CHAPTER - 3

MHD VISCOUS FLOW OVER A STRETCHING SHEET
3.1. Introduction

The boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet is often occurs in the several engineering and industrial processes. It has attracted considerable interest during the last few decades. Such flows have important applications in industries, for example in the aerodynamic extrusion of a polymer sheet from a die, hot rolling, the cooling of an infinite metallic plate in a cooling bath, the boundary layer along a liquid film in condensation process, glass-fiber production and so on [157-159]. Many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. The mechanical properties of the final product strictly depend on the stretching and cold drawing rates in the process. The pioneering work in this area was conducted by Sakiadis [160,161] and the boundary layer flow on a continuously stretching surface with a constant speed was investigated by several researchers in the field. Specifically Crane [162] found a closed form solution for flow of an incompressible viscous fluid past a stretching plate. Gupta and Gupta [163] added suction or injection on the surface which models condensation or evaporation, a uniform transverse magnetic field, when the fluid is electrically conducting, by Anderson [164]. The joint effect of viscoelasticity and magnetic field on Crane’s problem has been investigated by Ariel [165]. The flow past a stretching sheet need not be necessarily two-dimensional because the stretching of the sheet can take place in a number of ways. It can be three-dimensional when the stretching is radial. The three-
dimensional and axisymmetric stretching flat surface was studied by Wang [166]. The flow inside a stretching channel or tube was considered by Brady and Acrivos [167] and the flow outside a stretching tube by Wang [168]. The unsteady stretching sheet was investigated by Wang [169] and Usha and Sridharan [170]. A non-iterative solution for the MHD flow has been given by Ariel [171] using the technique of Samuel and Hall [172], Ariel et al. [173] and Ariel [174] applied HPM and extended HPM to derive the analytical solution to the axisymmetric flow past a stretching sheet.

Magnetohydrodynamics (MHD) is the study of the interaction of conducting fluid with electromagnetic phenomena. The flow of an electrically conducting fluid in the presence of magnetic field is important in several areas of science and engineering such as MHD power generation, MHD flow generators, MHD pumps etc. Most boundary-layer models can be reduced to systems of nonlinear ordinary differential equations which are solved by numerical and analytical methods. Analytical methods have significant advantages over numerical methods in providing analytic, verifiable, rapidly convergent approximation. There exists a wide class of literature dealing with the problems of approximate solutions to nonlinear differential equations with various different methodologies, called perturbation methods. The perturbation methods have some limitations e.g., the approximate solution involves series of small parameters which posses difficulty since majority of nonlinear problems have no small parameters at all. Although appropriate
choices of small parameters some time leads to ideal solution but in most of the cases unsuitable choices lead to serious effects in the solutions. Therefore, an analytical method is welcome which does not require a small parameter in the equation modeling the phenomenon. In this paper we, present a simple recursive algorithm based on the HPSTM which produces the series solution of MHD boundary-layer equations for stretching sheet problem.

### 3.2. Mathematical Model

The MHD boundary layer flow over a flat plate is governed by the continuity and the Navier-Stokes equations for an incompressible viscous fluid. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the stretching sheet. The induced magnetic field is neglected. The resulting boundary-layer equations are:

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (3.2.1)
\]

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \nu \frac{\partial^2 U}{\partial y^2} - \frac{\sigma B^2(x)}{\rho} U, \quad (3.2.2)
\]

where $U$ and $V$ are the velocity components in the $x$ and $y$ directions, respectively. Also $\nu$, $\rho$ and $\sigma$ are the kinematic viscosity, density and electrical conductivity of the fluid. The $B(x)$ is taken as $B(x) = B_0 x^{n-1/2}$.

The boundary conditions applicable to the present flow are

\[
U(x,0) = cx^n, \quad V(x,0) = 0, \quad \text{and} \quad U(x,\infty) = 0. \quad (3.2.3)
\]

To solve the problem, momentum and energy equations are firstly nondimensionalized by introducing the following dimensionless variables:
Using Eqs. (3.2.4)–(3.2.6), the governing equations can be reduced to a nonlinear differential equation of the form:

\[ f^{\prime\prime\prime} + ff^{\prime\prime} - \beta(f^{\prime})^2 - Mf^{\prime\prime} = 0, \quad f(0) = 0, \quad f^{\prime}(0) = 1 \quad \text{and} \quad f^{\prime}(\infty) = 0, \]

where

\[ \beta = \frac{2n}{n+1}, \quad \quad M = \frac{2\sigma\beta^2}{\rho c(1+n)}. \]  

### 3.3. HPSTM Solution and Discussion

In this section, we apply the HPSTM to obtain an approximate analytical solution of (3.2.7)-(3.2.8). By applying the Sumudu transform on both sides of Eq. (3.2.7), we have

\[ S[f(\eta)] = u + c u^2 + u^3 S[\beta(f^{\prime})^2 - ff^{\prime\prime} + Mf^{\prime\prime}], \]  

where \( f^{\prime\prime}(0) = \alpha \).

The inverse Sumudu transform implies that

\[ f(\eta) = \eta + \frac{c \eta^2}{2} + S^{-1} \left\{ \frac{u^3}{u^3 S[(f^{\prime})^2 - ff^{\prime\prime} + Mf^{\prime\prime}].} \right\} \]
Now applying the HPM, we get

\[
\sum_{m=0}^{\infty} p^m f_m(\eta) = \eta + \frac{\alpha \eta^2}{2} + p \left( S^{-1} \beta \left( \sum_{m=0}^{\infty} \eta^m H_m(\eta) \right) - \left( \sum_{m=0}^{\infty} \eta^m H_m^{(\prime)}(\eta) \right) + M \left( \sum_{m=0}^{\infty} \eta^m f_m^{(\prime)}(\eta) \right) \right),
\]

where \( H_m(\eta) \) and \( H_m^{(\prime)}(\eta) \) are He’s polynomials that represents the nonlinear terms. So, He’s polynomials are given by

\[
\sum_{m=0}^{\infty} \eta^m H_m(\eta) = (f^{(\prime)})^2(\eta).
\]

The first few components of He’s polynomials, are given by

\[
H_0(\eta) = (f_0^{(\prime)})^2(\eta),
\]

\[
H_1(\eta) = 2 f_0^{(\prime)}(\eta) f_1^{(\prime)}(\eta),
\]

\[
H_2(\eta) = (f_1^{(\prime)})^2(\eta) + 2 f_0^{(\prime)}(\eta) f_2^{(\prime)}(\eta),
\]

\[ \vdots \]

and for \( H_m(\eta) \) we find that

\[
\sum_{m=0}^{\infty} \eta^m H_m^{(\prime)}(\eta) = f(\eta) f^{(\prime)}(\eta),
\]

\[
H_0(\eta) = f_0(\eta) f_0^{(\prime)}(\eta),
\]

\[
H_1(\eta) = f_0(\eta) f_1^{(\prime)}(\eta) + f_1(\eta) f_0^{(\prime)}(\eta),
\]

\[
H_2(\eta) = f_0(\eta) f_2^{(\prime)}(\eta) + f_1(\eta) f_1^{(\prime)}(\eta) + f_2(\eta) f_0^{(\prime)}(\eta),
\]

\[ \vdots \]
Comparing the coefficients of like powers of $p$, we have

\[ p^0 : f_0 = \eta + \frac{\alpha \eta^2}{2}, \quad (3.3.8) \]

\[ p^1 : f_1 = \frac{(2\beta - 1)}{120} \alpha^2 \eta^5 + \frac{(M - 1 + 2\beta)}{24} \alpha \eta^4 + \frac{(M + \beta)}{6} \eta^3, \quad (3.3.9) \]

\[ p^2 : f_2 = \frac{(20\beta^2 - 32\beta + 11)}{40320} \alpha^3 \eta^8 \]
\[ + \frac{(11 + 10M\beta + 20\beta^2 - 8M - 32\beta)}{5040} \alpha^2 \eta^7 \]
\[ + \frac{(3 - 12\beta - 8M + 10\beta^2 + 10M\beta + M^2)}{720} \alpha \eta^6 \]
\[ + \frac{(M^2 + 2\beta^2 - 2\beta - 2M + 3M\beta)}{120} \eta^5, \quad (3.3.10) \]

\[ p^3 : f_3 = \frac{(600\beta^3 - 1596\beta^2 + 1398\beta - 375)}{39916800} \alpha^4 \eta^{11} \]
\[ + \frac{(-375 - 546\beta M + 300M\beta^2 + 600\beta^3 - 1596\beta^2 + 1398\beta + 243M)}{3628800} \alpha^3 \eta^{10} \]
\[ (-129 + 42\beta M^2 + 300\beta^3 - 39M^2 + 548\beta - 710\beta^2 + 243M) \]
\[ + \frac{-546\beta M + 300M\beta^2}{362880} \alpha^2 \eta^9 \]
\[ (-15 + 120M\beta^2 + M^3 + 80\beta^3 - 166\beta^2 + 103\beta + 80M) \]
\[ + \frac{-39M^2 + 42\beta M^2 - 196\beta M}{40320} \alpha \eta^8 \]
\[ (M^3 - 10M^2 + 11\beta M^2 + 20\beta^2 M - 26\beta M + 8M + 8\beta) \]
\[ + \frac{-16\beta^2 + 10\beta^3}{5040} \eta^7, \quad (3.3.11) \]
and so on. In this way, the remaining terms of the HPSTM series solution can be calculated.

The series solution is given by

\[ f = f_0 + f_1 + f_2 + f_3 + \cdots, \quad (3.3.12) \]

Substituting Eqs. (3.3.8) - (3.3.11) into Eq. (3.3.12), we obtain the following series solution

\[
\begin{align*}
  f(\eta) &= \eta + \frac{\alpha \eta^2}{2} + \frac{(2 \beta - 1) \alpha^2 \eta^5}{24} + \frac{(M - 1 + 2 \beta) \eta^4}{6} + \frac{(M + \beta) \eta^3}{6} \\
  &+ \frac{(20 \beta^2 - 32 \beta + 11) \alpha^3 \eta^8}{40320} + \frac{(11 + 10M \beta + 20 \beta^2 - 8M - 32 \beta) \alpha^2 \eta^7}{5040} \\
  &+ \frac{(3 - 12 \beta - 8M + 10 \beta^2 + 10M \beta + M^2) \alpha \eta^6}{720} \\
  &+ \frac{(M^2 + 2 \beta^2 - 2 \beta - 2M + 3M \beta) \eta^5}{120} + \frac{(600 \beta^3 - 1596 \beta^2 + 1398 \beta - 375) \alpha^4 \eta^{11}}{39916800} \\
  &+ \frac{(-375 - 546 \beta M + 300 \beta M^2 + 600 \beta^3 - 1596 \beta^2 + 1398 \beta + 243M) \alpha^3 \eta^{10}}{3628800} \\
  &+ \frac{(-129 + 42 \beta M^2 + 300 \beta^3 - 39 M^2 + 548 \beta - 710 \beta^2 + 243M \eta^9}{362880} \\
  &+ \frac{-546 \beta M + 300 \beta M^2 \alpha^2 \eta^8}{362880} \\
  &+ \frac{-15 + 120M \beta^2 + M^3 + 80 \beta^3 - 166 \beta^2 + 103 \beta + 80M - 39M^2 \alpha \eta^7}{40320} \\
  &+ \frac{+42 \beta M^2 - 196 \beta M \alpha \eta^6}{5040} \\
  &+ \frac{(M^3 - 10M^2 + 11 \beta M^2 + 20 \beta^2 M - 26 \beta M + 8M + 8 \beta - 16 \beta^2 + 10 \beta^3) \eta^5 + \cdots}{5040}. \quad (3.3.13)
\end{align*}
\]
To select $f''(0) = \alpha$ and $\eta_\infty$, we begin with some initial guess of $\alpha$ and draw $f'$. The solution process is repeated with another values of $\alpha$ until $f'$ receive to zero. Also, $\eta_\infty$ is obtained from highest value of $\eta$ in $f'$.

Tables 3.1 and 3.2 clearly indicate that present solution of HPSTM shows excellent agreement with the HPM [176] and the exact solution and more convergent than the HPM [176]. This analysis shows that the new HPSTM is valid for MHD viscous flow problems.

<table>
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<tr>
<th>$\beta$</th>
<th>M</th>
<th>HPM [176]</th>
<th>HPSTM</th>
<th>Exact solution</th>
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</table>

*Table 3.1* Comparison of the values of $f''(0)$ obtained by HPM [176], HPSTM and the exact solution.
<table>
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<th>(\beta)</th>
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<th>HPM [176]</th>
<th>HPSTM</th>
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*Table 3.2* Variation in \(f''(0)\) at the different values of \(\beta\) and M obtained by HPM [176] and HPSTM.

### 3.4. Results and Discussion

Eq. (3.2.7) is solved analytically using the HPSTM. The analytical and the exact solutions of \(f\) for different values of M and \(\beta\) are presented in Figs. 3.1-3.2. Furthermore, the comparison between the HPSTM and the exact solutions of \(f'\) is performed in Fig. 3.3. The effects of the magnetic parameter M are shown in Fig. 3.3. It is observed from Fig. 3.3 that when M increases, the velocity profile is more and more far away from the wall, and the boundary layer thickness is more and more thicker. Therefore, it is concluded from Fig. 3.3 that \(f'\) decreases when M increases. A very good agreement is achieved between the results obtained by the HPSTM, the HPM [176] and the exact solutions for all values of \(\eta\). The results of HPSTM are better than HPM [176] in large amount of M.
(a)

(b)
Fig. 3.1 The results of $f$ obtained by HPSTM, HPM [176] and exact solutions for $\beta = 1$, when (a) $M = 0$, (b) $M = 1$, (c) $M = 50$, (d) $M = 500$. 
Fig. 3.2 The results of $f$ obtained by HPSTM and HPM [176] for $\beta = 1.5$, when (a) $M = 0$, (b) $M = 1$, (c) $M = 50$, (d) $M = 500$. 
3.5. Conclusions

In this chapter, a simple algorithm based on HPSTM has been applied for solving the magnetohydrodynamics (MHD) viscous flow due to stretching sheet. An excellent agreement is achieved by comparing the present solution with the HPM [176] and exact solution. The method is applied here in direct manner without the use of linearization, transformation, discretization, perturbation, or restrictive assumptions. The approach gives more realistic series solutions that converge very rapidly in physical problems.