CHAPTER - 4

TWO-DIMENSIONAL VISCOUS FLOW WITH A SHRINKING SHEET
4.1. Introduction

The boundary layer viscous flow induced by stretching surface moving with a certain velocity in an otherwise quiescent fluid medium often occurs in the several engineering processes. It has attracted considerable interest during the last few decades. Such flows have important applications in industries, for example in the extrusion of a polymer sheet from a die or in the drawing of plastic films. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. The mechanical properties of the final product strictly depend on the stretching and cold drawing rates in the process. The pioneering work in this area was conducted by Sakiadis [160,161] and the boundary layer flow on a continuously stretching surface with a constant speed was investigated by several researchers in the field. Specifically Crane [162] found a closed form solution for flow of an incompressible viscous fluid past a stretching plate. Furthermore, the basic stretching solutions (which differ appreciably form Crane's) are as follows. Gupta and Gupta [163] added suction or injection on the surface. The flow inside a stretching channel or tube was considered by Brady and Acrivos [167] and the flow outside a stretching tube by Wang [168]. The three-dimensional and axisymmetric stretching flat surface was studied by Wang [166]. The unsteady stretching sheet was investigated by Wang [169] and Usha and Sridharan [170].

In recent years, the boundary layer flow due to a shrinking sheet has attracted considerable attention. The unsteady viscous flow induced by a shrinking
sheet was first studied by Wang [169]. The proof of the existence and
(non)uniqueness, the exact solutions, both numerical and in closed form, are
given by Miklavcic and Wang [177] for the steady viscous hydrodynamic flow
due to a shrinking sheet for a specific value of the suction parameter.
Miklavcic and Wang [177] concluded that the solution for shrinking sheets
may not be unique at certain suction rates for both two-dimensional and
axisymmetric flows. Wang [178] investigated the stagnation flow towards a
shrinking sheet and found for the first time that non-alignment of the
stagnation flow and the shrinking of the sheet destroys the symmetry and
complicates the flow field. Furthermore, Faraz et al. [179] obtained the
analytical solution of a two-dimensional viscous flow due to a shrinking sheet
by using variational iteration algorithm-II (VIM-II) and Adomian
decomposition method (ADM).

In this chapter, the HPSTM basically illustrates how the Sumudu transform
can be used to approximate the solutions of the nonlinear equations by
manipulating the homotopy perturbation method. The perturbation methods
which are generally used to solve nonlinear problems have some limitations
e.g., the approximate solution involves series of small parameters which
possess difficulty since majority of nonlinear problems have no small
parameters at all. Although appropriate choices of small parameters some time
leads to ideal solution but in most of the cases unsuitable choices lead to
serious effects in the solutions.
The objective of this chapter is to present a simple recursive algorithm based on the HPSTM which produces the series solution of the two-dimensional viscous flow due to shrinking sheet. The difficulty of the condition at infinity is overcome by the use of padé approximants. The velocity profiles given by HPSTM are in good agreement with the VIM-II solutions given in [179]. For best approximation, the resulting series is best manipulated by padé approximants. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

4.2. Padé Approximants

Padé approximants constitute the best approximation of a function by a rational function of a given order. The Padé approximants developed by Henri Padé, often provide better approximation of a function than does truncating its Taylor Series, and they may still work in cases in which the Taylor Series does not converge. Due to these reasons, Padé approximants are used extensively in computer calculations, and it is now well known that these approximants have the advantage of being able to manipulate polynomial approximation into the rational functions of polynomials. In addition, power series are not useful for
large value of a variable, say $\eta \to \infty$, which can be attributed to the possibility of the radius of convergence not being sufficiently large to contain the boundaries of the domain. To provide an effective tool that can handle boundary value problems on an infinite or semi-infinite domain, it is therefore essential to combine the series solution, which is obtained by the iteration method or any other series solution method, with the Padé approximants \[180].

### 4.3. Mathematical Formulation

In this section, we consider the two basic equations of fluid mechanics in Cartesian coordinates. The continuity equation and momentum equations for viscous flow are

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{4.3.1}
\]

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + V \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right), \tag{4.3.2}
\]

\[
U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + V \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right), \tag{4.3.3}
\]

\[
U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + V \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right). \tag{4.3.4}
\]

where $\nu = \mu/\rho$ is the kinematic viscosity.

The boundary conditions applicable to the present flow are

\[
U = -ax, \quad V = -a(m-1)y, \quad W = -w \text{ at } y = 0,
\]

\[
U \to 0 \text{ as } y \to \infty. \tag{4.3.5}
\]
For shrinking phenomenon, $a > 0$ is the shrinking constant and $w$ is the suction velocity. $m = 1$ when the sheet shrinks in the $x$-direction and $m=2$ when it shrinks axisymmetrically. Introducing the following similarity transformations.

$$U = axf' (\eta), \quad V = a(m-1)yf' (\eta), \quad \eta = \sqrt{\frac{a}{v}} z. \quad (4.3.6)$$

Eq. (4.3.1) is identically satisfied. Eq. (4.3.4) can be integrated to give

$$\frac{p}{\rho} = \sqrt{\frac{\partial W}{\partial z}} - \frac{W^2}{2} + \text{Cons} \tan t. \quad (4.3.7)$$

Eqs. (4.3.2), (4.3.3) and (4.3.4) are reduced to the boundary value problem:

$$f'' - (f')^2 + mff' = 0, \quad (4.3.8)$$

$$f = s, \quad f' = -1 \text{ at } \eta = 0, \quad f' \to 0 \text{ as } \eta \to \infty, \quad (4.3.9)$$

where $s = w/m\sqrt{4\alpha v}$.

### 4.4. HPSTM Solution and Discussion

In this section, we apply the homotopy perturbation Sumudu transform method (HPSTM) to obtain an approximate analytical solution of (4.3.8)-(4.3.9). By applying the Sumudu transform on the both sides of equation (4.3.8), we have

$$S[f(\eta)] = 2 - u + \alpha u^2 + u^3 S[(f')^2 - 2ff''], \quad (4.4.1)$$

where $f''(0) = \alpha, m = 2$ and $s = 2$. 
The inverse Sumudu transform implies that

\[ f(\eta) = 2 - \eta + \frac{\alpha \eta^2}{2} + S^{-1} \left( 3 S \left( f' \right)^2 - 2 f' f'' \right). \]  

(4.4.2)

Now applying the homotopy perturbation method, we get

\[ \sum_{n=0}^{\infty} p^n f_n(\eta) = 2 - \eta + \frac{\alpha \eta^2}{2} + p \left( S^{-1} \left( u^3 S \left[ \sum_{n=0}^{\infty} p^n H_n(\eta) \right] - 2 \left( \sum_{n=0}^{\infty} p^n H_n(\eta) \right) \right) \right). \]  

(4.4.3)

where \( H_n(\eta) \) and \( H_n'(\eta) \) are He’s polynomials that represents the nonlinear terms. So, He’s polynomials are given by

\[ \sum_{n=0}^{\infty} p^n H_n(\eta) = (f')^2(\eta). \]  

(4.4.4)

The first few components of He’s polynomials, are given by

\[ H_0(\eta) = (f_0')^2(\eta). \]

\[ H_1(\eta) = 2 f_0'(\eta) f_1'(\eta). \]

\[ H_2(\eta) = (f_1')^2(\eta) + 2 f_0'(\eta) f_2'(\eta). \]

\[ H_3(\eta) = 2 f_1'(\eta) f_2'(\eta) + 2 f_0'(\eta) f_3'(\eta). \]  

(4.4.5)

\[ \vdots \]

\[ H_n(\eta) = \sum_{i=0}^{n} f_i'(\eta) f_{n-i}'(\eta). \]

and for \( H_n'(\eta) \) we find

\[ \sum_{n=0}^{\infty} p^n H_n'(\eta) = f(\eta) f'(\eta). \]  

(4.4.6)
\[ H_0(\eta) = f_0(\eta) f_0^\prime(\eta), \]

\[ H_1(\eta) = f_0(\eta) f_1^\prime(\eta) + f_1(\eta) f_0^\prime(\eta), \]

\[ H_2(\eta) = f_0(\eta) f_2^\prime(\eta) + f_1(\eta) f_1^\prime(\eta) + f_2(\eta) f_0^\prime(\eta), \quad (4.4.7) \]

\[ H_3(\eta) = f_0(\eta) f_3^\prime(\eta) + f_1(\eta) f_2^\prime(\eta) + f_2(\eta) f_1^\prime(\eta) + f_3(\eta) f_0^\prime(\eta), \]

\[ \vdots \]

\[ H_n(\eta) = \sum_{i=0}^{n} f_i(\eta) f_{n-i}^\prime(\eta). \]

Comparing the coefficients of like powers of \( p \), we have

\[ p^0 : f_0 = 2 - \eta + \frac{\alpha \eta^2}{2}, \quad (4.4.8) \]

\[ p^1 : f_1 = \frac{\eta^3}{6} - \frac{2\alpha \eta^3}{3}, \quad (4.4.9) \]

\[ p^2 : f_2 = -\frac{\eta^4}{6} + \frac{2\alpha \eta^4}{3} + \frac{\eta^5}{60} - \frac{\alpha \eta^5}{15} + \frac{\alpha \eta^6}{360} - \frac{\alpha^2 \eta^6}{90}, \quad (4.4.10) \]

\[ p^3 : f_3 = \frac{2\eta^5}{15} - \frac{8 \alpha \eta^5}{30} + \frac{\eta^6}{90} - \frac{2\alpha \eta^6}{15} + \frac{\eta^7}{504} - \frac{\alpha \eta^7}{315} - \frac{2\alpha^2 \eta^7}{105}, \quad (4.4.11) \]

\[ \quad - \frac{\alpha \eta^8}{5040} + \frac{\alpha^2 \eta^8}{1260} - \frac{\alpha \eta^9}{9072} + \frac{\alpha^2 \eta^9}{2268}. \]

\[ p^4 : f_4 = -\frac{4\eta^6}{45} + \frac{16 \alpha \eta^6}{45} + \frac{4 \eta^7}{105} - \frac{16 \alpha \eta^7}{105} - \frac{\eta^8}{360} + \frac{\alpha \eta^8}{126} + \frac{8 \alpha^2 \eta^8}{105} + \frac{\eta^9}{9072} + \frac{37 \alpha \eta^9}{11340} - \frac{73 \alpha^2 \eta^9}{5670} - \frac{47 \alpha \eta^{10}}{30400} + \frac{37 \alpha^2 \eta^{10}}{5670} - \frac{\alpha^3 \eta^{10}}{8100} - \frac{\alpha^2 \eta^{11}}{178200} - \frac{\alpha^3 \eta^{11}}{44550} + \frac{\alpha^3 \eta^{12}}{213840} - \frac{\alpha^4 \eta^{12}}{53460}, \quad (4.4.12) \]

\[ \vdots \]
and so on. In this way, the remaining terms of the HPSTM series solution can be calculated.

The series solution is given by

$$ f = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots. $$ (4.4.13)

Substituting Eqs. (4.4.8)- (4.4.12) into Eq. (4.4.13), we obtain the following series solution

$$ f(\eta) = 2 - \eta + \frac{\alpha \eta^2}{2} + \frac{\eta^3}{6} - \frac{2\alpha \eta^3}{3} - \frac{\eta^4}{6} + \frac{2\alpha \eta^4}{3} + \frac{3\eta^5}{20} - \frac{3\alpha \eta^5}{5} $$

$$ - \frac{11\eta^6}{90} + \frac{59\alpha \eta^6}{120} - \frac{\alpha^2 \eta^6}{90} + \frac{101\eta^7}{2520} - \frac{7\alpha \eta^7}{45} - \frac{2\alpha^2 \eta^7}{105} $$

$$ - \frac{\eta^8}{360} + \frac{41\alpha \eta^8}{5040} + \frac{97\alpha^2 \eta^8}{1260} - \frac{\eta^9}{90720} + \frac{37\alpha \eta^9}{11340} $$

$$ - \frac{589\alpha^2 \eta^9}{45360} + \frac{\alpha^3 \eta^9}{2268} - \frac{47\alpha \eta^{10}}{302400} + \frac{37\alpha^2 \eta^{10}}{56700} - \frac{\alpha^3 \eta^{10}}{8100} $$

$$ - \frac{\alpha^2 \eta^{11}}{178200} + \frac{\alpha^3 \eta^{11}}{44550} + \frac{\alpha^3 \eta^{12}}{213840} - \frac{\alpha^4 \eta^{12}}{53460} + \cdots. $$ (4.4.14)

Software packages such as Maple or Mathematica can be used to solve the polynomials $f'(\eta)$ to calculate the value of $\alpha$ with the help of boundary condition $f'(\eta) \to 0$, for $\eta \to \infty$. By using the table above, we can choose the value of $\alpha = f'(\eta) \to 0 = 0.249243$ for HPSTM and VIM-II solutions, which is an average value of $[5/5]$ Padé approximation (Table 4.1).

The numerical results of HPSTM are depicted in Fig. 4.1. Fig. 4.2 compares the solutions obtained by the present method and VIM-II [179].
Table 4.1 The numerical values for $f'' = \alpha$ using Padé approximation.

<table>
<thead>
<tr>
<th>Padé approximation</th>
<th>$f'' = \alpha$ for VIM-II [179]</th>
<th>$f'' = \alpha$ for HPSTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1/1]</td>
<td>0.224748</td>
<td>0.292893</td>
</tr>
<tr>
<td>[2/2]</td>
<td>Complex number</td>
<td>Complex number</td>
</tr>
<tr>
<td>[3/3]</td>
<td>0.294748</td>
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</tr>
<tr>
<td>[4/4]</td>
<td>0.308086</td>
<td>0.247723</td>
</tr>
<tr>
<td>[5/5]</td>
<td>0.249556</td>
<td>0.24893</td>
</tr>
</tbody>
</table>

**Fig. 4.1.** Graphical presentation of HPSTM solution, when $\alpha = 0.249243$, $m = 2$ and $s = 2$. 
4.5. Conclusions

In this chapter, a simple algorithm based on the HPSTM-Padé approach has been applied for solving the viscous flow due to shrinking sheet. The method is applied here in direct manner without the use of linearization, transformation, discretization, perturbation, or restrictive assumptions. This study has considered only an axisymmetrically shrinking sheet by taking $m = 2$. Our results compare very well with the results obtained by VIM-II [179]. The results show that the HPSTM is a powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations.

![Graph showing comparison of HPSTM solution and VIM-II solution](image)