CHAPTER 0

INTRODUCTION

This thesis is a study of fuzzy topological semigroups.

FUZZY MATHEMATICS

Most of the initial work in the development of the concept of fuzzy set was theoretical in nature. Now a days the theory of fuzzy sets have wider scope and applicability in diverse areas like Linguistics, Robotics, Computer languages, Military control, Artificial intelligence, Law, Psychology, Taxonomy, Economics and Medical and Social sciences.

The notion of fuzziness was formally introduced by Lotfi A. Zadeh in 1965 [77]. In a very limited and specific context the concept and the term "fuzzy set" were already used by K. Menger in 1951 [52]. It was primarily intended to be applied in areas of pattern classification and information processing, but later it was found to be useful and applicable in various areas of knowledge.

When Zadeh introduced fuzzy subset he used \( (0,1) \) as the membership set. But while developing the theory of fuzzy sets many mathematicians used different lattice structures for the membership set. In 1987, Goguen introduced \( T \)-norm lattice structures. The fuzzy subset is defined by a membership function \( \mu_A(x) \), which associates with each point \( x \) in \( X \), a real number in the interval \( [0,1] \) with the value of \( \mu_A(x) \) representing the grade of membership of \( x \) in \( A \). That is the nearer the value of \( \mu_A(x) \) is to 1, the higher is the grade of membership of \( x \) in \( A \). No further mathematical meaning is given to the value that a fuzzy set attains at a point \( x \). That is basically a fuzzy set is a class in which there may...
be a continuum of grades of memberships in comparison to the situation that only two grades of memberships are possible in ordinary set theory, with this description fuzzy sets can be used to model many concepts.

Most of the initial work in the development of the concept of fuzzy set was theoretical in nature. Now a days the theory of fuzzy sets have wider scope of applicability than classical set theory. It has been found applicable in so diverse areas like Linguistics, Robotics, Computer languages, Military control, Artificial intelligence, Law, Psychology, Taxonomy, Economics and Medical and Social sciences.

When Zadeh introduced fuzzy subset he used [0,1] as the membership set. But while developing the theory of fuzzy sets many mathematicians used different lattice structures for the membership set. In 1967, Goguen introduced L-fuzzy set, where L is an arbitrary lattice with both a minimal and a maximal element, 0 and 1 respectively. Some of the other lattice structures that are used are the following:-

1) Complete and distributive lattice with 0 and 1 by T.E.Gantner [23]

2) Complete and completely distributive lattice equipped
with order reversing involution by Bruce Hutton [28]

Later R. Lowen in 1976 [37] modified this definition by taking the set of all constant maps instead of \( \mathcal{C} \) and \( X \) in axiom (1) of Chang's definition. Either has disadvantages or advantages over the other; we are following throughout the thesis, definition nearer to Chang's rather than to Lowen's.

3) Complete and completely distributive non-atomic Boolean algebra by Mira Sarkar [63]

4) Complete Brouwerian lattice with its dual also Brouwerian by Ulrich Hohle [27]

5) Completely distributive lattice by S. E. Rodabaugh [59]

General topology is one of the first branches of pure Mathematics to which the notion of fuzzy sets has been applied systematically. In 1968, that is three years after Zadeh's paper had appeared, C.L. Chang [8] first brought together the notion of fuzzy set and General topology. He introduced the notion that we now call Chang fuzzy space and made an attempt to develop basic topological notions for such spaces. This paper was followed by others in which Chang fuzzy space and other topological type structures for fuzzy set systems were considered.

According to Chang, a fuzzy topological space is a pair \((X, F)\), where \( X \) is any set and \( F \subseteq \mathcal{P}(X) \) satisfying the following axioms:

i) \( \phi, X \in F \)

ii) If \( A, B \in F \), then \( A \cap B \in F \)

iii) If \( A_i \in F \) for each \( i \in I \), then \( \bigvee_i A_i \in F \)

Separation axioms in fuzzy topological spaces were studied by R. Lowen [37-41], Fu and Liu [57-58], S.P. Dossak [65], T.E. Gentner [23], S.R. Hailgian [46], K.K. Asaad [2], S.S. Rodabaugh [59], B. Hutton [29] and R. Srivastava [66]. The important separation property namely, Hausdorffness, has been defined and studied by many Mathematicians from different viewpoints. At present not less than 10 approaches to the definition of Hausdorff fuzzy topological spaces are known. Some of these differ negligibly, but others do basically. The relation between some of them are available in [54].

Compactness property of a fuzzy topological space is one of the most important notions in fuzzy topology. The idea of compactness for fuzzy topological spaces was proposed in 1968 by C.L. Chang. Goguen [24] extended it for the case of \( \alpha \)-fuzzy topological spaces.
Later R. Lowen in 1976 [37] modified this definition by taking the set of all constant maps instead of $\emptyset$ and $X$ in axiom (i) of Chang's definition. Either has disadvantages or advantages over the other; we are following throughout the thesis, definition nearer to Chang's rather than to Lowen's.

Separation axioms in fuzzy topological spaces were studied by R. Lowen [37-41], Pu and Liu [57-58], A.P. Shostak [65], T.E. Gantner [23], S.R. Malghan [46], K.K. Azad [2], S.E. Rodabaugh [59], B. Hutton [29] and R. Srivastava [66]. The important separation property namely Hausdorffness Concept has been defined and studied by many Mathematicians from different view points. At present not less than 10 approaches to the definition of Hausdorff fuzzy topological spaces are known. Some of them differ negligibly; but others do basically. The relation between some of them are available in [54].

Compactness property of a fuzzy topological space is one of the most important notions in Fuzzy topology. The first definition of compactness for fuzzy topological spaces was proposed in 1968 by C. L. Chang. Goguen [24] extended it into the case of L-fuzzy topological spaces.
T.E. Gantner, R.C. Steinlage and R.H. Warren [23] proposed α-Compactness and observed that it is possible to have degrees of compactness. R. Lowen [37], Wang Guojun [69], S. Ganguli and S. Saha [21] and J.T. Chadwick [7] have different definitions for compactness property in fuzzy topological spaces. Fuzzy topology on groups and other algebraic objects was studied by D.H. Foster [20], A.K. Katsaras [30-31], Chun Hai Yu [13], Ma Ji liang [43], B.T. Lerner [35], Ulrich Höhle [27], Ahsanullah [1] etc.

The study of topological semigroups was started in 1953. During these years the subject has developed in many directions and the literature is so vast and it would be difficult to give a brief survey of the developments in this area. Some of the early contributors of this area are A.D. Wallace [On the structure of topological semigroups, Bull. Amer. Math. Soc. 61 (1955a), 95-112.]


By definition a topological semigroup $S$ is a Hausdorff space with continuous associative multiplication $(x, y) \rightarrow xy$ of $S \times S$ into $S$, and if the multiplication is continuous in each variable separately, $S$ is called a semi topological semigroup. Some of the major areas of developments in the theory of semitopological semigroups are the theory of compact semi topological semigroups, structure theory of
compact semigroups, semigroup compactification and almost periodic and weakly almost periodic compactification. For details regarding these, one may refer to [3].

SUMMARY OF THE THESIS

chapter 1

In the first chapter we introduce a category of L-fuzzy topological spaces, where L stands for a complete completely distributive lattice with minimal element 0 and maximal element 1 and also the order reversing involution \( a \overset{c}{\rightarrow} a^c \) is fixed (\( a \leq b \), \( a, b \in L \). \( \overset{c}{\rightarrow} b^c \leq a^c \)).

[THROUGHOUT THE THESIS WE USE L IN THE ABOVE SENSE]

In section 1 we define an L-fuzzy topological space \((X, \mu, F)\). A subspace of \((X, \mu, F)\), an induced fuzzy space of \((X, \mu, F)\), product of family of L-fuzzy topological spaces \(\{(X_i, \mu_i, F_i) | i \in I\}\), the quotient of \((X, \mu, F)\) are obtained in this section.

In section 2 we define the category FTOP of L-fuzzy topological spaces, Subcategories, subobjects, terminal objects and initial objects of FTOP are also described in this section.
CHAPTER 2

This chapter is a study of L-fuzzy topological semigroups. In section 2 we identify a semigroup object in FTOP as an L-fuzzy topological semigroup. Also we define an induced L-fuzzy topological semigroup and obtain a relation between L-fuzzy topological semigroups and induced L-fuzzy topological semigroups. In section 3 of this chapter we specialize L-fuzzy topological semigroups, for $L=[0,1]$ and $\mu=\chi_X$, and consider $(X,\mu_X,F)$ as a fuzzy topological semigroup. We define a fuzzy topological semigroup in terms of neighbourhoods and Q-neighbourhoods. We observe an association between the classes of topological semigroups and fuzzy topological semigroups in this section.

CHAPTER 3

We proceed to study the analogous concept namely L-fuzzy semi topological semigroups in chapter 3. We define an L-fuzzy semi topological semigroup in the categorical point of view. In section 2 we specialize an L-fuzzy semi topological semigroup to a fuzzy topological semigroup and it is defined in terms of neighbourhoods and Q-neighbourhoods. In the last section we give some examples to observe relations between fuzzy topological semigroups and fuzzy semi topological semigroups.
CHAPTER 4

We study fuzzy topology on function spaces in chapter 4. We define point open fuzzy topology and compact open fuzzy topology for a family of fuzzy continuous functions between two fuzzy topological spaces. A few separation properties of compact open fuzzy topological spaces are studied and also we obtain some relations between compact open topology and compact open fuzzy topology for a family of functions between two topological spaces.

CHAPTER 5

In chapter 5 we study homomorphism and Isomorphism between two L-fuzzy (semi) topological semigroups. We define F-morphism and F-isomorphism between two L-fuzzy (semi) topological semigroups. B.T.Lerner studied the homomorphic images and inverse images of fuzzy right topological semigroups. We prove the analogous results for L-fuzzy topological semigroups. In section 2 we prove that the product of a family of L-fuzzy topological semigroups 

\[ \left\{ X_i \cdot \mu_i F_i \mid i \in I \right\} \]

is an L-fuzzy topological semigroup and each factor space is an F-morphic image of the product space.
Also we prove that the quotient of an L-fuzzy topological semigroup is also an L-fuzzy topological semigroup. In the last section we consider some categories of fuzzy (semi) topological semigroups.

CHAPTER 6

In chapter 6 we study the problem of fuzzy space compactification on fuzzy topological semigroups. In section 2 we prove that the Bohr fuzzy compactification exists for a fuzzy topological semigroup. We define F-semigroup compactification for a fuzzy topological semigroup and obtain an association with the Bohr fuzzy compactification. In the last section we define an order relation on the set of all F-semigroup compactifications of a fuzzy topological semigroup and prove that it is a complete lattice.

Definition 0.1.2

Let $f$ be a mapping from a fuzzy set in $X$ to $Y$. Let $B$ be a fuzzy set in $Y$ with membership function $\mu_B$. Then the inverse image of $B$ is the fuzzy set in $X$ whose membership function is defined by $\mu^{-1}_B(x) = \max \{ \mu_B(y) | y \in f^{-1}(x) \}$ for all $x \in X$, Conversely let
CHAPTER 0.1

PRELIMINARY DEFINITIONS USED IN THE THESIS

Definition 0.1.1

Let X be a set and A, B be two fuzzy sets in X. Then

i) \( A = B \) \( \iff \) \( \mu_A(x) = \mu_B(x) \) for all \( x \in X \)

ii) \( A \subseteq B \) \( \iff \) \( \mu_A(x) \leq \mu_B(x) \) for all \( x \in X \)

iii) \( \mathcal{A} = A \cup B \) \( \iff \) \( \mu_{\mathcal{A}}(x) = \max(\mu_A(x), \mu_B(x)) \) for all \( x \in X \)

iv) \( D = A \cap B \) \( \iff \) \( \mu_D(x) = \min(\mu_A(x), \mu_B(x)) \) for all \( x \in X \)

More generally for a family of fuzzy sets \( \mathcal{A} = \{A_i, i \in I\} \),
the union \( C = \bigcup_{i \in I} A_i \) and the intersection \( D = \bigcap_{i \in I} A_i \) are
defined by \( \mu_C(x) = \sup_{i \in I} \mu_{A_i}(x) \), \( x \in X \) and \( \mu_D(x) = \inf_{i \in I} \mu_{A_i}(x) \), \( x \in X \).

Definition 0.1.2

Let \( f \) be a mapping from a set \( X \) to a set \( Y \). Let \( B \) be a fuzzy set in \( Y \) with membership function \( \mu_B \). Then the inverse image of \( B \) is the fuzzy set in \( X \) whose membership function
is defined by \( \mu_{f^{-1}}(B)(x) = \mu_B(f(x)) \) \( \forall x \in X \). Conversely let
A be a fuzzy set in $X$ with membership function $\mu_A$, then the image of $A$ is the fuzzy set in $Y$ whose membership function is defined by $\mu_f(A)(y) = \text{Sup}_{z \in f^{-1}(y)} \mu_A(z)$ if $f^{-1}(y)$ is nonempty.

$= 0$ otherwise $\forall y \in Y$ where $f^{-1}(y) = \{x \mid f(x) = y\}$

**Definition 0.1.3**

Let $(X,F), (Y,U)$ be two fuzzy topological spaces. A mapping $f$ of $(X,F)$ into $(Y,U)$ is fuzzy continuous if and only if for each open fuzzy set $v$ in $U$, the inverse image $f^{-1}(v)$ is in $F$.

**Definition 0.1.4**

A bijective mapping $f$ of a fuzzy topological space $(X,F)$ into a fuzzy topological space $(Y,U)$ is fuzzy homeomorphism if and only if both $f$ and $f^{-1}$ are fuzzy continuous.

**Definition 0.1.5**

Let $(X,T)$ be a topological space and $I = [0,1]$ equipped with the usual topology, a mapping $\mu: (X,T) \rightarrow I$ is said to be lower semi continuous (l.s.c) if $\mu^{-1}(\alpha,1]$ is open in $X$ for every $\alpha \in I$.

Let $(X,T)$ be a topological space in the above sense, then $(X,\mathcal{W}(T))$, where $\mathcal{W}(T) = \{f:X \rightarrow [0,1] \mid f \text{ is a l.s.c map}\}$,
is called the associated fuzzy space or topologically generated fuzzy space of \((X,T)\).

**Definition 0.1.6**

A category consists of

i) A collection \(\mathcal{R}\) of objects \(X,Y,\ldots\).

ii) A set \(\text{mor}(X,Y)\) of morphisms associated with every two objects \(X\) and \(Y\) of \(\mathcal{R}\) (written \(f:X\to Y\) or \(X \xleftarrow{f} Y\) if \(f\) is an element of \(\text{mor}(X,Y)\)).

iii) A map \(\text{mor}(X,Y) \times \text{mor}(Y,Z) \to \text{mor}(X,Z)\) for every three objects \(X,Y\) and \(Z\) of \(\mathcal{R}\) such that

1) if \(f:X\to Y\), \(g:Y\to Z\), \(h:Z\to T\) then \(h \circ (g \circ f) = (h \circ g) \circ f\).

2) to each \(X\) of \(\mathcal{R}\) there exists identity morphism \(e_X:X\to X\) such that \(f \circ e_X = f\) and \(e_X \circ g = g\) for every \(f:X\to Y\) and \(g:Y\to X\).

**Definition 0.1.7**

A subcategory \(\mathcal{K}\) of a category \(\mathcal{C}\) is a collection of some of the objects and some of the morphisms of \(\mathcal{C}\) such that

i) if object \(X\) is in \(\mathcal{K}\) so is the identity morphism \(e_X\).

ii) if morphism \(f:X\to Y\) is in \(\mathcal{K}\) so are \(X\) and \(Y\).

iii) if \(f:X\to Y\), \(g:Y\to Z\) are in \(\mathcal{K}\) so is \(g \circ f\).
Definition 0.1.8

Let $\mathcal{C}$ be a category, $A, B, C$ be objects in $\mathcal{C}$. A morphism $f: A \to B$ is a monomorphism in $\mathcal{C}$ whenever for any two morphisms $g_1, g_2: C \to A$ the equality $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$.

A morphism $h: A \to B$ is an epimorphism in $\mathcal{C}$ whenever for any two morphisms $g_1, g_2: B \to C$ the equality $g_1 \circ h = g_2 \circ h \Rightarrow g_1 = g_2$.

Definition 0.1.9

A subobject of an object $B$ is a pair $(A, f)$ where $f: A \to B$ is a monomorphism. Subobjects $(A, f)$ & $(C, g)$ are said to be isomorphic if there exists a unique isomorphism $g: A \to C$ such that $g \circ h = f$.

Definition 0.1.10

An object $T$ is a terminal object in $\mathcal{C}$ if to each object $C$ in $\mathcal{C}$ there exists exactly one morphism $g: C \to T$. And an object $S$ is initial in $\mathcal{C}$ if to each object $C$ in $\mathcal{C}$ there exists exactly one morphism $h: S \to C$.

Definition 0.1.11

Let $\mathcal{C}$ be a category with finite products. An ordered pair $(X, m)$ is called a semigroup object in $\mathcal{C}$ if:

1) $X$ is an object of $\mathcal{C}$.
2) $m: X \times X \to X$ is a morphism in $\mathcal{C}$.
3) $m$ is associative.