (X_1, X_2) \) is bivariate exponential of Gumbel (1960).

Proof:

When \( (X_1, X_2) \) is of Gumbel's form,

\[
r_i(x_1, x_2) = (\lambda_i + \theta x_j)^{-1}.
\]

So that from (4.33) \( r_i(x_1, x_2) = K_i(x_1, x_2) \), establishing (4.36). Conversely, if (4.36) holds differentiating the identity (4.33) with respect to \( x_i \),

\[
x_i \frac{\partial K_i(x_1, x_2)}{\partial x_i} + K_i(x_1, x_2) = r_i(x_1, x_2)
\]

or

\[
x_i \frac{\partial K_i(x_1, x_2)}{\partial x_i} = 0
\]

giving \( K_i(x_1, x_2) = r_i(x_1, x_2) = P_i(x_j) \).
Thus \( r_i(x_1, x_2) = (p_1(x_2), p_2(x_1)) \) and hence the result follows from Nair and Nair (1988).

**Corollary:**

\[
K(x_1, x_2) = (p_1(x_2), p_2(x_1))
\]

if and only if \((X_1, X_2)\) has Gumbel's bivariate exponential distribution.

Adopting the same logic, but with a little different algebra, it can be seen that the following theorems hold.

**Theorem 4.2**

\[
L(x_1, x_2) = r(x_1, x_2) \quad \text{for every } x_1, x_2 > 0 \quad \text{if and only if } (X_1, X_2) \text{ has Gumbel's distribution.}
\]

**Theorem 4.3**

\[
M(x_1, x_2) = r(x_1, x_2) \quad \text{for every } x_1, x_2 > 0 \quad \text{if and only if } (X_1, X_2) \text{ has Gumbel's distribution.}
\]

**Theorem 4.4**

A necessary and sufficient condition for \((X_1, X_2)\) to be
an absolutely continuous random vector in the support of $R_2^+$ satisfies any one of the following conditions

1. $K_i(x_1, x_2) H_i(x_1, x_2) = C$

2. $L_i(x_1, x_2) G_i(x_1, x_2) = C$

3. $M_i(x_1, x_2) A_i(x_1, x_2) = C$

for $i = 1, 2$, every $x_1, x_2 > 0$ and some positive real $c$ is that $(X_1, X_2)$ is distributed either as Gumbel's bivariate exponential distribution for $c = 1$ or a bivariate Pareto type in (4.26) for $c > 1$, or as bivariate finite range with survival function

$$P(X_1 > x_1, X_2 > x_2) = (1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)^d$$

(4.37)

$p_1, p_2, d > 0$, $0 < x_1 < p_1^{-1}$, $0 < x_2 < (1 - p_1 x_1)$, $1 - d < q p_1^{-1} p_2^{-1} < 1$

for $0 < C < 1$.

Proof:

Suppose (1) holds. Then for $i = 1$

$$\left(\frac{1}{x_1} \int_0^{x_1} r_1(t, x_2) dt\right) \left(\frac{1}{x_1} \int_0^{x_1} \frac{dt}{h_1(t, x_2)}\right)^{-1} = C$$
or

\[
\int_{0}^{x_1} r_1(t, x_2) \, dt = C \int_{0}^{x_1} \frac{dt}{h_1(t, x_2)}
\]

Differentiating with respect to \(x_1\),

\[
r_1(x_1, x_2) \cdot h_1(x_1, x_2) = C
\]

Similarly for \(i = 2\)

\[
r_2(x_1, x_2) \cdot h_2(x_1, x_2) = C
\]

This gives the form of

\[
r_i(x_1, x_2) = A x_1 + B_i(x_j) \quad i, j = 1, 2 \quad i \neq j
\]

which characterizes the models in the Theorem for the specified values of \(C\) as given in Sankaran and Nair (1992).

When (2) holds, for \(i = 1\),

\[
\exp \left\{ \frac{1}{x_1} \int_{0}^{x_1} \log r_1(t, x_2) \, dt \right\} \exp \left\{ \frac{1}{x_1} \int_{0}^{x_1} \log h_1(t, x_2) \, dt \right\} = C
\]

which gives the same expression for \(r_1(x_1, x_2)\) as in the case of assumption (1). The proof for case (3) follows suit and this establishes the Theorem.
Theorem 4.5

An AM MRLF of the form,

\[ K_1(x_1, x_2) = ax_1 + b_1(x_2) \]

characterizes the Gumbel's bivariate law for a = 0 Pareto II distribution for \( a > \frac{1}{2} \) and the finite range distribution for \( 0 < a < \frac{1}{2} \).

Proof:

\[ K_1(x_1, x_2) = ax_1 + b_1(x_2) \]

\[ \leftrightarrow \frac{1}{x_1} \int_0^{x_1} \log r_1(t, x_2) dt = ax_1 + b_1(x_2) \]

\[ \leftrightarrow r_1(x_1, x_2) = 2ax_1 + b_1(x_2). \]

4.7 CONCLUSION

The present study has considered three general families of distributions, each bringing a class of bivariate distributions under a uniform framework. They provide new derivations for some well known distributions as
well as certain new bivariate continuous distributions. Derivation of all the models are based on extensions of concepts that have found acceptance among a large audience.

We have presented characterization theorems that will enable identification of the member which will suit the observations in a practical situation.

In view of the general functional form appearing in the survival function in each family, general characterization theorems were hard to establish, as in many cases the assumed properties lead to functional equations that are difficult to solve, by the existing methods. However, characterization theorems based on basic reliability concepts have been established, where the models are most apt to use. More characteristics of the families are being investigated and is hopefully expected to be presented in a future work.