Chapter I
INTRODUCTION

1.1 CONSTRUCTION OF MULTIVARIATE MODELS

The early developments in statistical distribution theory were dominated by the normal distribution due to various facts such as the pleasing solutions the assumptions of normality can produce, the belief of users that in many practical situations normal observations will result either naturally or as a very good approximations by invoking limit theorems and the large volume of theoretical results that justified normal approximations. A comprehensive study of multivariate distributions including mechanisms to generate them began only from the sixties of the century, and most of the developments in this connection are only in formative stages and is therefore a fertile area of research. One of the main problems associated with the evolution of multivariate models is that there is no unique way of extending a corresponding univariate law and this has
created a large body of distributions in the multivariate set up.

Of the several standard methods to generate multivariate distributions, one is to generalise a system or equation defining a univariate distribution into the multivariate case. This includes generalization of the differential equation representing the Pearson family by Van Uven (1925, 1926, 1929, 1947), Steyn (1960), Elderton and Johnson (1969) and the translation equation system by Johnson (1949). A second method is to construct a multivariate distribution by specifying the form of its marginal distributions. Work in this connection can be seen in the papers of Frechet (1951), Mongestern (1956), Farlie (1960), Mardia (1970). However, it is to be noted that there exist several multivariate forms with the same type of marginals and therefore, the basis of adopting a particular method in this category has to be accompanied by a proper justification based on physical considerations. When we replace marginal distributions by conditionals in the above
specification, the picture is more rosy as in many cases it is possible to extract unique joint distribution, at least in the bivariate cases. Considerable interest has been aroused recently in deriving bivariate models which have pre-designated forms for their conditional distributions and the present study reviews the important work in this area and works out some new results in this direction.

Another approach results when one starts with the univariate density and then postulate a functional form for its multivariate analogue. For multivariate distributions generated in this fashion, we refer to Mardia (1962), Gumbel (1960), Bildikar and Patil (1968), Day (1969).

The main limitation of this approach is that, in many cases, it is difficult to give a physical interpretation to the model and its parameters. In contrast, the modelling approach takes into consideration the physical states of the system, the relationships between the variables and the parameters involved etc. so that when real world situations that conform to these properties, are
encountered the model becomes the right choice. The distributions of Fruend (1961), Marshall and Olkin (1967), Block and Basu (1974) are typical examples. An amalgamation of these two types of model generation can be affected successfully, if the models belonging to the first category can be derived using the well-known system properties. The present study envisages some interpretations to some of the existing models.

A multivariate model that inherits the essential features of the corresponding univariate version is often a reasonable requirement for constructing the former. This can be easily accomplished by identifying a characteristic property that is of interest of the univariate law, which needs extension to higher dimension and then to find its multivariate analogue, wherever possible, uniquely. The construction of the desired multivariate distribution is complete, once the law is characterized by the newly identified property. Galambos and Kotz (1977) considers this as the soundest approach to develop multivariate distributions.
In the current investigation also, it is proposed to work along these lines in deriving certain classes of distributions, the main basis of which is an extension of the lack of memory property to the bivariate case. To prepare the background for the research problem in this study we briefly survey the important results in literature that is of concern.

1.2 SURVEY OF LITERATURE.

The importance of the exponential distribution among the class of continuous probability models is next only to that of the normal distribution. The reasons for the popularity of the exponential distribution in theoretical and applied investigations can be attributed mainly to the lack of memory property, its relation with the Poisson process and the properties enjoyed by the order statistics. Of these, the lack of memory property is perhaps the best studied and widely applied among the properties of the model, and the one that lends itself to
extensions in various directions. From a theoretical point of view lack of memory property is justified from the fact that there are many real life situations where it holds good, while in applied work, it is extensively used as a characteristic property of the exponential distribution. In view of its implications in reliability and life testing, the property is best expressed in terms of the lifelength of a component or device, although the meaning conveyed by it can be shared among the class of other duration variables as well.

If $X$ is a non-negative random variable possessing absolutely continuous distribution with respect to Lebesgue measure, we say that the random variable $X$ or its distribution has lack of memory property if for all $x,y \geq 0$ such that $P(X \geq y) > 0$,

$$P(X \geq x+y \mid X \geq y) = P(X \geq x)$$

(1.1)

or equivalently, if $P(X = 0) \neq 1$ for all $x,y \geq 0$

$$P(X \geq x+y) = P(X \geq x) P(X \geq y).$$

(1.2)
In terms of the survival function of the random variable

\[ R(x) = P(X \geq x), \quad (1.6) \]

(1.2) is restated as \( R(x+y) = R(x) R(y) \). \( (1.3) \)

The characterization of the exponential distribution, using any one of the equivalent forms (1.1) to (1.3) arises from the functional equation explored by Cauchy (1821) and Darboux (1875)

\[ U(x+y) = U(x)U(y) \quad (1.4) \]

whose solution, is either \( U(x) = 0 \) for all \( x \) or \( U(x) = e^{-\lambda x} \) for some constant \( \lambda \), whenever \( U(x) \) is a solution defined for \( x > 0 \).

For an absolutely continuous survival function \( R(x) \), its failure rate \( h(x) \) is defined as

\[ h(x) = \frac{-d \log R(x)}{dx}. \quad (1.5) \]

The lack memory property is equivalent to the statement

\[ h(x) = a \text{ constant}. \]

Further, the truncated mean or mean residual life defined as
\[ r(x) = \mathbb{E}(X-x \mid X \geq x) \]

\[ = [R(x)]^{-1} \int_{x}^{\infty} R(t) dt, \quad (1.6) \]

often interpreted as the average life time remaining to a component at age \( z \), is related to the failure rate through the equation

\[ h(x) = [r(x)]^{-1} \left( 1 + \frac{dr(x)}{dx} \right). \quad (1.7) \]

It is given in Cox (1962) that for the exponential distribution \( r(x) = \text{a constant} \), Galambos and Kotz (1977) establishes the equivalence of lack of memory property, constancy of the failure rate, and constancy of the mean residual life.

The extension of the lack of memory property is often envisaged to serve one or both of the following objectives.

a. to extend the domain of the values of \( x \) and \( y \) for which equation (1.1) is true.

b. to provide a larger family of distributions that includes the exponential model as a special case.
By finite induction we obtain from (1.1) that

$$R(x_1 + x_2 + \ldots + x_n) = R(x_1) R(x_2) \ldots R(x_n)$$  \hspace{1cm} (1.8)

Setting $x_1 = x_2 = \ldots = x_n = x \geq 0$ and requiring the resulting equation to hold for all integers $n \geq 1$ results in a characterization of the exponential model, in modification of the condition that (1.1) holds for all $x, y \geq 0$. Further Fortet (1977) considered the assumption that (1.7) is true almost everywhere with respect to Lebesgue measure for $(x,z)$ in $[0,\infty)$, is sufficient to guarantee that the distribution is exponential. Another result in this direction due to Sethuraman (1965) shows that $x_1 = x_2 = \ldots = x_n = x, x \geq 0$

$$\log n_1$$

in (1.8), together with $\frac{\log n_1}{\log n_2}$ is irrational, where $n_1$ and $n_2$ are integers satisfying (1.8) characterizes the distribution. Alternatively, a survival function satisfying (1.1) for two values $y_1$ and $y_2$ of $y$ such the $y_1|y_2$ is irrational and for all non-negative values of $x$, is equivalent to the lack of memory property.

Obviously, the values $x$ and $y$ in (1.1) can be replaced by random variables $Y$ and $Z$ with degenerate
distributions to produce equivalent characterizations. Once the assumption of degenerate distribution is removed, the equality in (1.1) changes to the inequality

\[ R(x+y) \geq R(x) \cdot R(y) \]

where \( x \) and \( y \) are random variables hailing from two families \( D_1 \) and \( D_2 \) respectively, with the following properties.

1. Every member of \( D_1 \) and \( D_2 \) is independent of \( X \).
2. Every member of \( D_1 \) is independent of every member of \( D_2 \)
   and
3. \( P(a < Y < b) > 0 \) and \( P(a < Z < b) > 0 \) for every \( [a,b] \).

A variant approach to extension of the lack of memory property takes advantages of the equivalence of lack of memory property and the constancy of residual life function for all \( x \geq 0 \). Often this approach provide a larger class of distributions than the exponential.

The choice in most cases will be a function \( g(x) \) for which either

\[ E(h(x) \mid X \geq x) = g(x); \ x \geq 0 \quad (1.9) \]
or

\[ E(h(X-x) \mid X \geq x) = g(x) \]  \hspace{1cm} (1.10)

where \( h(.) \) and \( g(.) \) are known functions ending up with the solutions that are proper survival functions. For details we refer to Kotlarski (1972), Laurent (1972) Shanbhag and Rao (1975), Gupta (1976) and Dallas (1976).

An attempt along some what different lines by Muliere and Scarsini (1987) to extend (1.1) in generating a class of probability distribution, uses the extension of the LMP by the following equation,

\[ P(X > x*y) = P(X > x) \ P(X > y). \]  \hspace{1cm} (1.11)

In equation (1.11) '*' is used to represent a binary operation that is associative, and reducible (\( x * y = x*z \rightarrow y = z \)). When the last equation is read as

\[ R(x*y) = R(x) \ R(y) \]  \hspace{1cm} (1.12)

its only continuous solution is

\[ x * y = g^{-1}(g(x) + g(y)) \]  \hspace{1cm} (1.13)

with \( g(.) \) continuous and strictly monotonic. In this case, the only continuous solution of (1.12) is
\[ R(x) = \exp(-\lambda g(x)), \lambda > 0 \quad (1.14) \]

\[ u = g^{-1}(0) < x < g^{-1}(\infty). \] With appropriate choice for \( g(x) \) the authors characterize class of probability distribution that includes the exponential, Pareto Type I, Weibull models. As a bivariate extension of the above functional equation Muliere and Scarsini (1987) also derives Marshall Olkin (1967) type class of distributions also.

The concepts and methods so far reviewed extends to a multivariate setup. Since our interest in the present investigation concerns only bivariate distributions, the important developments in this area are now presented. An obvious extension of the LMP in the bivariate case is defined by the relationship

\[ R(x_1 + y_1, x_2 + y_2) = R(x_1, x_2) R(y_1, y_2) \quad (1.15) \]

for all \( x_1, x_2, y_1, y_2 > 0 \), where

\[ R(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) \]

is the survival function of the random vector \( X = (x_1, x_2) \) in the support of \( R_2^+ = \{ (x_1, x_2) \mid x_1, x_2 > 0 \} \).
A serious limitation of defining bivariate LMP, by equation (1.15) is that its unique solution turns out to be

\[ R(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2} \]

(1.16)

which is the trivial bivariate exponential distribution which is the product of its marginals. In the reliability context this amounts to the distribution of the life times of a two component system, in which the life time of each component is independent of the other. This severe restriction that prevents consideration of two-component systems where there is dependency among individual components, has led to a relaxation of the requirements on the values of \( y_1 \) and \( y_2 \). One way of doing this is to consider the equation,

\[ R(x_1 + t, x_2 + t) = R(x_1, x_2).R(t, t) \]

(1.17)

for all \( x_1, x_2, t > 0 \). Marshall and Olkin (1967) derived a solution of (1.17) by requiring the marginals of \( X_1 \) and \( X_2 \) to be exponential distribution with parameters \( \lambda_1 \) and \( \lambda_2 \) in the form,

\[ R(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)]. \]

(1.18)
In this setup $P(X=Y)$ is positive so that the simultaneous failure of the two components is a physical characteristic concerning the system, although this produces a singular component in the distribution. This is unavoidable in the sense that the assumptions of LMP, absolute continuity and exponential marginals can result only bivariate distribution with independent exponential marginals. It is therefore apparent that to arrive at a meaningful bivariate distribution, one has to abandon anyone of the three conditions mentioned just above. By preserving LMP and absolute continuity, Block and Basu (1974) derived bivariate exponential distribution, in which the marginals are mixture of exponentials. In spite of other extensions proposed by Friday and Patil (1977), Arnold (1975). Esary and Marshall (1974) based on reliability considerations a detailed assessment of these distributions vis-a-vis their usefulness through characterization by reliability concepts are yet to be established.

A fruitful alternative way of looking at the equipment behaviour can be accomplished by investigating the
behaviour of one of the components, when life time of the other is pre-assigned. The first work in this direction concerning the bivariate system, appears to be that of Johnson and Kotz (1975) who defined the vector valued failure rate of a device with component lifetimes \((X_1, X_2)\) as

\[
h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))
\]

where

\[
h_i(x_1, x_2) = \frac{-\partial \log R(x_1, x_2)}{\partial x_i}, \quad i = 1, 2
\]

Drawing parallels from the univariate theory, they considered the situation when

\[
h(x_1, x_2) = (c_1, c_2)
\]

where \(c_i\)'s are constants independent of \(x_1\) and \(x_2\), and established that such a case exist only when the joint distribution has independent exponential marginals. Accordingly, they considered the situation where the components of the failure rate are locally constant, in the form,

\[
h(x_1, x_2) = (A_1(x_2), A_2(x_1))
\]
and characterized the Gumbel's (1960) bivariate exponential distribution with survival function,

\[ R(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2), \quad (1.23) \]

\( \lambda_1, \lambda_2 > 0, \ 0 \leq \theta \leq \lambda_1 \lambda_2 \) with such a property.

In proving this result, they used the representation

\[ R(x_1, x_2) = \exp\left[ - \int_{0}^{x_1} h_1(t, 0) dt - \int_{0}^{x_2} h_2(x_1, t) dt \right] \quad (1.24) \]

that connects uniquely, the failure rate and the survival function. The fact that, unlike in the univariate case, the failure rate can be defined in a multiplicity of ways, leaves open the question of constructing multivariate distributions that can serve us models of specific equipment behaviour. This will be further discussed in the sequel.

Taking specific clues from the definition of bivariate failure rate, it is possible to look for representation for the mean residual life in higher dimensions. Defining the mean residual life function as the vector,
\[ r(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2)) \] (1.25)

where

\[ r_i(x_1, x_2) = \mathbb{E}(X_i - x_i \mid X_1 > x_1, X_2 > x_2) \] (1.26)

\[ i = 1, 2 \]

and using the unique representation

\[ r_1(0,0).r_2(x_1,0) R(x_1', x_2) r_1(x_1,0).r_2(x_1,x_2) \]

\[ \exp \left\{ -\int_0^{x_1} \frac{dt_1}{r_1(t,0)} - \int_0^{x_2} \frac{dt_2}{r_2(x_1,t_2)} \right\} \] (1.27)

Nair and Nair (1988 a) established a characterization of the Gumbel's distribution (1.23) using the local constancy of

\[ r(x_1, x_2) = (B_1(x_2), B_2(x_1)) \]

This result was further extended by Nair and Nair (1988b) by showing that the local constancy of the truncated moments,

\[ \mathbb{E}((X_i - x_i)^r \mid X_1 > x_1, X_2 > x_2) = B_i(x_j) \] (1.28)

\[ i, j = 1, 2, i \neq j \]

for every \( r = 1, 2, 3, \ldots \) is a characteristic property of the
same distribution. Nair and Nair (1991) further defined the local lack of memory property of the random vector $X$ by the relations,

$$P(X_i > x_i + y_i \mid X_1 > x_1, X_2 > x_2) = P(X_i > x_i \mid X_j > x_j),$$

$$i,j = 1,2 \text{ } i \neq j \quad (1.29)$$

and established the equivalence of (1.29) and (1.11).

Analogous to the univariate definition of LMP, one can think about a similar property for the conditional distribution arising from a bivariate distribution. In this way Nair and Nair (1991) defined the notion of conditional lack of memory for a random vector in the support of $R^+_2$ by the relationship

$$P(X_i \geq t_i + s_i \mid X_i \geq s_i, X_j = x_j) = P(X_i \geq t_i \mid X_j = x_j),$$

$$i,j = 1,2 \text{ } i \neq j \quad (1.30)$$

for all $x_j, t_i, s_i > 0$.

The equations (1.30) are satisfied if and only if the distribution of $X_i$ given $X_j = x_j$ has density,
\[ f(x_i | x_j) = \lambda_i(x_j) \exp(-\lambda_i(x_j)x_i) \]

which corresponds to the exponential form. Accordingly, the bivariate distribution that possesses conditional LMP is one for which the conditional densities are exponential. Arnold and Strauss (1988) has shown that there exist a unique bivariate distribution with exponential conditionals and obtained its density function as

\[
f(x_1, x_2) = \lambda_1 \lambda_2 \theta \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \delta \lambda_1 \lambda_2 x_1 x_2]
\]

\[ \lambda_1, \lambda_2 > 0, \delta \geq 0, x_1, x_2 > 0 \quad (1.31) \]

where \( \theta = \theta(\delta) = \delta e^{-1/\delta} / \left[ -E_1(1/\delta) \right] \)

and \( E_1 \) is the well known exponential integral function

\[
- E_1(u) = \int_u^{\infty} e^{-w} w^{-1} dw
\]

The above discussions provide the main results that are required to identify our research problem. References needed to supplement the specific problems will be provided in the appropriate chapters.
1.3 RESEARCH PROBLEM AND SUMMARY OF THE THESIS.

It is already shown that basically, there are two approaches to extend the definition of the LMP to the bivariate case; One given in Marshall and Olkin (1967) and the other designated as local LMP in Nair and Nair (1990).

The generalised version of LMP proposed by Muliere and Scarsini (1987) in the univariate case, naturally allows extension to the bivariate case. In the same manner as the LMP of the univariate exponential distribution was extended to generate bivariate exponential model of Gumbel (1960) through the local LMP, there is scope for attempting a localized bivariate version of the relation specified by equation (1.1.6i). Accordingly in Chapter II we consider the equations,

\[ P(X_i > G(x_i, y_i) \mid X_i > x_i, i=1,2) = P(X_i > y_i \mid X_j > x_j) \]

where \(G(\ldots)\) is a function satisfying certain algebraic properties. The family of continuous distributions
characterized by such a property is derived. It is shown that by appropriate choices of $G(.,.)$ the bivariate exponential, Pareto, Weibull etc. can be reached as special cases. It is well known that by monotone transformations of the exponential variable, it is possible to obtain several other continuous distributions as Weibull, Pareto, Logistic extreme value, Uniform etc. but there exist no meaningful single defining property that embraces all these models. The above mentioned defining property forms a basis from which all such related distributions can be brought under a uniform framework. After deriving the basic model giving rise to what is called the bivariate exponential type family, some of its important members are identified. The fundamental objective in studying a family of distributions is to derive the global properties enjoyed by it, so that individual investigation of each of its members can be avoided. Thus we look at the properties of the family, like, the constituent marginals, conditional distributions, moments regression and correlation etc. Since the major application is envisaged in the analysis of reliability, the
basic quantities required in such cases for modelling equipment behaviour such as failure rate, mean residual life and residual life distributions are derived.

Although the Bivariate exponential type (BET) family itself originated from a characteristic property, we look into other possible characterizations in Chapter III. The main objective of this investigation is to find out interesting properties by which specific members of the family can be identified in a practical situation. Characterizations through conditional distributions, functional forms of failure rates, conditional expectation and mean failure rates are presented.

Following the logic involved in the extension of local LMP, we consider extension of the conditional LMP defined in equation (1.42) in Chapter IV. In effect, this provides bivariate distributions that are determined uniquely, by their conditional distributions. The results in the Chapter supplements the efforts recently made by researchers in addressing the problem of finding bivariate
distributions that are compatible with conditional distributions with pre-designated forms. The members of the BET family of Chapter II are viewed to represent the distribution of life lengths in two-component systems assessed in the tests in laboratory where the components are built. When they work outside the laboratory, the operating condition might be different. A new system of models that can accommodate such changes are also discussed in Chapter IV. This approach provides several new additions to the class of continuous bivariate distributions.