CHAPTER – 2

TWO-DIMENSIONAL AND AXISYMMETRIC UNSTEADY FLOWS BETWEEN PARALLEL PLATES
2.1. Introduction

The problem of unsteady squeezing of a viscous incompressible fluid between two parallel plates in motion normal to their own surfaces independent of each other and arbitrary with respect to time is a fundamental type of unsteady flow which is met frequently in many hydrodynamical machines and apparatuses. Some practical examples of squeezing flow include polymer processing, compression, and injection molding. In addition, the lubrication system can also be modeled by squeezing flows. The pioneering work on squeezing flow by using lubrication approximation was conducted by Stefan [140]. Further, Reynolds [141] derived a solution for elliptic plates, and Archibald [142] investigated this problem for rectangular plates. The theoretical and experimental studies of squeezing flows have been conducted by many research workers [143–153]. Earlier studies of squeezing flow are based on Reynolds equation. The inadequacy of Reynolds equation in the analysis of porous thrust bearings and squeeze films involving high velocity has been demonstrated by Jackson [153], Ishizawa [154], and others. The general study of the problem with full Navier-Stokes equations involves extensive numerical study requiring more computer time and larger memory. However, many of the important features of this problem can be grasped by prescribing the relative velocity of the plates suitably. If the relative normal velocity is proportional to \( (1 - \alpha t)^{1/2} \), where \( t \) is the time and \( \alpha \) a constant of dimension...
which characterizes unsteadiness, then the unsteady Navier–Stokes equations admit similarity solution.

In this chapter, we present a simple recursive algorithm based on the homotopy perturbation Sumudu transform method (HPSTM) which produces the series solution of the two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates.

2.2. Mathematical Formulation

Let the solution of the two plates be at \( z = \pm \ell (1 - \alpha t)^{1/2} \), where \( \ell \) is the position at time \( t = 0 \) as depicted in Fig. 2.1. We consider that the length \( \ell \) (in the two-dimensional case) or the diameter \( D \) (in the axisymmetric case) is much larger than the gap width \( 2z \) at any time such that the end effects can be neglected. When \( \alpha \) is positive, the two plates are

![Fig. 2.1 Schematic diagram of the problem.](image-url)
squeezed until they touch at \( t = 1/\alpha \). When \( \alpha \) is negative, the two plates are separated. Let \( U, V \) and \( W \) be the velocity components along \( x, y \) and \( z \) axis respectively. For two-dimensional flow, Wang [155] introduce the following transform

\[
U = \frac{\alpha x}{[2(1-\alpha t)]} f'(\eta),
\]

\[
W = -\frac{\alpha \ell}{[2(1-\alpha t)^{1/2}]} f(\eta),
\]

where

\[
\eta = \frac{z}{\ell(1-\alpha t)^{1/2}}.
\]

Substituting Eq. (2.2.2) into the unsteady two-dimensional Navier-Stokes equations transform nonlinear differential equation in the following form

\[
f'' + S(-\eta f'' - 3f' - f'f'' + ff') = 0,
\]

where \( S = at^2 / 2\nu \) (squeeze number) is the nondimensional parameter. The flow is characterized by this parameter. The boundary conditions are such that on the plates, the lateral velocity is zero and the normal velocity is equal to the velocity of the plate, that is,

\[
f(0) = 0, \quad f'(0) = 0,
\]

\[
f(1) = 1, \quad f'(1) = 0.
\]

Similarly, Wang’s transform [155] for axisymmetric flow are

\[
U = \frac{\alpha x}{[4(1-\alpha t)]} f'(\eta),
\]
Using transforms (2.2.5), unsteady axisymmetric Navier-Stokes equation reduces to
\[ f'' + S\left[ -\eta f''' - 3f'' + ff'' \right] = 0, \] (2.2.6)
subjected to the boundary conditions (2.2.4).

Consequently, we should solve the nonlinear boundary differential equation
\[ f'' + S\left[ -\eta f'' - 3f' + ff' \right] = 0, \] (2.2.7)
\[ \beta = \begin{cases} 0, & \text{Axisymmetric,} \\ 1, & \text{Two - dimensional,} \end{cases} \] (2.2.8)
and subject to the boundary conditions (2.2.4). The two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates have also been studied by Dinarvand and Moradi [156].

2.3. Solution of the Problem

In this section, we apply HPSTM to obtain an approximate solution of the Eq. (2.2.7). By applying Sumudu transform on the both sides of Eq. (2.2.7), we have
\[ S[F(\eta)] = au + bu^3 - u^4 S\left[ -\eta f'' - 3f' + ff' \right]. \] (2.3.1)
The inverse Sumudu transform gives
\[ F(\eta) = a\eta + \frac{1}{6} b\eta^3 - S^{-1}\left[ u^4 S\left[ -\eta f'' - 3f' + ff' \right] \right]. \] (2.3.2)
Now applying the HPM, we get

\[
\sum_{m=0}^{\infty} p^m F_m(\eta) = a\eta + \frac{1}{6} b\eta^3 - p \left[ S^{-1} \left(u^4 S \left[ -\eta \sum_{m=0}^{\infty} p^m f''_m(\eta) - 3 \sum_{m=0}^{\infty} p^m f''_m(\eta) \right]\right] \right),
\]

where \( H_m(\eta) \) and \( H'_m(\eta) \) are He’s polynomials that represents the nonlinear terms. So, He’s polynomials are given by

\[
\sum_{m=0}^{\infty} p^m H_m(\eta) = f'(\eta)f''(\eta).
\] (2.3.4)

The first few components of He’s polynomials, are given by

\[
H_0(\eta) = f'_0(\eta)f''_0(\eta),
\]

\[
H_1(\eta) = f'_0(\eta)f''_1(\eta) + f'_1(\eta)f''_0(\eta),
\] (2.3.5)

\[
H_2(\eta) = f'_0(\eta)f''_2(\eta) + f'_1(\eta)f''_1(\eta) + f'_2(\eta)f''_0(\eta),
\]

\[\vdots\]

and for \( H'_m(\eta) \) we find that

\[
\sum_{m=0}^{\infty} p^m H'_m(\eta) = f(\eta)f''(\eta),
\] (2.3.6)

\[
H'_0(\eta) = f'_0(\eta)f'''_0(\eta),
\]

\[
H'_1(\eta) = f'_0(\eta)f'''_1(\eta) + f'_1(\eta)f'''_0(\eta),
\] (2.3.7)

\[
H'_2(\eta) = f'_0(\eta)f'''_2(\eta) + f'_1(\eta)f'''_1(\eta) + f'_2(\eta)f'''_0(\eta),
\]

\[\vdots\]
Comparing the coefficients of like powers of $p$, we have

$$p^0 : f_0 = a \eta + \frac{b}{6} \eta^3,$$  \hspace{1cm} (2.3.8)

$$p^1 : f_1 = -\frac{b s(a - a \beta - 4)}{120} \eta^5 + \frac{b^2 s(3 \beta - 1)}{5040} \eta^7,$$  \hspace{1cm} (2.3.9)

$$p^2 : f_2 = -\frac{s^2 (11a - 42 \beta - 4 - 2a^2 + 2a^2 \beta)b}{5040} \eta^7 - \frac{s^2 (5a - 5a \beta - 20 - 9b \beta - 53b - 2ab \beta + 10ab)b}{362880} \eta^9 + \frac{s^2 (28 - 42 \beta + 18b \beta - 6b)b^2}{39916800} \eta^{11},$$  \hspace{1cm} (2.3.10)

where $a = f'(0)$ and $b = f''(0)$ are to be determined from the boundary conditions. The solutions of the Eq. (2.2.7), when $p \to 1$, will be as follows:

$$f(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta) + \cdots.$$  \hspace{1cm} (2.3.11)
Fig. 2.2 The comparison of the solution \( f' (\eta) \) obtained by HPM [156] and HPSTM for the two-dimensional case when \( \beta = 1 \) and \( s = 5 \).

Fig. 2.3 The influence of positive \( s \) on \( f' (\eta) \) for the two-dimensional case for \( \beta = 1 \).
Fig. 2.4 The influence of negative $s$ on $f'(\eta)$ for the axisymmetric case for $\beta = 0$.

Fig. 2.5 The skin fraction ($f'(l)$) for the axisymmetric and two-dimensional cases.
2.4. Results and Discussion

In this Chapter the HPSTM is used for solving two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. Our main purpose is to find the various values of \( f(\eta) \) and \( f'(\eta) \). These values describe the flow behavior. The comparison of the solution \( f'(\eta) \) obtained by HPM [156] and HPSTM for the two-dimensional case when \( \beta = 1 \) and \( s = 5 \) is described by Fig. 2.2. Fig. 2.2 shows that the numerical results obtained with the help of HPSTM are in an excellent agreement with the results derived by using HPM [156]. Fig. 2.3 depicts the variation of \( f'(\eta) \) with the change in the positive values of \( s \) for the two-dimensional case. Fig. 2.4 presents the effect of negative \( s \) on \( f'(\eta) \) for the axisymmetric flow. It is important that for the large values of \( s \), the results of similarity analysis are not consistent. \( f''(l) \) gives skin friction, \( f''(l) \) represents the pressure gradient and are shown as a function of \( s \) in Fig. 2.4 and Fig. 2.5 respectively.

2.5. Conclusions

In this chapter, the HPSTM has been applied for solving two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. Graphical results are presented to investigate the effect of squeeze number on the velocity, skin friction and pressure gradient. The result shows that the HPSTM is powerful and efficient technique in finding exact
and approximate solutions for nonlinear differential equations. The HPSTM requires less computational work as compared to other analytical methods. In conclusion, the HPSTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.