CHAPTER - 1

SUMUDU TRANSFORM TECHNIQUES
1.1. Introduction

In the last several years with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems in solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic that’s need new methods for finding the exact and approximate solutions. Most of new nonlinear equations do not have a precise analytic solution; so, numerical methods have largely been used to handle these equations. There are also analytic techniques for nonlinear equations. Some of the classic analytic methods are Lyapunov’s artificial small parameter method [92], δ-expansion method [93] and Hirota bilinear method [94,95]. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM) [96-101], He’s semi-inverse method [102] and the variational iteration method (VIM) [103-114]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian’s polynomials, the Lagrange multiplier, divergent results and huge computational work. He [32-45] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied
without any discretization, restrictive assumption or transformation and is free from round off errors. The HPM was also studied by many authors to handle nonlinear equations arising in science and engineering [115–121]. One of the highly applicable analytical techniques is homotopy analysis method (HAM), which was introduced and developed by Liao [66-70]. This method is applied to solve many nonlinear problems [71-77] and the references therein to handle a wide variety of scientific and engineering applications: linear and nonlinear as well as homogeneous and inhomogeneous.

In this chapter, we introduce the three new analytic techniques namely homotopy perturbation Sumudu transform method (HPSTM) [122], Sumudu decomposition method (SDM) [123] and homotopy analysis Sumudu transform method (HASTM) [124]. The HPSTM is an elegant combination of the Sumudu transformation, the homotopy perturbation method and He’s polynomials and is mainly due to Ghorbani [125,126]. The use of He’s polynomials in the nonlinear term was first introduced by Ghorbani [125,126]. The SDM is an elegant combination of the Sumudu transform and decomposition method which was first introduced by Adomian [96-108], and the HASTM is combined form of Sumudu transform method and homotopy analysis method. The proposed algorithms provide the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of these methods is their capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. These
techniques provide the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions.

1.2. Sumudu Transform

In a recent work Watugala [127] introduced a new integral transform, called the Sumudu transform defined for functions of exponential order. Over the set of functions

\[ A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \; |f(t)| < M e^{\|f\|_1}, \; \text{if} \; t \in (-1)^j \times [0, \infty) \} \]

the Sumudu transform is defined by

\[ \tilde{f}(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} \, dt, \; u \in (-\tau_1, \tau_2). \]  \hspace{1cm} (1.2.1)

The discrete analog of the Sumudu transform (1.2.1) is defined for power series functions \( f(t) = \sum_{k=0}^\infty a_k t^k \), having an interval of convergence containing \( t = 0 \), as follows:

\[ \tilde{f}(u) = \sum_{k=0}^\infty k! a_k u^k \quad \text{for} \quad u \in (-\tau_1, \tau_2). \]  \hspace{1cm} (1.2.2)

So, the linear function \( f(t) = a_0 + a_1 t \) transforms to itself, \( \tilde{f}(u) = a_0 + a_1 u = f(u) \). However, the power series

\[ f(t) = \sum_{k=0}^\infty (-1)^k (at)^k / k! = e^{-at} \]  \hspace{1cm} (1.2.3)

transforms to the geometric series
\[ \tilde{f}(u) = \sum_{k=0}^{\infty} (-1)^k (au)^k = \frac{1}{1 + au}, \]  
\hspace{1cm} (1.2.4) 

with \( u \) in \((-1/a,1/a)\).

Equations (1.2.2), (1.2.3) and (1.2.4) reveal that the Sumudu transform amplifies the coefficients of the power series according to their order, without changing the initials units of the series. Therefore, a signal with increasingly decaying higher-order coefficients \( a_n \) transforms to another with much more prominent tail end. So, the power series of \( e^t \) which converges throughout \( \mathbb{R} \) transforms to the geometric series of \( 1/(1-t) \) which converges only in the interval \((-1,1)\). Moreover, the discrete version of the Sumudu transform gives us the insight of how to obtain \( f(t) \) from \( \tilde{f}(u) \). We simply divide the coefficients of the power series for \( \tilde{f}(u) \) by the respective \( n! \) value to obtain the power series for \( f(t) \).

While it is difficult to compute at times, the integral transform in (1.2.1) is clearly much more general than its discrete counterpart defined in (1.2.2). In the problems of differential, integral, or engineering control nature, the Sumudu transform can handle all problems that are usually treated by the well-known and extensively used Laplace transform defined for \( \text{Re}(s) > 0 \) by

\[ F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt. \]  
\hspace{1cm} (1.2.5) 

Some of the properties of the Sumudu transform were established by Asiru \[128\]. Further, fundamental properties of this transform were established by
Belgacem et al. [129] and Belgacem and Karaballi [130]. In fact it was shown that there is strong relationship between Sumudu and other integral transform, see Kilicman et al. [131]. In particular the relation between Sumudu transform and Laplace transforms was proved in Kilicman and Eltayeb [132]. Indeed, as the theorem obtained by Belgacem et al. [129] have shown that Sumudu transform is closely connected with the Laplace transform.

**Theorem 1.2.1.** [129]. Let $f(t) \in A$ with Laplace transform $F(s)$. Then the Sumudu transform $\tilde{f}(u)$ of $f(t)$ is given by

$$\tilde{f}(u) = \frac{F(1/u)}{u}. \quad (1.2.6)$$

**Proof.** Let $f(t) \in A$, then for $-\tau_1 < u < \tau_2$,

$$\tilde{f}(u) = \int_{0}^{\infty} e^{-ut} f(t) dt. \quad (1.2.7)$$

If we set $w = ut (t = w/u)$, then the right-hand side can be written as

$$\tilde{f}(u) = \int_{0}^{\infty} e^{-w/u} f(w) dw. \quad (1.2.8)$$

The integral on right-hand is clearly $F(1/u)$, thus yielding (1.2.6).

We observe that $\tilde{f}(1) = F(1)$ so that both the Sumudu and Laplace transforms must coincide at $u = s = 1$. Furthermore, since for $x > 0$, the Gamma function

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt \quad (1.2.9)$$
is the Laplace transform of \( t^{s-1} \{ L(t^{s-1}) \} \) when \( s = 1 \), then \( \Gamma(x) \) must also be the Sumudu transform \( \{ S(t^{s-1}) \} \) when \( u = 1 \).

### 1.3. Homotopy Perturbation Sumudu Transform Method (HPSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation of the form:

\[
LU + RU + NU = g(x),
\]

(1.3.1)

where \( L \) is the highest order linear differential operator, \( R \) is the linear differential operator of less order than \( L \), \( N \) represents the general nonlinear differential operator and \( g(x) \) is the source term. By applying the Sumudu transform on both sides of Eq. (1.3.1), we get

\[
S[U] = u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + u^n S[g(x)] - u^n S[U + NU] = 0.
\]

(1.3.2)

Now applying the inverse Sumudu transform on both sides of Eq. (1.3.2), we get

\[
U = G(x) - S^{-1} \left[ \int S[RU + NU] \right],
\]

(1.3.3)

where \( G(x) \) represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM

\[
U = \sum_{m=0}^{\infty} p^m U_m
\]

(1.3.4)

and the nonlinear term can be decomposed as

\[
NU = \sum_{m=0}^{\infty} p^m H_m,
\]

(1.3.5)
for some He's polynomials [126,133] that are given by

\[ H_m(U_0, U_1, ..., U_m) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \left[ N \left( \sum_{i=0}^{\infty} p^i U_i \right) \right], \quad m = 0, 1, 2, 3, ... \]  

(1.3.6)

Substituting Eqs. (1.3.4) and (1.3.5) in Eq. (1.3.3), we get

\[ \sum_{m=0}^{\infty} p^m U_m = G(x) - p \left( S^{-1} \left[ u^S \left[ R \sum_{m=0}^{\infty} p^m U_m + \sum_{m=0}^{\infty} p^m H_m \right] \right] \right), \]  

(1.3.7)

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficient of like powers of \( p \), the following approximations are obtained

\[ p^0 : U_0(x) = G(x), \]

\[ p^1 : U_1(x) = -S^{-1} \left[ u^S [RU_0(x) + H_0(U)] \right], \]

\[ p^2 : U_2(x) = -S^{-1} \left[ u^S [RU_1(x) + H_1(U)] \right], \]  

(1.3.8)

\[ p^3 : U_3(x) = -S^{-1} \left[ u^S [RU_2(x) + H_2(U)] \right], \]

\[ \vdots \]

Proceeding in this same manner, the rest of the components \( U_m \) can be completely obtained and the series solution is thus entirely determined. Finally, we approximate the analytical solution \( U \) by truncated series

\[ U = \lim_{N \to \infty} \sum_{m=0}^{N} U_m. \]  

(1.3.9)
The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [134].

1.3.1. Numerical Experiments

In order to elucidate the solution procedure of the homotopy perturbation Sumudu transform method, we first consider the nonlinear advection equations. The advection equation is the partial differential equation that governs the motion of a conserved scalar field as it is advected by a known velocity vector field. It is derived using the scalar field's conservation law, together with Gauss's theorem, and taking the infinitesimal limit. One easily-visualized example of advection is the transport of ink dumped into a river. As the river flows, ink will move downstream in a "pulse" via advection, as the water's movement itself transports the ink. If added to a lake without significant bulk water flow, the ink would simply disperse outwards from its source in a diffusive manner, which is not advection. Note that as it moves downstream, the "pulse" of ink will also spread via diffusion. The sum of these processes is called convection. Several techniques including the Laplace decomposition method [82] and the weighted finite difference techniques [135] have been used to handle advection equations.

**Example 1.3.1.** Consider the following homogenous advection problem:

\[ U_t + U U_x = 0, \]  \hspace{1cm} (1.3.10)

\[ U(x, 0) = -x. \]
Taking the Sumudu transform on both sides of Eq. (1.3.10) subject to the initial condition, we have

\[ S[U(x,t)] = -x - u S[U_x]. \]  
(1.3.11)

The inverse of Sumudu transform implies that

\[ U(x,t) = -x - S^{-1}[S[U_x]]_x. \]  
(1.3.12)

Now applying the homotopy perturbation method, we get

\[ \sum_{m=0}^{\infty} p^m U_m(x,t) = -x - p \left( S^{-1}\left[ u S \left[ \sum_{m=0}^{\infty} p^m H_m(U) \right] \right] \right), \]  
(1.3.13)

where \( H_m(U) \) are He’s polynomial \([126,133]\) that represents the nonlinear terms. The first few components of He’s polynomials, are given by

\[ H_0(U) = U_0 U_{0x}, \]
\[ H_1(U) = U_0 U_{1x} + U_1 U_{0x}, \]  
(1.3.14)
\[ H_2(U) = U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x}, \]
\[ \vdots \]

Comparing the coefficients of like powers of \( p \), we have

\[ p^0 : U_0(x,t) = -x, \]
\[ p^1 : U_1(x,t) = -S^{-1}\left[ S[H_0(U)]_x \right] = -xt, \]  
(1.3.15)
\[ p^2 : U_2(x,t) = -S^{-1}\left[ S[H_1(U)]_x \right] = -xt^2. \]

Proceeding in a similar manner, we have

\[ p^3 : U_3(x,t) = -xt^3. \]
Therefore the series solution $U(x,t)$ is given by

$$U(x,t) = -x(1 + t + t^2 + t^3 + t^4 + t^5 + \cdots).$$

This solution is equivalent to the exact solution in closed form

$$U(x,t) = \frac{x}{t-1}.$$  \hfill (1.3.18)

The homogenous advection equation considered in Example 1.3.1 is graphically represented through Figs. 1.1-1.2. The numerical results of the exact solution $U(x,t)$ vs. $x$ and also those vs. $t$ are depicted through Figs. 1.1 and 1.2. It is observed from Fig. 1.1 that $U(x,t)$ increases with the increase in $x$. Its can also be noted from Fig. 1.2 that $U(x,t)$ increases with the increase in $t$. 

\begin{align*}
p^4 : U_4(x,t) &= -xt^4, \quad \text{(1.3.16)} \\
p^5 : U_5(x,t) &= -xt^5,
\end{align*}
Fig. 1.1 Plot of exact solution $U(x,t)$ vs. $x$ at $t = 5$ for Example 1.3.1.

Fig. 1.2 Plot of exact solution $U(x,t)$ vs. $t$ at $x = 1$ for Example 1.3.1.
Example 1.3.2. Now, consider the following nonhomogenous advection problem:

\[ U_t + U U_x = 2t + x + t^3 + xt^2, \quad (1.3.19) \]

\[ U(x, 0) = 0. \]

Taking the Sumudu transform on both sides of Eq. (1.3.19) subject to the initial condition, we have

\[ S[U(x, t)] = 2u^2 + xu + 6u^4 + 2xu^3 - u S[U U_x]. \quad (1.3.20) \]

The inverse of Sumudu transform implies that

\[ U(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - S^{-1}\left[S[U U_x]\right]. \quad (1.3.21) \]

Now applying the homotopy perturbation method, we get

\[ \sum_{m=0}^{\infty} p^m U_m(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - p\left[S^{-1}\left[u S\left[\sum_{m=0}^{\infty} p^m H_m(U)\right]\right]\right], \quad (1.3.22) \]

where \( H_m(U) \) are He’s polynomial that represents the nonlinear terms. The first few components of He’s polynomials, are given by

\[ H_0(U) = U_0 U_{0x}, \]

\[ H_1(U) = U_0 U_{1x} + U_1 U_{0x}, \quad (1.3.23) \]

\[ H_2(U) = U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x}, \]

\[ \vdots \]

Comparing the coefficients of like powers of \( p \), we have

\[ p^0 : U_0(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \]
\[ p^1: U_1(x,t) = -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8, \]

\[ p^2: U_2(x,t) = \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \frac{22}{315}xt^7 + \frac{7}{12}t^6 + \frac{2}{15}xt^5. \]  

(1.3.24)

It is important to recall here that the noise terms appear between the components \( U_0(x,t) \) and \( U_1(x,t) \), where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms \( \pm \frac{1}{4}t^4 \pm \frac{1}{3}xt^3 \) between the components \( U_0(x,t) \) and \( U_1(x,t) \) can be cancelled and the remaining terms of \( U_0(x,t) \) still satisfy the equation. Therefore, the exact solution is given by

\[ U(x,t) = t^2 + xt. \]  

(1.3.25)

The nonhomogenous advection equation considered in Example 1.3.2 is graphically represented through Figs. 1.3-1.5. The numerical results of the solution \( U(x,t) \) for different values of \( x \) and \( t \) are shown through Fig. 1.3. The numerical results of the exact solution \( U(x,t) \) vs. \( x \) and also those vs. \( t \) are depicted through Figs. 1.4 and 1.5. It is observed from Fig. 1.4 that \( U(x,t) \) increases with the increase in \( x \). It can be also noted from Fig. 1.5 that \( U(x,t) \) increases with the increase in \( t \).
Fig. 1.3 Plot of exact solution $U(x,t)$ w.r. to x and t for Example 1.3.2.

Fig. 1.4 Plot of exact solution $U(x,t)$ vs. x at $t = 1$ for Example 1.3.2.
1.4. Sumudu Decomposition Method (SDM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation of the form:

\[ LU + RU + NU = g(x), \]  

(1.4.1)

where L is the highest order linear differential operator, R is the linear differential operator of less order than L, N represents the general nonlinear differential operator and \( g(x) \) is the source term. By applying the Sumudu transform on both sides of Eq. (1.4.1), we get

\[ S[U] = u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + u^n S[g(x)] - u^n S[RU + NU] = 0. \]  

(1.4.2)

Now applying the inverse Sumudu transform on both sides of Eq. (1.4.2), we get

\[ U(x,t) = \frac{1}{t} g(x) \]  

Fig. 1.5 Plot of \( U(x,t) \) vs. \( t \) at \( x = 1 \) for Example 1.3.2.
where \( G(x) \) represents the term arising from the source term and the prescribed initial conditions. The second step in Sumudu decomposition method is that we represent solution as an infinite series given below

\[
U = \sum_{m=0}^{\infty} U_m \quad (1.4.4)
\]

and the nonlinear term can be decomposed as

\[
NU = \sum_{m=0}^{\infty} A_m, \quad (1.4.5)
\]

where \( A_m \) are Adomian polynomials [96-98] of \( U_0, U_1, U_2, \ldots, U_m \) and it can be calculated by formula given below

\[
A_m = \frac{1}{n!} \frac{d^n}{dx^n} \left[ \sum_{i=0}^{\infty} \lambda^i U_i \right], \quad m = 0, 1, 2, \ldots \quad (1.4.6)
\]

Using Eqs. (1.4.4) and (1.4.5) in Eq. (1.4.3), we get

\[
\sum_{m=0}^{\infty} U_m = G(x) - S^{-1} \left[ u^n S \left[ R \sum_{m=0}^{\infty} U_m + \sum_{m=0}^{\infty} A_m \right] \right] \quad (1.4.7)
\]

On comparing both sides of the Eq. (1.4.7), we get

\[U_0 = G(x),\]

\[U_1 = -S^{-1} \left[ u^n S [RU_0 + A_0] \right],\]

\[U_2 = -S^{-1} \left[ u^n S [RU_1 + A_1] \right].\]

In general the recursive relation is given by

\[U_{m+1} = -S^{-1} \left[ u^n S [RU_m + A_m] \right], \quad m \geq 0.\]
Now first of all applying the Sumudu transform of the right hand side of Eq. (1.4.8) then applying the inverse Sumudu transform, we get the values of \( U_0, U_1, U_2, \ldots, U_m \) respectively. Finally, we approximate the analytical solution \( U \) by truncated series

\[
U = \lim_{N \to \infty} \sum_{m=0}^{N} U_m.
\]  
(1.4.10)

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [134].

**1.4.1 Numerical Experiments**

In order to elucidate the solution procedure of the Sumudu decomposition method (SDM), we consider the nonlinear partial differential equations.

**Example 1.4.1.** Consider a nonlinear partial differential equation

\[
U_t + UU_x = x + xt^2,
\]  
(1.4.11)

with initial condition

\[
U(x,0) = 0.
\]  
(1.4.12)

Taking the Sumudu transform on both sides of Eq. (1.4.11) subject to the initial condition (1.4.12), we have

\[
S[U(x,t)] = xu + 2xu^3 - uS[UU_x].
\]  
(1.4.13)

The inverse of Sumudu transform implies that

\[
U(x,t) = xt + \frac{xt^3}{3} - S^{-1} \left[ S[UU_x] \right].
\]  
(1.4.14)
Following the technique, if we assume an infinite series solution of the form (1.4.4), we obtain

$$\sum_{n=0}^{\infty} U_m(x,t) = xt + \frac{xt^{3}}{3} - S^{-1}\left[ u \left( \sum_{n=0}^{\infty} A_m(U) \right) \right], \quad (1.4.15)$$

where $A_m(U)$ are Adomian polynomials that represent the nonlinear terms.

The first few components of Adomian polynomials, are given by

$$A_0(U) = U_0 U_{0x},$$

$$A_1(U) = U_0 U_{1x} + U_1 U_{0x}, \quad (1.4.16)$$

: 

The recursive relation is given below

$$U_0(x,t) = xt + \frac{xt^{3}}{3},$$

$$U_1(x,t) = -S^{-1}\left\{ S[A_0(U)] \right\}_-, \quad (1.4.17)$$

$$U_{m+1}(x,t) = -S^{-1}\left\{ S[A_m(U)] \right\}_-, m \geq 0.$$ 

The other components of the solutions can be easily found by using above recursive relation

$$U_1(x,t) = -S^{-1}\left\{ S[A_0(U)] \right\}_-$$

$$= -S^{-1}\left\{ S[U_0 U_{0x}] \right\}_-$$

$$= -\frac{xt^{3}}{3} - \frac{xt^{7}}{63} - \frac{2xt^{5}}{15}, \quad (1.4.18)$$

$$U_2(x,t) = -S^{-1}\left\{ S[A_1(U)] \right\}_-$$

$$= -S^{-1}\left\{ S[U_0 U_{1x} + U_1 U_{0x}] \right\}_-$$
\[
\frac{2xt^5}{15} + \frac{22xt^7}{315} + \frac{38xt^9}{2835} + \frac{2xt^{11}}{2079},
\]  

(1.4.19)

It is important to recall here that the noise terms appear between the components \( U_0(x,t) \) and \( U_1(x,t) \), where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms \( \pm \frac{1}{3}xt^3 \) between the components \( U_0(x,t) \) and \( U_1(x,t) \) can be cancelled and the remaining terms of \( U_0(x,t) \) still satisfy the equation. Therefore, the exact solution is given by

\[
U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) = xt. 
\]  

(1.4.20)

The nonlinear partial differential equation given in Example 1.4.1 is graphically described through Figs. 1.6-1.8. The numerical results of the solution \( U(x,t) \) for different values of \( x \) and \( t \) are shown through Fig. 1.6. The numerical results of the exact solution \( U(x,t) \) vs. \( x \) and also those vs. \( t \) are depicted through Figs. 1.7 and 1.8. It is observed from Figs. 1.6-1.8 that \( U(x,t) \) increases with the increase in both \( x \) and \( t \).
Fig. 1.6 Plot of $U(x,t)$ w.r. to $x$ and $t$ for Example 1.4.1.

Fig. 1.7 Plot of $U(x,t)$ vs. $x$ at $t = 1$ for Example 1.4.1.
Example 1.4.2. Now, consider the following nonlinear partial differential equation
\[ U_t - U U_{xx} = -t + U, \tag{1.4.21} \]
with initial conditions
\[ U(x,0) = \sin x, \tag{1.4.22} \]
\[ U_x(x,0) = 1. \tag{1.4.23} \]
Taking the Sumudu transform on both sides of Eq. (1.4.21) subject to the initial conditions (1.4.22) and (1.4.23), we have
\[ S[U(x,t)] = u + \sin x - u^3 + u^2 S[U + U U_{xx}]. \tag{1.4.24} \]
The inverse of Sumudu transform implies that
\[ U(x,t) = t + \sin x - \frac{t^3}{6} + S^{-1} \left[ 2 S[U + U U_{xx}] - \right] \tag{1.4.25} \]
Now, applying the same procedure as in previous example we arrive at recursive relation given below

\[ U_0(x,t) = t + \sin x - \frac{t^3}{6}, \]

\[ U_1(x,t) = S^{-1} \left\{ \int_0^t 2 S[U_0 + B_0(U)] \right\} \]

\[ U_{m+1}(x,t) = S^{-1} \left\{ \int_0^t 2 S[U_m + B_m(U)] \right\}, \quad m \geq 1, \]

where \( B_m(U) \) are Adomian polynomials that represent the nonlinear terms in the above equation (1.4.26). The other components of the solutions can be easily found by using above recursive relation

\[ U_1(x,t) = S^{-1} \left\{ \int_0^t 2 S[U_0 + B_0(U)] \right\} \]

\[ = S^{-1} \left\{ \int_0^t 2 S[U_0 + U_0 U_{0x}] \right\} \]

\[ = \frac{t^3}{6} - \frac{t^5}{120} + \frac{t^4}{24} \sin x, \]

\( \vdots \)

It is important to recall here that the noise terms appear between the components \( U_0(x,t) \) and \( U_1(x,t) \), where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms \( \pm \frac{t^3}{6} \) between the components \( U_0(x,t) \) and \( U_1(x,t) \) can be cancelled and the remaining terms of \( U_0(x,t) \) still satisfy the equation. Therefore, the exact solution is given by
The nonlinear partial differential equation considered in Example 1.4.2 is graphically represented through Figs. 1.9-1.11. The numerical results of the solution $U(x,t)$ for different values of $x$ and $t$ are shown through Fig. 1.9. The numerical results of the exact solution $U(x,t)$ vs. $x$ and also those vs. $t$ are depicted through Figs. 1.10 and 1.11. It is observed from Figs. 1.9-1.11, that $U(x,t)$ increases with the increase in $t$.  

![Graph of $U(x,t)$ vs. $x$ and $t$](image-url)

**Fig. 1.9** Plot of $U(x,t)$ w.r. to $x$ and $t$ for Example 1.4.2.
Fig. 1.10 Plot of $U(x,t)$ vs. $x$ at $t = 1$ for Example 1.4.2.

Fig. 1.11 Plot of $U(x,t)$ vs. $t$ at $x = 1$ for Example 1.4.2.
1.5. Homotopy Analysis Sumudu Transform Method (HASTM)

To illustrate the basic idea of this method, we consider an equation

\[ N[U(x)] = g(x), \]

where \( N \) represents a general nonlinear ordinary or partial differential operator including both linear and nonlinear terms. The linear terms are decomposed into \( L+R \), where \( L \) is the highest order linear operator and \( R \) is the remaining of the linear operator. Thus, the equation can be written as

\[ LU + RU + NU = g(x), \]

where \( NU \) indicates the nonlinear terms.

By applying the Sumudu transform on both sides of equation (1.5.1), we get

\[ S[LU] + S[RU] + S[NU] = S[g(x)]. \]

Using the differentiation property of the Sumudu transform, we have

\[ \frac{S[U]}{u^n} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + S[RU] + S[NU] = S[g(x)]. \]

On simplifying

\[ S[U] - u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + u^n [S[RU] + S[NU] - S[g(x)]] = 0. \]

We define the nonlinear operator

\[ N[\phi(x; q)] = S[\phi(x; q)] - u^n \sum_{k=0}^{n-1} \frac{\phi^{(k)}(x; q)(0)}{u^{(n-k)}} \\
+ u^n [S[R\phi(x; q)] + S[N\phi(x; q)] - S[g(x)]] \]
where \( q \in [0,1] \) and \( \phi(x; q) \) is a real function of \( x \) and \( q \). We construct a homotopy as follows

\[
(1 - q) S[\phi(x; q) - U_0(x)] = h q H(x) N[U(x)],
\]

(1.5.6)

where \( S \) denotes the Sumudu transform, \( q \in [0,1] \) is the embedding parameter, \( H(x) \) denotes a nonzero auxiliary function, \( h \neq 0 \) is an auxiliary parameter, \( U_0(x) \) is an initial guess of \( U(x) \) and \( \phi(x; q) \) is a unknown function. Obviously, when the embedding parameter \( q = 0 \) and \( q = 1 \), it holds

\[
\phi(x; 0) = U_0(x), \quad \phi(x; 1) = U(x),
\]

(1.5.7)

respectively. Thus, as \( q \) increases from 0 to 1, the solution \( \phi(x; q) \) varies from the initial guess \( U_0(x) \) to the solution \( U(x) \). Expanding \( \phi(x; q) \) in Taylor series with respect to \( q \), we have

\[
\phi(x; q) = U_0(x) + \sum_{m=1}^{\infty} U_m(x) q^m,
\]

(1.5.8)

where

\[
U_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \bigg|_{q=0}.
\]

(1.5.9)

If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \), and the auxiliary function are properly chosen, the series (1.5.8) converges at \( q = 1 \), then we have
\[
U(x) = U_0(x) + \sum_{m=1}^{\infty} U_m(x),
\]

which must be one of the solutions of the original nonlinear equations. According to the definition (1.5.10), the governing equation can be deduced from the zero-order deformation (1.5.6). Define the vectors

\[
\bar{U}_m = \{U_0(x), U_1(x), \ldots, U_m(x)\}.
\]

Differentiating the zeroth-order deformation equation (1.5.6) \(m\)-times with respect to \(q\) and then dividing them by \(m!\) and finally setting \(q = 0\), we get the following \(m\)th-order deformation equation:

\[
S[U_m(x) - \chi_m U_{m-1}(x)] = \hbar H(x) \mathcal{R}_m(\bar{U}_{m-1}).
\]

Applying the inverse Sumudu transform, we have

\[
U_m(x) = \chi_m U_{m-1}(x) + \hbar S^{-1}[H(x) \mathcal{R}_m(\bar{U}_{m-1})],
\]

where

\[
\mathcal{R}_m(\bar{U}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^{m-1}} \right|_{q=0},
\]

and

\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\]

1.5.1. Numerical Experiments

In this section, we use the HASTM to solve linear and nonlinear Fokker-Planck equations. The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [136]. The Fokker-
Planck equation arises in different fields in natural sciences such as quantum optics, solid state physics, chemical physics, theoretical biology and circuit theory. Fokker-Planck equations describe the erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic behavior of exchange rates. Several techniques including Adomian decomposition method (ADM), the variational iteration method (VIM) and the homotopy perturbation method (HPM) are also applied for solving the Fokker-Planck equation, see [137-139] and references therein.

**Example 1.5.1.** Consider the following linear Fokker-Planck equation

\[ U_t = U_x + U_{xx}, \quad (1.5.16) \]

with the initial condition

\[ U(x,0) = x. \quad (1.5.17) \]

According to the HASTM, we take the initial guess as

\[ U_0(x,t) = x. \quad (1.5.18) \]

By applying the aforesaid method subject to initial condition, we have

\[ S[U] - x - u \left[ S[U_x]\right] - S[U_{xx}] = 0. \quad (1.5.19) \]

The nonlinear operator is

\[ N[\phi(x,t;q)] = S[\phi(x,t;q)] - x - u \left[ S\left[ \frac{\partial \phi(x,t;q)}{\partial x} \right] \right] + S\left[ \frac{\partial^2 \phi(x,t;q)}{\partial x^2} \right] \quad (1.5.20) \]

and thus
The $m$th-order deformation equation is given by
\begin{equation}
\mathfrak{R}_m(\dot{U}_{m-1}) = S[U_{m-1}] - (1 - \chi_m)x - u \left[ S \left( \frac{\partial U_{m-1}}{\partial x} \right) + S \left( \frac{\partial^2 U_{m-1}}{\partial x^2} \right) \right],
\end{equation}
\begin{equation}
(1.5.21)
\end{equation}

Applying the inverse Sumudu transform, we have
\begin{equation}
U_m(x,t) = \chi_m U_{m-1}(x,t) + h S^{-1}[\mathfrak{R}_m(\bar{U}_{m-1})].
\end{equation}
\begin{equation}
(1.5.22)
\end{equation}

Solving above equation (1.5.23), for $m = 1, 2, 3, \ldots$, we get
\begin{align*}
U_1(x,t) &= -ht, \\
U_2(x,t) &= -h(1 + h)t, \\
U_3(x,t) &= -h(1 + h)^2t, \\
\vdots
\end{align*}
\begin{equation}
(1.5.24)
\end{equation}
and so on.

Taking $h = -1$, the solution is given by
\begin{equation}
U(x,t) = \sum_{m=0}^{\infty} U_m(x,t) = x + t,
\end{equation}
\begin{equation}
(1.5.25)
\end{equation}
which is the exact solution.

The linear Fokker-Planck equation studied in Example 1.5.1 is graphically depicted through Figs. 1.12-1.14. The numerical results of the solution $U(x,t)$ for different values of $x$ and $t$ are shown through Fig. 1.12. The numerical results of the exact solution $U(x,t)$ vs. $x$ and also those vs. $t$ are depicted
through Figs. 1.13 and 1.14. It can be seen from Figs. 1.12-1.14, that $U(x,t)$ increases with the increase in both $x$ and $t$.

**Fig. 1.12** Plot of $U(x,t)$ w.r. to $x$ and $t$ for Example 1.5.1.

**Fig. 1.13** Plot of $U(x,t)$ vs. $x$ at $t = 1$ for Example 1.5.1.
Example 1.5.2. Consider the following linear Fokker-Planck equation

\[ U_t = \frac{\partial}{\partial x} A(x,t) U + \frac{\partial^2}{\partial x^2} B(x,t) U, \tag{1.5.26} \]

where \( A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x \), \( B(x,t) = e^t \cosh x \), with the initial condition

\[ U(x,0) = \sinh x, \quad x \in \mathbb{R}. \tag{1.5.27} \]

According to the HASTM, we take the initial guess as

\[ U_0(x,t) = \sinh x. \tag{1.5.28} \]

By applying the aforesaid method subject to initial condition, we have

\[ S[U] - \sinh x - u S \left[ -\frac{\partial}{\partial x} A(x,t) U + \frac{\partial^2}{\partial x^2} B(x,t) U \right] = 0. \tag{1.5.29} \]
The nonlinear operator is
\[ N[\phi(x,t; q)] = S[\phi(x,t; q)] - \sinh x \]
\[ - uS \left[ - \frac{\partial}{\partial x} A(x,t)\phi(x,t; q) + \frac{\partial^2}{\partial x^2} B(x,t)\phi(x,t; q) \right] \quad (1.5.30) \]
and thus
\[ \mathfrak{R}_m(\tilde{U}_{m-1}) = S[U_{m-1}] - (1 - \chi_m) \sinh x \]
\[ - uS \left[ - \frac{\partial}{\partial x} A(x,t)U_{m-1} + \frac{\partial^2}{\partial x^2} B(x,t)U_{m-1} \right]. \quad (1.5.31) \]
The m\(^{th}\)-order deformation equation is given by
\[ S[U_n(x,t) - \chi_n U_{m-1}(x,t)] = h \mathfrak{R}_m(\tilde{U}_{m-1}). \quad (1.5.32) \]
Applying the inverse Sumudu transform, we have
\[ U_m(x,t) = \chi_m U_{m-1}(x,t) + h S^{-1}[\mathfrak{R}_m(\tilde{U}_{m-1})]. \quad (1.5.33) \]
Solving above equation (1.5.33), for \( m = 1, 2, 3 \ldots \), we get
\[ U_1(x,t) = -h \, t \, \sinh x, \]
\[ U_2(x,t) = -h \, (1 + h) \, t \, \sinh x + \frac{h^2 t^3 \, \sinh x}{2}, \quad (1.5.34) \]
\[ U_3(x,t) = -h \, (1 + h)^2 \, t \, \sinh x + h^2 (1 + h) \, t^2 \, \sinh x - \frac{h^3 t^3 \, \sinh x}{6}, \]
\[ \vdots \]
and so on.
Taking \( h = -1 \), the series solution is given by
\[ U(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) \sinh x. \] (1.5.35)

This solution is equivalent to the exact solution in closed form

\[ U(x,t) = e^t \sinh x. \] (1.5.36)

The linear Fokker-Planck equation considered in Example 1.5.2 is graphically described through Figs. 1.15 and 1.16. The numerical results of the approximate solution (1.5.35) and exact solution (1.5.36) are shown through Figs. 1.15 and 1.16 respectively. It can be seen from the Figs. 1.15 and 1.16 that the solution obtained by the HASTM is nearly identical with the exact solution. It is observed from Figs. 1.15 and 1.16 that \( U(x,t) \) increases with the increase in both \( x \) and \( t \). It is to be noted that only third order term of the HASTM was used in evaluating the approximate solutions for Fig. 1.15. It is evident that the efficiency of the present method can be dramatically enhanced by computing further terms of \( U(x,t) \) when the HASTM is used.
Fig. 1.15 Plot of approximate solution $U(x, t)$ w.r. to $x$ and $t$ for Example 1.5.2.

Fig. 1.16 Plot of exact solution $U(x, t)$ w.r. to $x$ and $t$ for Example 1.5.2.
Example 1.5.3. Consider the Backward Kolmogorov equation

$$U_t = (x+1)U_x + x^2 e^t U_{xx},$$

with the initial condition

$$U(x,0) = x + 1, \quad x \in \mathbb{R}.$$  \hspace{1cm} (1.5.38)

According to the HASTM, we take the initial guess as

$$U_0(x,t) = (x+1).$$  \hspace{1cm} (1.5.39)

By applying the aforesaid method subject to initial condition, we have

$$S[U] - (x+1) - uS \left[ x+1 U_x + x^2 e^t U_{xx} \right] = 0.$$  \hspace{1cm} (1.5.40)

The nonlinear operator is

$$N[\phi(x,t;q)] = S[\phi(x,t;q)] - (x+1)$$

$$- uS \left[ (x+1) \frac{\partial \phi(x,t;q)}{\partial x} + x^2 e^t \frac{\partial^2 \phi(x,t;q)}{\partial x^2} \right]$$

and thus

$$\mathcal{R}_m(\tilde{U}_{m+1}) = S[U_{m+1}] - (1 - \chi_m)(x+1)$$

$$- uS \left[ (x+1) \frac{\partial U_{m+1}}{\partial x} + x^2 e^t \frac{\partial^2 U_{m+1}}{\partial x^2} \right].$$  \hspace{1cm} (1.5.42)

The $m^{th}$-order deformation equation is given by

$$S[U_m(x,t) - \chi_m U_{m+1}(x,t)] = h \mathcal{R}_m(\tilde{U}_{m+1}).$$  \hspace{1cm} (1.5.43)

Applying the inverse Sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m+1}(x,t) + h S^{-1}[\mathcal{R}_m(\tilde{U}_{m+1})].$$  \hspace{1cm} (1.5.44)
Solving above equation (1.5.44), for \( m = 1, 2, 3, \ldots \), we get

\[
U_1(x, t) = -h(x+1)t,
\]

\[
U_2(x, t) = -h(1+h)(x+1)t + \frac{h^2(x+1)t^2}{2},
\]

\[
U_3(x, t) = -h(1+h)^2(x+1)t + \frac{h^2(1+h)(x+1)t^2}{2} - \frac{h^3(x+1)t^3}{6},
\]

\[\vdots\]

and so on.

Taking \( h = -1 \), the series solution is given by

\[
U(x, t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)(x+1).
\]

(1.5.46)

This solution is equivalent to the exact solution in closed form

\[
U(x, t) = e^t(x+1).
\]

(1.5.47)

The linear Fokker-Planck equation considered in Example 1.5.3 is graphically described through Figs. 1.17 and 1.18. The numerical results of the approximate solution (1.5.46) and exact solution (1.5.47) are shown through Figs. 1.17 and 1.18 respectively. It can be seen from the Figs. 1.17 and 1.18 that the solution obtained by the HASTM is nearly identical with the exact solution. It is observed from Figs. 1.17 and 1.18 that \( U(x, t) \) increases with the increase in both \( x \) and \( t \).
Fig. 1.17 Plot of approximate solution $U(x,t)$ w.r. to $x$ and $t$ for Example 1.5.3.

Fig. 1.18 Plot of exact solution $U(x,t)$ w.r. to $x$ and $t$ for Example 1.5.3.
Example 1.5.4. Consider the following nonlinear Fokker-Planck equation

\[ U_t = \frac{\partial}{\partial x} \left( \frac{xU}{3} - \frac{4}{x} U^2 \right) + \frac{\partial^2}{\partial x^2} (U^2), \]

subject to the initial condition

\[ U(x,0) = x^2, \quad x \in \mathbb{R}. \]

According to the HASTM, we take the initial guess as

\[ U_0(x,t) = x^2. \]

By applying the aforesaid method subject to initial condition, we have

\[ S[U] - x^2 - uS\left[ \frac{\partial}{\partial x} \left( \frac{xU}{3} - \frac{4}{x} U^2 \right) + \frac{\partial^2}{\partial x^2} (U^2) \right] = 0. \]

The nonlinear operator is

\[ N[\phi(x,t; q)] = S[\phi(x,t; q)] - x^2 \]

\[ - uS\left[ \frac{\partial}{\partial x} \left( \frac{x\phi(x,t; q)}{3} - \frac{4\phi^2(x,t; q)}{x} \right) + \frac{\partial^2}{\partial x^2} (\phi^2(x,t; q)) \right] \]

and thus

\[ \mathfrak{N}_m(\bar{U}_{m-1}) = S[U_{m-1}] - (1 - \chi_m)x^2 \]

\[ - uS\left[ \frac{\partial}{\partial x} \left( \frac{xU_{m-1}}{3} - \frac{4}{x} \left( \sum_{r=0}^{m-1} U_{m-1-r} \right) \right) + \frac{\partial^2}{\partial x^2} \left( \sum_{r=0}^{m-1} U_{m-1-r} \right) \right]. \]

The \( m \)-th order deformation equation is given by

\[ S[U_m(x,t) - \chi_m U_{m-1}(x,t)] = \mathfrak{h} \mathfrak{N}_m(\bar{U}_{m-1}). \]

Applying the inverse Sumudu transform, we have
\[ U_m(x,t) = \chi_m U_{m-1}(x,t) + h S_m^{-1}[\mathcal{R}_m(\tilde{U}_{m-1})]. \] (1.5.55)

Solving above equation (1.5.55), for \( m = 1, 2, 3, \ldots \), we get

\[ U_1(x,t) = -hx^2t, \]
\[ U_2(x,t) = -h(1+h)x^2t + \frac{h^2x^2t^2}{2}, \] (1.5.56)
\[ U_3(x,t) = -h(1+h)x^2t + \frac{h^2(1+h)x^2t^2}{2} - \frac{h^3x^2t^3}{6}, \]
\[ \vdots \]

and so on.

Taking \( h = -1 \), the solution is given by

\[ U(x,t) = x^2 \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) \] (1.5.57)

This solution is equivalent to the exact solution in closed form

\[ U(x,t) = x^2 e^t. \] (1.5.58)

The nonlinear Fokker-Planck equation considered in Example 1.5.4 is graphically described through Figs. 1.19 and 1.20. The numerical results of the approximate solution (1.5.57) and exact solution (1.5.58) are shown through Figs. 1.19 and 1.20 respectively. It can be seen from the Figs. 1.19 and 1.20 that the solution obtained by the HASTM is nearly identical with the exact solution. It is observed from Figs. 1.19 and 1.20 that \( U(x,t) \) increases with the increase in both \( x \) and \( t \).
Fig. 1.19 Plot of Approximate solution $U(x,t)$ w.r. to $x$ and $t$ for Example 1.5.4.

Fig. 1.20 Plot of Exact solution $U(x,t)$ w.r. to $x$ and $t$ for Example 1.5.4.
1.6. Conclusions

In this chapter, three new analytic techniques namely homotopy perturbation Sumudu transform method (HPSTM), Sumudu decomposition method (SDM) and homotopy analysis Sumudu transform method (HASTM) are proposed for solving nonlinear equations. The applicability and efficiency of these methods is shown to obtain the solutions of various linear and nonlinear partial equations. It is worth mentioning that the proposed techniques are capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. It is shown that the techniques are very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear partial differential equations.