CHAPTER - 8

JEFFERY-HAMEL FLOW
8.1. Introduction

Internal flow between two plates is one of the most applicable cases in mechanics, civil and environmental engineering. In simple cases, the one-dimensional flow through tube and parallel plates, this is known as Couette-Poisseuille flow, have exact solution, but in general, like most of fluid mechanics equations, a set of nonlinear equations must be solved which make some problems for analytical solution.

The flow between two planes that meet at an angle was first analyzed by Jeffery [201] and Hamel [202] and so, it is known as Jeffery-Hamel flow, too. They worked mathematically on incompressible viscous fluid flow through convergent-divergent channels. They presented an exact similarity solution of the Navier-Stokes equations. In the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex and with a source or sink at the vertex and have been studied extensively by several authors and discussed in many textbooks e.g. [203-213]. Sadri [214] has denoted that Jeffery-Hamel used as a asymptotic boundary conditions to examine steady two-dimensional flow of a viscous fluid in a channel. But, here some symmetric solutions of the flow has been considered, although asymmetric solutions are both possible and of physical interest [215].

In this chapter, we present two new analytical techniques namely the homotopy perturbation Sumudu transform method (HPSTM) and homotopy analysis Sumudu transform method (HASTM) to obtain the approximate solutions of nonlinear equation governing Jeffery-Hamel flow.
8.2. Mathematical Model

Consider the steady unidirectional flow of an incompressible viscous fluid flow from a source or sink at the intersection between two rigid plane walls that the angle between them is $2\alpha$ as it is shown in the Fig. 8.1.

![Fig. 8.1 Schematic figure of the problem.](image)

The velocity is assumed only along radial direction and depends on $r$ and $\theta$.

Conservation of mass and momentum for two-dimensional flow in the cylindrical coordinate can be expressed as following [216]

$$
\frac{1}{r} \frac{\partial}{\partial r} (r U_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (r U_\theta) = 0,
$$

(8.2.1a)
where \( \tau \) is the pressure term, \( U_r \) and \( U_{\theta} \) are the velocities in \( r \) and \( \theta \) directions respectively. Stress components are defined as follows:

\[
\tau_{rr} = \mu \left( 2 \frac{\partial U_r}{\partial r} - \frac{2}{3} \text{div}(\mathbf{U}) \right),
\]

(8.2.2a)

\[
\tau_{\theta\theta} = \mu \left( 2 \left( \frac{1}{r} \frac{\partial U_{\theta}}{\partial \theta} + \frac{U_r}{r} \right) - \frac{2}{3} \text{div}(\mathbf{U}) \right),
\]

(8.2.2b)

\[
\tau_{r\theta} = \mu \left( 2 \frac{\partial}{\partial r} \left( \frac{U_{\theta}}{r} \right) + \frac{1}{r} \left( \frac{\partial U_r}{\partial \theta} \right) \right).
\]

(8.2.2c)

Considering \( U_{\theta} = 0 \) for purely radial flow leads to continuity and Navier-Stokes equations in polar coordinates become

\[
\frac{\rho}{r} \frac{\partial}{\partial r} (r U_r) = 0,
\]

(8.2.3a)

\[
U_r \frac{\partial U_r}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r} \frac{\partial U_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U_r}{\partial \theta^2} - \frac{U_r}{r^2} \right],
\]

(8.2.3b)
\[- \frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{2\nu}{r^2} \frac{\partial U_r}{\partial \theta} = 0. \quad (8.2.3c)\]

The boundary conditions are

At centerline of the channel: \[\frac{\partial U_r}{\partial \theta} = 0,\]

On the wall of the channel:

\[U_r = 0. \quad (8.2.4)\]

From Eq. (8.2.3a)

\[g(\theta) \equiv r U_r, \quad (8.2.5)\]

using dimensionless parameter

\[f(x) \equiv \frac{g(0)}{g_{\text{max}}}, \quad x = \frac{\theta}{\alpha}, \quad (8.2.6)\]

and with eliminating P from Eqs. (8.2.3b) and (8.2.3c), an ordinary differential equation is obtained for the normalized function profile \(f(x)\):

\[f'' - 2\alpha \Re f(x)f'(x) + 4\alpha^2 f'(x) = 0. \quad (8.2.7)\]

According to the relation (8.2.4)-(8.2.6), the boundary conditions will be

\[f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0. \quad (8.2.8)\]
The Reynolds number is
\[
\text{Re} = \frac{g_{\text{max}} \alpha}{\nu} = \frac{U_{\text{max}} \rho \alpha}{\nu} \left( \begin{array}{l}
\text{Divergent Channel : } \alpha > 0 \\
\text{Convergent Channel : } \alpha < 0
\end{array} \right).
\] (8.2.9)

where \( U_{\text{max}} \) is the velocity at the centre of the channel \((r = 0)\).

### 8.3. Solution by Homotopy Perturbation Sumudu Transform Method (HPSTM)

In this section, we apply the HPSTM to obtain an approximate analytical solution of Eq. (8.2.7). By applying the Sumudu transform on both sides of Eq. (8.2.7), we have

\[
S[f(x)] = 1 + au^2 - u^3 S[2\alpha \text{Re } f \cdot + 4\alpha^2 f \cdot].
\] (8.3.1)

Taking inverse Sumudu transform on both sides of Eq. (8.3.1), we get

\[
f(x) = 1 + \frac{1}{2} \alpha x^2 - S^{-1}[u^3 S[2\alpha \text{Re } f \cdot + 4\alpha^2 f \cdot]].
\] (8.3.2)

Now applying the HPM, we get

\[
\sum_{m=0}^{\infty} p^m f_m(x) = 1 + \frac{1}{2} \alpha x^2 - \left( \sum_{m=0}^{\infty} p^m H_m(x) \right)
+ 4\alpha^2 \left( \sum_{m=0}^{\infty} p^m f'_m(\eta) \right),
\] (8.3.3)

where \( H_m \) are He’s polynomials that represents the nonlinear terms. So He’s polynomials are given by

\[
\sum_{m=0}^{\infty} p^m H_m(x) = f(x) f \cdot (x).
\] (8.3.4)
The first few components of He’s polynomials, are given by

$$H_0(x) = f_0(x) f'_0(x),$$

$$H_1(x) = f_0(x) f'_1(x) + f_1(x) f'_0(x), \quad (8.3.5)$$

$$H_2(x) = f_0(x) f'_2(x) + f_1(x) f'_1(x) + f_2(x) f'_0(x), \quad :$$

Comparing the coefficients of like powers of p, we have

$$p^0: f_0(x) = 1 + \frac{1}{2} a x^2, \quad (8.3.6)$$

$$p^1: f_1(x) = -a \left[ \frac{a^2}{120} \text{Re} x^6 + \frac{1}{12} (\text{Re} a + 2 \alpha a) x^4 \right], \quad (8.3.7)$$

$$p^2: f_2(x) = -\frac{4}{15} \alpha^2 \left[ -\frac{a^3}{2880} \text{Re}^2 x^8 - \frac{a^2}{1344} (9 \text{Re}^2 + 18 \alpha \text{Re}) x^6 \right. \left. - \frac{a}{240} (5 \text{Re}^2 + 20 \alpha \text{Re} + 20 \alpha^2) x^4 \right], \quad (8.3.8)$$

where $a = f''(0)$ to be determined from the boundary conditions. The solutions of the Eq. (8.2.7), when $p \to 1$, will be as follows:

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots. \quad (8.3.9)$$
8.4. Solution by Homotopy Analysis Sumudu Transform Method (HASTM)

We take the initial guess as

\[ f_0(x) = 1 + \frac{ax^2}{2}. \]  

(8.4.1)

Applying the Sumudu transform on both sides of Eq. (8.2.7), we have

\[ S[f] - 1 - au^2 + u^3 S\left[2\alpha \text{Re}(f) f' + 4\alpha^2 f'\right] = 0. \]  

(8.4.2)

We define the nonlinear operator

\[ N[\phi(x; q)] = S[\phi(x; q)] - 1 - au^2 + u^3 S\left[2\alpha \text{Re}(\phi(x; q)\phi'(x; q)) + 4\alpha^2 \phi'(x; q)\right] \]

(8.4.3)

and thus

\[ \mathcal{R}_m(\tilde{f}_{m-1}) = S[f_{m-1}] - (1 - \chi_m)(1 + au^2) + u^3 S\left[2\alpha \text{Re} \sum_{k=0}^{m-1} f_{m-1-k} f_k' + 4\alpha^2 f_{m-1}'\right]. \]  

(8.4.4)

The mth -order deformation equation is given by

\[ S[f_m(x) - \chi_m f_{m-1}(x)] = \hbar \mathcal{R}_m(\tilde{f}_{m-1}). \]  

(8.4.5)

Applying the inverse Sumudu transform, we have

\[ f_m(x) = \chi_m f_{m-1}(x) + \hbar S^{-1}[\mathcal{R}_m(\tilde{f}_{m-1})]. \]  

(8.4.6)

Solving the above Eq. (8.4.6), for \( m = 1, 2, 3, \ldots \), we get

\[ f_1(x) = \alpha h \left[ \frac{a^2}{120} \text{Re} x^6 + \frac{1}{12} (\text{Re} a + 2\alpha a)x^4 \right]. \]  

(8.4.7)
\[ f_2(x) = \alpha h(1 + h) \left[ \frac{a^2}{120} \text{Re} x^6 + \frac{1}{12} (\text{Re} a + 2\alpha a) x^4 \right] + \frac{4}{15} \alpha^2 h \left[ -\frac{a^3}{2880} \text{Re}^2 x^8 \right. \]

\[ \left. - \frac{a^2}{1344} (9 \text{Re}^2 + 18\alpha \text{Re}) x^8 - \frac{a}{240} (5 \text{Re}^2 + 20\alpha \text{Re} + 20\alpha^2) x^6 \right], \quad (8.4.8) \]

where \( a = f''(0) \) to be determined from the boundary conditions. The solutions of the Eq. (8.2.7), when \( p \to 1 \), will be as follows:

\[ f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots. \quad (8.4.9) \]

**Table 8.1**

The results of HPSTM and HASTM for \( f(x) \) when \( \text{Re} = 80 \) and \( \alpha = 3^\circ \).

<table>
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<tr>
<th>x</th>
<th>HPSTM</th>
<th>HASTM</th>
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<tbody>
<tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9865647246</td>
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<tr>
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Table 8.2

The comparison between the RKHSM (with H = 0) [213] and HPSTM and HASTM for \( f(x) \) when \( \text{Re} = 80 \) and \( \alpha = -5^\circ \).

<table>
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<tr>
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<th>HASTM</th>
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8.5. Results and Discussion

Eq. (8.2.7) is solved analytically using the HPSTM and HASTM. Tables 8.1 and 8.2 show numerical data for HPSTM and HASTM for the validity of two analytical methods when \( \text{Re} = 110, \alpha = 3^\circ \) and \( \text{Re} = 80, \alpha = -5^\circ \). It is observed from Table 8.2 that the results obtained by using HPSTM and HASTM are in a very good agreement with the results obtained by using reproducing kernel Hilbert space method (RKHSM) [213]. Figs. 8.2-8.5 illustrate the effects of Reynolds number and steep angle of the channel on velocity profile.
Fig. 8.2 Velocity diagram via HASTM for different values of Re when $\alpha = 3^\circ$.

Fig. 8.3 Velocity diagram via HASTM for different values of Re when $\alpha = -5^\circ$. 
8.6. Conclusions

In this chapter, the HASTM and HPSTM are applied successfully to find the analytical solution of Jeffery-Hamel flow. The results of the present methods are in excellent agreement and the obtained solutions are revealed graphically.

In this chapter, we use Maple Package to calculate the He’s polynomials. Also from figures, we can find some results as follows:

(1) When $\alpha > 0$ and steep of the channel is divergent, increase in the values of Reynolds number decreases the velocity as shown in Fig. 8.2 when $\alpha = 3^\circ$. 

Fig. 8.4 Velocity diagram via HASTM for different values of $\alpha$ when Re = 50.
(2) When $\alpha < 0$ and the steep of the channel is convergent, the velocity increases with the increase in Reynolds number as depicted in Fig. 8.3 when $\alpha = -5^\circ$.

(3) When Reynolds number is fixed, there is an inverse relation between divergence angle of the channel and the velocity of the fluid as shown in Fig. 8.4.