CHAPTER 1

THE CATEGORY FTOP

Introduction

Categorical approach in fuzzy set theory was initiated by J.A. Goguen [25], R. Lowen [42], C.K. Wong [76], S.E. Rodabaugh [59], P. Eklund [17-18] and U. Cerruti [6]. Goguen in [25] gave the first categorical definition of fuzzy sets. He constructed the category Set-L, where the objects are \((X, \mu)\) where \(\mu : X \rightarrow L\) is an \(L\)-fuzzy subset of \(X\) and the morphisms\(f : (X, \mu) \rightarrow (Y, \beta)\) are functions \(f : (X, \mu) \rightarrow (Y, \beta)\) such that for every \(x \in X\), \(\mu(x) \leq \beta(f(x))\). C.K. Wong [76] defined two categories \(\mathcal{R}_1\) and \(\mathcal{R}_2\) of fuzzy sub sets as follows:

Let \((X, \alpha)\) denote the ordered pair of a set \(X\) and a fuzzy sub set \(\alpha\) of \(X\). Then \(\mathcal{R}_1\) is the collection of all \((X, \alpha), (Y, \beta)\).... For each pair of objects \((X, \alpha), (Y, \beta)\) of \(\mathcal{R}_1\) define a set \(\text{mor} \{X, \alpha), (Y, \beta)\} = \{f, f\}_{\alpha \beta}\) where \(f\) ranges over all possible set mappings from \(X\) to \(Y\) and \(f\) is the induced mapping from fuzzy set \(\alpha\) to fuzzy set \(\beta\) defined by \(f(\mu(x)) = \mu(f(x))\). Also \(\mathcal{R}_2\) is the collection of all \((X, \mathcal{A})\) where \(\mathcal{A}\) denotes the collection of all fuzzy subsets of \(X\). For any two objects \((X, \mathcal{A}), (Y, \mathcal{B})\) of \(\mathcal{R}_2\) define \(\text{mor} \{X, \mathcal{A}), (Y, \mathcal{B})\} = \{f, f\}_{\mathcal{A} \mathcal{B}}\) where \(f\) ranges over all possible...
set mappings from \( X \) to \( Y \) and \( f \) is the collection of all mappings from any fuzzy set \( A \) in \( X \) to any fuzzy set \( B \) in \( Y \) induced by \( f \).

In the case of fuzzy topology, there are various interesting categories of fuzzy topological spaces available in the literature. The collection of all fuzzy topological spaces and fuzzy continuous maps form a category. Since C.L.Chang [8], R.Lowen [37] and J.A.Goguen [24] have defined fuzzy topology in different ways, each of them defines a different category of fuzzy topological spaces.

In 1983 S.E.Rodabaugh [59] defined a new fuzzy topological category FUZZ. It is a significant generalization of all previous approaches to fuzzy topology. The objects of FUZZ are of the form \((X,L,T)\) where \((X,T)\) is an L-fuzzy topological space where \( L \) is a complete distributive lattice with universal bounds and order reversing involution. A morphism from \((X_1,L_1,T_1)\) to \((X_2,L_2,T_2)\) is a pair \((f,\phi)\) satisfying the following conditions.

i) \( f:X_1 \rightarrow X_2 \) is a function

ii) \( \phi^{-1}:L_2 \rightarrow L_1 \) is a function preserving \( \cap, \cup \)

iii) \( V \in T_2 \mapsto \phi^{-1} \circ V \circ f \in T_1 \)
In this chapter we introduce a new category FTOP, which appears to be the best framework to define fuzzy topological semigroups. In section 1 of this chapter we define an L-fuzzy topological space \((X, \mu, F)\) and obtain some of its basic properties.

In section 2 we define FTOP. The objects of FTOP are "L-fuzzy topological spaces" \((X, \mu, F)\) and the morphisms are the "fuzzy continuous maps" between two L-fuzzy topological spaces. The subcategories, subobjects, initial objects and final objects in FTOP are obtained. Also we find relations between FTOP and some other categories of fuzzy topological spaces.

1.1 L-fuzzy topological spaces

Definition 1.1.1

Let \(X\) be a set, \(\mu : X \rightarrow \mathbb{L}\) be an L-fuzzy subset of \(X\) and \(F\) be a subset of \(X\) satisfying the following conditions:

i) \(g \in F \implies g(x) \leq \mu(x) \ \forall \ x \in X\)

ii) \(\{g_i \text{ where } i \in I\} \subseteq F \implies \bigcup\{g_i \text{ where } i \in I\} \in F\)

iii) \(g_1, g_2 \in F \implies g_1 \cap g_2 \in F\)

iv) \(1, \mu \in F\) where \(1, \mu\) is a constant map from \(X\) to \(\mathbb{L}\) which takes the value 0 for every point \(x \in X\).
The triple \((X, \mu, F)\) is called an \(L\)-fuzzy topological space subordinate to \(\mu\) (or where there is no chance for confusion just \(L\)-fuzzy topological space). The members of \(F\) are called \(L\)-fuzzy open sets and the complements of members of \(F\) are called \(L\)-fuzzy closed sets.

If \(F\) consists of all \(L\)-fuzzy sub sets of \(X\) which are less than \(\mu\), it is called discrete \(L\)-fuzzy topology and if it consists of 1 and \(\mu\) only, it is called indiscrete \(L\)-fuzzy topology.

**Remark**

When \(L=[0,1]\) and \(\mu=\chi\) such that \(\chi(x)=1\) \(\forall x \in X\), an \(L\)-fuzzy topological space is nothing but a Chang's fuzzy space.

**Definition 1.1.2**

Let \((X_1, \mu_1, F_1)\) and \((X_2, \mu_2, F_2)\) be two \(L\)-fuzzy topological spaces. A mapping \(g\) of \((X_1, \mu_1, F_1)\) into \((X_2, \mu_2, F_2)\) is fuzzy continuous if:

i. \(\mu_1(x) \leq \mu_2(g(x)), \forall x \in X_1\)

ii. \(\mu_1 \cap g^{-1}(u) \in F_1, \forall u \in F_2\)
Remark

When \( L = [0,1] \), \( \mu_1 = \mu_1^X \) and \( \mu_2 = \mu_2^X \) this definition coincides with the definition 0.1.3.

Proposition 1.1.3

If \( f : (X_1, \mu_1^X, F_1) \longrightarrow (X_2, \mu_2^X, F_2) \) and \( g : (X_2, \mu_2^X, F_2) \longrightarrow (X_3, \mu_3^X, F_3) \) are fuzzy continuous then \( g \circ f \) is also fuzzy continuous.

Proof

Since \( f \) and \( g \) are fuzzy continuous,

\[
\mu_1(x) \leq \mu_2(f(x)), \forall x \in X_1
\]

Let \((X, \mu, F)\) be an \( L \)-fuzzy topological space, \( Y \subset X \). Let \( Y \subset X \) and \( Y = \{ U \mid U \subset F \} \), then \((Y, \gamma, \mathcal{V})\) is a subspace of \((X, \mu, F)\).

and

\[
\mu_2(x) \leq \mu_3(g(x)), \forall x \in X_2
\]

If \( Y \subset X \) and \( f : (X_1, \mu_1^X, F_1) \longrightarrow (X_2, \mu_2^X, F_2) \) is fuzzy continuous, then \( f \) restricted to the subspace \((Y, \gamma, \mathcal{V})\) is fuzzy continuous.

By (1) \( \mu_1(x) \leq \mu_2((f(x)) \forall x \in X_1 \)

\[
\leq \mu_3(g(f(x)) \forall x \in X_1 \quad \text{by (2)}
\]

\[
= \mu_3(g \circ f)(x) \forall x \in X_1
\]
That is $\mu_1 \cap f^{-1}(\mu_2 \cap g^{-1}(V)) \in F_1$ by (1) & (2)

That is $\mu_1 \cap f^{-1}(\mu_2) \cap f^{-1}(g^{-1}(V)) \in F_1$

That is $\mu_1 \cap f^{-1}(g^{-1}(V)) \in F_1 \{\mu_1(x) \leq \mu_2(f(x)) \forall x \in X\}$

That is $f^{-1}(u) \cap \{\mu_1(x) \leq f^{-1}(\mu_2)(x) \forall x \in X\}$

That is $\mu_1 \cap (g \circ f)^{-1}(V) \in F_1 \forall V \in F_3$

Therefore $g \circ f$ is fuzzy continuous.

**Definition 1.1.4**

Let $(X, \mu, F)$ be an L-fuzzy topological space, $\mu \in \mathcal{U}$ be any fuzzy subset of $X$. Then the induced fuzzy topology on $Y$ is the family of fuzzy subsets of $Y$ which are the intersections with $\mu$ of L-fuzzy open subsets of $X$. The induced L-fuzzy topology is denoted by $F_Y$ and the triple $(X, \mu, F_Y)$ is called a subspace of $(X, \mu, F)$.

**Proposition 1.1.5**

If $Y \subseteq X$ and $f:(X_1, \mu_1, F_1) \rightarrow (X_2, \mu_2, F_2)$ is fuzzy continuous, then $f$ restricted to the subspace $(Y, \gamma, \mathcal{U})$ is fuzzy continuous.

**Proof:**

We have $\gamma = \mu_1|_Y$ for $i = 1, 2$ then $f$ is fuzzy continuous from $(X_1, \mu_1, F_1)$ if $\mu_1(x) \leq \mu_2(f(x)) \forall x \in X_1$. Therefore $f$ is fuzzy continuous on $(Y, \gamma, \mathcal{U})$. 

Therefore $g \circ f$ is fuzzy continuous.
Let \( U \in F_2 \), then \( f^{-1}(U) \cap \mu_1 \in F_1 \)

That is \( \left\{ f^{-1}(U) \cap \mu_1 \right\} \) \( \in \mathcal{U} \)

That is \( (f^{-1}(U) \cap 1_Y) \cap (\mu_1 \cap 1_Y) \in \mathcal{U} \)

That is \( (f \mid_1)^{-1}(U) \cap Y \in \mathcal{U} \)

Therefore \( f \mid_1 \) is fuzzy continuous.

**Definition 1.1.6**

Let \( (X, \mu, F) \) be an \( L \)-fuzzy topological space, \( \mu \subseteq \mu \) be any fuzzy subset of \( X \). Then the induced fuzzy topology on \( \mu \) is the family of fuzzy subsets of \( X \) which are the intersections with \( \mu \) of \( L \)-fuzzy open subsets of \( X \). The induced \( L \)-fuzzy topology is denoted by \( F_{\mu'} \) and the triple \( (X, \mu', F_{\mu'}) \) is called induced fuzzy sub space of \( (X, \mu, F) \).

**Proposition 1.1.7**

If \( (X_1, \mu_1', F_{\mu_1'}) \longrightarrow (X_2, \mu_2', F_{\mu_2'}) \) is fuzzy continuous and \( \mu_i \subseteq \mu_i \) for \( i = 1,2 \) then \( f \) is fuzzy continuous from \( (X_1, \mu_1', F_{\mu_1'}) \) to \( (X_2, \mu_2', F_{\mu_2'}) \) if \( \mu_1'(x) \leq \mu_2'(f(x)) \forall x \in X_1 \).
We have to show that \( \mu_1 \cap f^{-1}(U') \in F \), \( \forall U' \in F' \).

Let \( U' \in F' \).

That is \( U' = \mu_2' \cap U \) for some \( U \in F_2 \).

Therefore \( \mu_1' \cap f^{-1}(U') = \mu_1' \cap f^{-1}(\mu_2') \cap f^{-1}(U) \)

\[ = \mu_1' \cap f^{-1}(\mu_2') \cap f^{-1}(U) \]

Since \( \mu_1 \preceq f^{-1}(\mu_2) \)

\[ \in F_{\mu_1'} \]

**Definition 1.1.8**

Let \( \left\{(X_i, \mu_i, F_i) \mid i \in I \right\} \) be a family of L-fuzzy topological spaces. We define their product \( \prod_{i \in I} (X_i, \mu_i, F_i) \) to be the L-fuzzy topological space \( (X, \mu, F) \), where \( X = \prod_{i \in I} X_i \) is the usual set product, \( \mu \) the product fuzzy set in \( X \) whose membership function is defined by

\[ \mu(x) = \inf_{i \in I} \{ \mu_i(x_i) \mid x = (x_i) \in X \} \]

and \( F \) is generated by the sub basis \( \beta = \left\{ p_i^{-1}(U_i) \cap \mu \mid i \in I \right\} \).
Proposition 1.1.9

i) For each $\alpha \in I$ the projection map $p_\alpha$ is fuzzy continuous.

ii) The product $L$-fuzzy topology is the smallest $L$-fuzzy topology for $X$ such that i) is true.

iii) Let $(Y, \gamma, \mathcal{U})$ be an $L$-fuzzy topological space and let $f$ be a function from $(Y, \gamma, \mathcal{U})$ to $(X, \mu, F)$, then $f$ is fuzzy continuous if and only if $\forall \alpha \in I$, $p_\alpha \circ f$ is fuzzy continuous.

Proof

(i) & (ii) follows from the definition of product $L$-fuzzy topology.

(iii) Suppose $f: (Y, \gamma, \mathcal{U}) \rightarrow (X, \mu, F)$ is fuzzy continuous since $p_\alpha$ is fuzzy continuous the composition $p_\alpha \circ f$ is fuzzy continuous.

Conversely suppose $p_\alpha \circ f$ is fuzzy continuous $\forall \alpha \in I$ then $\gamma \cap (p_\alpha \circ f)^{-1}(U) \in \mathcal{U}$ $\forall U \in F$ $\forall \alpha$ which is $\gamma \cap f^{-1}(p_\alpha^{-1}(U)) \in \mathcal{U}$ $\forall U \in F$ $\forall \alpha$ that is $\gamma \cap f^{-1}(U) \in \mathcal{U}$ where $U = p_\alpha^{-1}(U)$ therefore $f$ is fuzzy continuous. (by using definition 1.1.8)

Definition 1.1.10

Let $(X, \mu, F)$ be an $L$-fuzzy topological space, $R$ be an equivalence relation on $X$. Let $X/R$ be the usual quotient set
and \( p: X \longrightarrow X/R \) be the usual quotient map. We define the quotient \( L \)-fuzzy topology as follows

let \( \nu = p(\mu) \) so that \( \nu \) is an \( L \)-fuzzy set in \( X/R \) and

\[
\mathcal{U} = \left\{ U: X/R \longrightarrow L \left| p^{-1}(U) \cap \mu \in \mathcal{F} \right. \right\}.
\]

Then \((X/R, \nu, \mathcal{U})\) is the quotient space of \((X, \mu, \mathcal{F})\).

Proposition 1.1.11

let \((X, \mu, \mathcal{F})\) be an \( L \)-fuzzy topological space and 
\((X/R, \nu, \mathcal{U})\) be the quotient space of \((X, \mu, \mathcal{F})\), then:

i) \( p: (X, \mu, \mathcal{F}) \longrightarrow (X/R, \nu, \mathcal{U}) \) is fuzzy continuous

ii) Let \((X_1, \mu_1, \mathcal{F}_1)\) be an \( L \)-fuzzy topological space and \( g \) be a function from the quotient fuzzy space \((X/R, \nu, \mathcal{U})\) to \((X_1, \mu_1, \mathcal{F}_1)\) then \( g \) is fuzzy continuous if and only if \( g \circ p \) is fuzzy continuous

Proof

i) It is trivial from the definition of quotient \( L \)-fuzzy topology.

ii) Suppose \( g \) is fuzzy continuous, then the composition \( g \circ p \) is fuzzy continuous

conversely suppose \( g \circ p \) is fuzzy continuous

that is \( \mu \cap (g \circ p)^{-1}(U) \in \mathcal{F} \) \( \forall \ U \in \mathcal{F}_1 \)
that is \( p^{-1}(g^{-1}(U)) \subset F \forall U \subset F_1 \)

that is \( g^{-1}(U) \in U \) (by the definition of quotient L-fuzzy topology)

hence \( \nu \cap (g^{-1}(U)) \) is open in \( X/R \).

Therefore \( g \) is fuzzy continuous.

1.2 Categories of L-fuzzy topological spaces

Definition 1.2.1

Let \( \mathcal{C} \) be the collection of all L-fuzzy topological spaces \((X, \mu, F)\). For each pair of objects \((X_1, \mu_1, F_1)\) and \((X_2, \mu_2, F_2)\) of \( \mathcal{C} \) let \( \text{mor} \left[ (X_1, \mu_1, F_1), (X_2, \mu_2, F_2) \right] \) be the set of all fuzzy continuous mappings from \((X_1, \mu_1, F_1)\) to \((X_2, \mu_2, F_2)\). Clearly \( \mathcal{C} \) constitutes a category; we denote this category by \( \text{FTOP} \).

Results. 1.2.2

In \( \text{FTOP} \) the monomorphisms are the injections in the usual sense.
For let \( f: (X_1, \mu_1, F_1) \longrightarrow (X_2, \mu_2, F_2) \) be an injection.

Consider two morphisms \( g_1, g_2: (X_3, \mu_3, F_3) \longrightarrow (X_1, \mu_1, F_1) \)

and suppose \( f \circ g_1 = f \circ g_2 \)

that is \( (f \circ g_1)(x) = (f \circ g_2)(x) \quad \forall x \in X_3 \)

that is \( f(g_1(x)) = f(g_2(x)) \quad \forall x \in X_3 \)

that is \( g_1(x) = g_2(x) \quad \forall x \in X_3 \)

that is \( g_1 = g_2 \)

Conversely, if \( f \) is not an injection, and let \( f \circ g_1 = f \circ g_2 \)

that is \( (f \circ g_1)(x) = (f \circ g_2)(x) \quad \forall x \in X_3 \)

that is \( f(g_1(x)) = f(g_2(x)) \quad \forall x \in X_3 \)

that is \( g_1(x) \neq g_2(x) \quad \forall x \in X_3 \)

that is \( g_1 \neq g_2 \)

Therefore \( f \) can not be a monomorphism.

Similarly, we can show that the epimorphisms of \( FTOP \)

are the surjections.

An object \( \{x\}, 1_{|x|}, F \) where \( |x| \) denotes any one point

set, \( 1_{|x|} \) is an L-fuzzy subset of \( |x| \) such that \( 1_{|x|} = 1 \) and \( \{x\} \)

\( F = \{0, 1_{|x|}\} \) is a terminal object of \( FTOP \).
For each object \((X, \mu, F)\) of \(\text{FTOP}\) the subspace \((Y, \gamma, \mathcal{U})\) (cf. definition 1.2.4) and the induced fuzzy subspace \((X, i(F))\) where \(i(F)\) denotes the smallest topology on \(X\) such that each member of \(F\) a l.s.c.map.

\(\mathcal{C}_1\) consists of those objects \((X, \mu, F)\) for a fixed \(X\) together with the morphisms and \(\mathcal{C}_2\) consists of those objects \((X, \mu, F)\) where \(\mu\) is fixed for a particular \(X\). Clearly \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are sub categories of \(\text{FTOP}\).

Relation between \(\text{FTOP}\) and other categories

Consider the following Functors \(\mathcal{F}_1, \mathcal{F}_2\)

1) where \(\mathcal{F}_1, \mathcal{F}_2 : \text{TOP} \rightarrow \text{FTOP}\) such that \(\mathcal{F}_1(X,T) = (X, 1, F)\) is generated by the sub basis \(\mathcal{S} = \{g_1 \times g_2 : g_1 \in F_1 \text{ and } g_2 \in F_2\}\) and \(\mathcal{F}_2(X,T) = (X, 1, \mathcal{U})\) where \(\mathcal{U} = \{\lambda_U : U \subseteq T \mid \lambda_U \text{ denote a characteristic map of } U\}\)

\(\mathcal{F}_2(f) = f\)

Clearly \(\mathcal{F}_1\) embedds \(\text{TOP}\) into a full subcategory of \(\text{FTOP}\).
2) Consider $F_3 : \text{FTOP} \longrightarrow \text{TOP}$

$$F_3(X, \mu, F) = (X, i(F))$$

where $i(F)$ denotes the smallest

topology on $X$ such that each member of $F$ a l.s.c.map.

and $F_3(f) = f$

Product objects in $\text{FTOP}$

Let $(X_1, \mu_1, F_1)$ and $(X_2, \mu_2, F_2)$ be two objects in $\text{FTOP}$.

In the categorical sense we define their product $(X_1, \mu_1, F_1)$

$x (X_2, \mu_2, F_2)$ as $(X_1 \times X_2, \mu_1 \times \mu_2, F_1 \times F_2, \pi_1, \pi_2)$ where $F_1 \times F_2$

is generated by the sub basis $S = \left\{ g_1 \times \mu_2 \mid g_1 \in F_1 \right\} \cup$

$\left\{ \mu_1 \times g_2 \mid g_2 \subset F_2 \right\}$ and $\pi_i : X_1 \times X_2 \longrightarrow X_i$ (i=1,2) are

ordinary projections.