CHAPTER V

A Study of Bingham Fluid Model of Blood Flow in Capillaries

Blood flow in capillaries of internal diameter equal to that of red cell is quite different from that in the large vessels. Since the size of RBC is not negligible as compared to vessels diameter, it is necessary to consider the flow of blood in capillaries as two phase non homogeneous flow.

The apparent viscosity of blood depends on several factors such as plasma viscosity, hematocrit, size of the vessel, shear rate, rate of flow, rigidity and deformability etc. Also the apparent viscosity in
capillaries is much lower than in large vessels (Dentifass). Also it is well known that apparent viscosity of blood decreases as the tube radius decreases. (Fahraeus Lindquist effect). Gupta and Seshadri, Lida and Murata, Charm et al have considered the two fluid models, in which both layers (Peripheral Plasma layer and core hematocrit layer) are of Newtonian fluids with different viscosities and yield stresses.

In the present paper we have considered two layer fluid model of blood with no slip velocity at the wall, both satisfying Bingham constitutive equation. In this paper velocity field, volume flow rate and apparent fluidity have been found. Flow parameters have been explained.

5.1 The Mathematical Analysis -

\[
\begin{align*}
\text{Peripheral region} & \quad \uparrow \quad R_p \quad \rightarrow v_p \\
\text{Core region} & \quad \downarrow \quad R_c \quad \rightarrow v_c \\
\text{Plug Flow} & \quad \rightarrow R_y \quad \rightarrow v_y \\
\text{Core region} & \quad \downarrow \quad \text{Peripheral Region}
\end{align*}
\]
The flow of blood is axially symmetric and is in the Z direction.

The equation of motion and continuity for steady, incompressible, laminar viscous flow under zero body forces are -

\[
\frac{\partial v}{\partial z} = 0
\]  

(1)

\[-\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r\tau \right) = 0\]  

(2)

\[
\frac{\partial p}{\partial r} = 0
\]  

(3)

Where \( \tau \) is shear stress normal to \( r \) in z direction, \( P \) is the pressure and \( V \) is velocity.

Let a two-fluid model of blood with a central core region of radius \( R_c \) and a plasma layer of thickness \( d = (R_p - R_c) \), satisfying Bingham plastic constitutive equation.

\[
\tau = \tau_y + \mu \dot{\gamma}, \quad \tau > \tau_y
\]  

(4)

\[
\dot{\gamma} = 0, \quad \tau \leq \tau_y
\]

Where \( \mu \) is Newtonian viscosity, \( \tau \) is shear stress, \( \tau_y \) is yield stress, \( \dot{\gamma} \) is shear strain.

Boundary conditions are:

\( \tau_p = \tau_c, \quad V_p = V_c \) at \( r = R_c \)
\[ V_P = 0 \text{ at } r = R_P \] (5)

\[ \gamma = 0, \tau_c = \tau_y, V_C = V_Y \text{ at } r = R_Y \]

\[ \tau \text{ is finite at } r = 0 \]

Where \( R_Y \) = Plug flow radius, \( V_Y \) = Plug flow velocity. Suffixes \( P \) and \( C \) denote the value of the corresponding parameter in peripheral layer and core layer, respectively.

From (2) & (5) we obtain

\[ \tau = \frac{r \frac{\partial p}{\partial z} - \tau_y}{2} \] (6)

Now using (6) in equ. (4)

we have

\[ \frac{1}{\mu \frac{\partial}{\partial z}} \left( \frac{r \frac{\partial p}{\partial z} - \tau_y}{2} \right) = \frac{\partial V}{\partial r} \] (7)

\[ \frac{\partial V_P}{\partial r} = -\left( \frac{1}{\mu p} \right) \left( \frac{\partial p}{\partial z} \frac{r}{2} - \tau_y \right), R_c \leq r \leq R_P \] (8)

\[ \frac{\partial V_C}{\partial r} = -\frac{1}{\mu_c} \left( \frac{\partial p}{\partial z} - \tau_c \right), R_Y \leq r \leq R_c \] (9)

\[ \frac{\partial V_Y}{\partial r} = 0, 0 \leq r \leq R_Y \] (10)

Integrating equations (8), (9) and (10)

we get

\[ V_P = \int -\frac{1}{\mu_P} \left( \frac{\partial p}{\partial z} - \tau_y \right) \, dr \]
or
\[ V_p = \frac{1}{\mu_p} \frac{\tau_w R_p}{2} \left[ 1 - \frac{r^2}{R_p^2} + 2\alpha_p \left( \frac{r}{R_p} - 1 \right) \right] \quad (11) \]

Similarly
\[ V_c = \frac{1}{\mu_p} \frac{\tau_w R_p}{2} \left[ 1 - h^2 + 2\alpha_p (h - 1) \right] \]
\[ + \frac{1}{\mu_c} \frac{\tau_w R_p}{2} \left[ h^2 - \frac{r^2}{R_p^2} + 2\alpha_c \left( \frac{r}{R_p} - h \right) \right] \quad (12) \]
\[ V_y = \frac{\tau_w R_p}{2\mu_p} \left[ 1 - h^2 + 2\alpha_p (h - 1) \right] + \frac{\tau_w R_p}{2\mu_c} \left[ h^2 + (\alpha_c - 2h) \right] \quad (13) \]

In non-dimensional form, we have
\[ V_p = 1 - \frac{r^2}{p} + 2\alpha_p \left( \frac{r}{R_p} - 1 \right) \quad (14) \]
\[ V_c = \left[ 1 - h^2 + 2\alpha_p (h - 1) + \frac{\mu_p}{\mu_c} \left( h^2 - \frac{r^2}{R_p^2} + 2\alpha_c \left( \frac{r}{R_p} - h \right) \right) \right] \quad (15) \]
\[ V_y = \left[ 1 - h^2 + 2\alpha_p (h - 1) + \frac{\mu_p}{\mu_c} \left( h^2 + \alpha_c (\alpha_c - 2h) \right) \right] \quad (16) \]

Where
\[ h = \left( \frac{R_c}{R_p} \right) \alpha_p = \left( \frac{\tau_{iy}}{\tau_w} \right), \quad \alpha_c = \left( \frac{\tau_{ic}}{\tau_w} \right), \quad \tau_w = \frac{\partial p}{\partial z}, \quad V_i = \left( \frac{V_i}{V_o} \right) \]
\[ V_o = \frac{\tau_w R_p}{2\mu_p} \quad i=p, \ c, \ y \]

i.e. \[ V_p = \frac{V_p}{V_o}, \ V_c = \frac{V_c}{V_o}, \ V_y = \frac{V_y}{V_o} \]
Volume flow rate is given by

\[ Q = Q_1 + Q_2 + Q_3 \]  \hspace{1cm} (17)

\[ Q_1 = 2\pi \int_{R_c}^{R_p} V_p \, rdr, \]

\[ Q_2 = 2\pi \int_{R_y}^{R_c} V_c \, rdr \quad \text{and} \]

\[ Q_y = 2\pi \int_{0}^{R_y} V_y \, rdr \]

Now, from equations (11), (12), (13) and (17) we have

\[ Q = \frac{\pi r_w R_p^3}{2\mu_p} \left[ \frac{1}{2} \frac{h^4}{2} + 2\alpha_p \alpha_c^2 - 2\alpha_p h \alpha_c^2 \right] \]

\[ -\alpha c^2 + \alpha c \alpha h^2 + \frac{\mu_e}{\mu_c} \left\{ \frac{h^4}{2} - \frac{2}{3} \alpha_c h^3 + \frac{\alpha_c^4}{6} \right\} \]  \hspace{1cm} (18)

Apparent fluidity is given by

\[ F_a = \frac{2Q}{\pi r_w R_p^3} \]  \hspace{1cm} (19)

Using equation (18) in (19) we get

\[ F_a = \frac{1}{\mu_p} \left[ \frac{1}{2} \frac{h^4}{2} + 2\alpha_p \alpha c^2 - 2\alpha_p h \alpha_c^2 - \alpha_c^2 + \alpha_c \alpha h^2 \right] \]

\[ + \frac{\mu_e}{\mu_c} \left\{ \frac{h^4}{2} - \frac{2}{3} \alpha_c h^3 + \frac{\alpha_c^4}{6} \right\} \]  \hspace{1cm} (20)
Putting $h = \left(1 - \frac{d}{R_p}\right)$ we have

$$F_a = 2 \frac{d}{R_p} \left(\frac{2 - \alpha_c^2 + \alpha_p \alpha_c^2}{\mu_p} - \frac{1 - \alpha_c}{\mu_c}\right)$$

$$+ \left(\frac{1 + \alpha_c^4}{6} - \frac{2}{3} \alpha_c\right) \frac{1}{\mu_c}$$

(21)
**Result and Discussion -**

There are many different models of blood flow through narrow vessels with different boundary conditions and flow parameters but the scope of their quantitative study is very limited because of experimental problems.

Here we have considered $\alpha_p = 0$ and $\alpha_p \neq 0$ i.e. peripheral layer is newtonian and non - Newtonian fluid respectively.

Variation of apparent fluidity with peripheral layer thickness and yield stress ($\alpha_c$) are given below in tables 1 and 2. Velocity distributions are also discussed in Table 3 and 4.

From fig.1 and 2, it is clear that apparent fluidity $f_a$ increases with $d$ (plasma layer thickness) and decreases with $\alpha_c$ (yield stress). This indicates that when hematocrit decreases, apparent fluidity increases. The results are composed with results of Gupta & Sehadri and found a good similarity in them. Also when $\alpha_p = 0$, $f_a$ is found. The values for $\alpha_p = 0$ are greater than those for $\alpha_p = 0.01$ and thus we found that apparent fluidity decreases with yield stress of plasma.

Fig 3 shows that velocity field decreases fastly with $r$ in the peripheral layer. In the core region,
Table 1

$$(\mu_p = 12, \mu_c = 22, \alpha_c = 0.2, \alpha_p = 0, 0.01)$$

<table>
<thead>
<tr>
<th>$\frac{d}{R_p}$</th>
<th>0.08</th>
<th>0.10</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
<th>0.18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fa ($\alpha_p = 0$)</td>
<td>0.37</td>
<td>0.421</td>
<td>0.471</td>
<td>0.522</td>
<td>0.558</td>
<td>0.624</td>
</tr>
<tr>
<td>Fa ($\alpha_p = 0.1$)</td>
<td>0.368</td>
<td>0.420</td>
<td>0.470</td>
<td>0.520</td>
<td>0.556</td>
<td>0.623</td>
</tr>
</tbody>
</table>

Variation of Fa with $\frac{d}{R_p}$ (Plasma layer thickness)

Table 2

$$\mu_p = 12; \mu_c = 22; H = 40\%; \frac{d}{R_p} = 0.15; \alpha_p = 0, 0.01$$

<table>
<thead>
<tr>
<th>$\alpha_s$</th>
<th>0.16</th>
<th>0.18</th>
<th>0.20</th>
<th>0.22</th>
<th>0.24</th>
<th>0.26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fa ($\alpha_p = 0$)</td>
<td>0.559</td>
<td>0.553</td>
<td>0.548</td>
<td>0.542</td>
<td>0.537</td>
<td>0.531</td>
</tr>
<tr>
<td>Fa ($\alpha_p = 0.1$)</td>
<td>0.558</td>
<td>0.552</td>
<td>0.547</td>
<td>0.541</td>
<td>0.536</td>
<td>0.530</td>
</tr>
</tbody>
</table>

Variation of apparent fluidity with yield stress
Table - 3

(h=40%, $\alpha/R_p=0.1$, $\alpha_c=0.2$, $\alpha_p=0.01$)

<table>
<thead>
<tr>
<th>$\frac{r}{R_p}$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>VP ($\alpha_p=0$)</td>
<td>1</td>
<td>0.96</td>
<td>0.84</td>
<td>0.64</td>
<td>0.36</td>
<td>0</td>
</tr>
<tr>
<td>VP ($\alpha_p=0.01$)</td>
<td>0.98</td>
<td>0.94</td>
<td>0.83</td>
<td>0.63</td>
<td>0.35</td>
<td>0</td>
</tr>
</tbody>
</table>

(Variation of $V_p$ with $r$)
Table 4

($\mu = 2.2; \ \mu_p = 1.2; \ \alpha_c = 0.2; \ h = 0.9, 0.85, \ \alpha_p = 0.01$)

<table>
<thead>
<tr>
<th>$\frac{r}{R_p}$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_c$</td>
<td>0.435</td>
<td>0.457</td>
<td>0.435</td>
<td>0.370</td>
<td>0.261</td>
<td>0.103</td>
</tr>
<tr>
<td>$\alpha_p = 0.0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| $h = 0.9$       |     |     |     |     |     |     |
| $V_c$           | 0.433 | 0.455 | 0.433 | 0.368 | 0.259 | 0.101 |
| $\alpha_p = 0.1$ |     |     |     |     |     |     |

| $h = 0.85$      |     |     |     |     |     |     |
| $V_c$           | 0.486 | 0.507 | 0.486 | 0.421 | 0.312 | 0.159 |
| $\alpha_p = 0$  |     |     |     |     |     |     |

| $h = 0.85$      |     |     |     |     |     |     |
| $V_c$           | 0.483 | 0.504 | 0.483 | 0.418 | 0.309 | 0.156 |
| $\alpha_p = 0.01$ |     |     |     |     |     |     |

(Velocity Distribution in core region)
velocity decreases slowly. On comparison of results with experimental results of Bugliarello and Sevilla we found a good similarity. When $\alpha_p = 0$, velocity is greater than those obtained for $\alpha_p = 0.01$. 
References


2] Chaturani, P. and Upadhyay, V.S: (1979), Bioorheology 16, 419


8] Charm, S.E. and Kurland, G.S.: (1967), Biorheology 4, 175

