CHAPTER 8
A NEW GENERALISED STABILITY TEST FOR
MULTI-DIMENSIONAL DIGITAL SYSTEMS

8.1 INTRODUCTION

8.2 NEW STABILITY TEST FOR MULTI-DIMENSIONAL DIGITAL SYSTEMS

8.3 ILLUSTRATIONS

8.4 SUMMARY

The part of this chapter has been published as under:

8.1 INTRODUCTION

In this chapter, a new algebraic scheme which is unique and generalised in application is presented for stability testing of multi-dimensional digital systems and is an extension of the work reported for 2-D digital systems in chapter 5.

8.2 NEW STABILITY TEST FOR MULTI-DIMENSIONAL DIGITAL SYSTEMS

The transfer function of an N-Dimensional digital system is represented as

\[ H(z_1, z_2, z_3, \ldots, z_N) = \frac{A(z_1, z_2, z_3, \ldots, z_N)}{B(z_1, z_2, z_3, \ldots, z_N)} \]  \hspace{1cm} (8.1)

where \( A(\ldots) \) and \( B(\ldots) \) are co-prime polynomials without non-essential singularities of the second kind.

From Jury [1] the sufficient and necessary conditions for an N-D characteristics polynomial \( B(z_1, z_2, z_3, \ldots, z_N) \) to be stable are given as under:

Necessary conditions (N-1)

(i) \( B(z_1, z_2, z_3, \ldots, z_{N-1}, 0) \neq 0, \left\{ \bigcap_{i=1}^{N-2} |z_i| = 1 \right\} \cap \left\{ |z_{N-1}| \leq 1 \right\} \) \hspace{1cm} (8.2)

(ii) \( B(z_1, z_2, z_3, \ldots, z_{N-2}, 0, 0) \neq 0, \left\{ \bigcap_{i=1}^{N-3} |z_i| = 1 \right\} \cap \left\{ |z_{N-2}| \leq 1 \right\} \) \hspace{1cm} (8.3)

\ldots

(N-2) \( B(z_1, z_2, 0, 0, \ldots, 0) \neq 0, \left\{ |z_1| = 1 \right\} \cap \left\{ |z_2| \leq 1 \right\} \) \hspace{1cm} (8.4)

(N-1) \( B(z_1, 0, \ldots, 0) \neq 0, |z_1| \leq 1 \) \hspace{1cm} (8.5)
Sufficiency condition:

\[ B(z_1, z_2, z_3, \ldots, z_n) \neq 0, \left\{ \bigcap_{i=1}^{N-1} |z_i| = 1 \right\} \cap \left\{ |z_N| \leq 1 \right\} \] (8.6)

All the above equations can be written in terms of the respective reciprocal polynomials in a similar way in which the above conditions are translated such that all the roots of the reciprocal polynomials are constrained to be within the polydisk for stability as follows:

The characteristic polynomial for sufficiency test is written as

\[(i) \quad B_1(z_1, z_2, z_3, \ldots, z_n) = 0, \left\{ \bigcap_{i=1}^{N-1} |z_i| = 1 \right\} \left\{ |z_N| \geq 1 \right\} \] (8.7)

Where \( B_1 (\ldots) \) is the reciprocal polynomial of \( B(\ldots) \)

In a similar way, the remaining polynomials defined for necessary condition can be translated to respective reciprocal polynomials and each polynomial can be tested by employing Pontryagin's inequality as adopted for 2-D stability test.

For instance in the reciprocal polynomial as shown in the equation (8.7) substitute \( z_1 = |\theta_1|, \quad z_2 = |\theta_2|, \quad z_3 = |\theta_3|, \ldots \) and \( z_{N-1} = |\theta_{N-1}| \), and let these angles be constant in the range \( 0 \leq \theta_k \leq 2\pi \), for \( k = 0, 1, 2, \ldots, N-1 \) and vary \( z_N = |\theta_N| \) with \( 0 \leq \theta_N \leq 2\pi \).

The characteristic polynomial is written in terms of real and imaginary parts as

\[ B_1(|\theta_1|, |\theta_2|, \ldots, |\theta_N|) = R(\theta_1, \theta_2, \ldots, \theta_N) + jI(\theta_1, \theta_2, \ldots, \theta_N) \]

For constant values of \( \theta_1, \theta_2, \ldots, \theta_{N-1} \) and for variable values of \( \theta_N \), the Pontryagin inequality can be adopted for stability testing. Similarly all other reciprocal polynomials can be tested for N-D stability.
The above testing scheme is employed for the same examples presented in chapter 5.

8.3 ILLUSTRATIONS [1]

Example 1: [1]

(1) Let \( B(z_1, z_2, z_3) = 4 + z_3 - z_2 + z_1 + z_1^2 z_2 \)

Necessary condition test:

The following polynomials are to be tested for this situation

(i) \( B(z_1, 0, 0) = 4 + z_1 \neq 0, \ |z_1| \leq 1 \)

(ii) \( B(z_1, z_2, 0) = 4 - z_2 + z_1 + z_1^2 z_2 \neq 0, \ |z_1| = 1, \ |z_2| \leq 1 \)

The inequalities for testing the necessary conditions are given as follows:

(i) \( T(\theta_1) = 16 + 4 \cos \theta_1 \)

(ii) \( T(\theta_1, \theta_2) = R - I \)

Where

\[ R = R(\theta_1, \theta_2) = 4 \cos \theta_2 + \cos(\theta_1 + \theta_2) + \cos 2\theta_1 - 1 \]

and

\[ I = I(\theta_1, \theta_2) = 4 \sin \theta_2 + \sin(\theta_1 + \theta_2) + \sin 2\theta_1 \]

\[ \dot{R} = R(\theta_1, \theta_2) = -4 \sin \theta_2 - \sin(\theta_1 + \theta_2) \]

\[ \dot{I} = I(\theta_1, \theta_2) = 4 \cos \theta_2 + \cos(\theta_1 + \theta_2) \]

Using computer approach, for constant values of \( \theta_1 \) in the range \( 0 \leq \theta_1 \leq 2\pi \), and variable values of \( \theta_2 \) in the range \( 0 \leq \theta_2 \leq 2\pi \), the
above two inequalities are found to be positive.

The sufficiency condition is tested using the following definition:

(iii) \[ B(z_1, z_2, z_3) \neq 0, \quad |z_1| = |z_2| = 1, \quad |z_3| \geq 1, \quad \text{then} \]

\[ B_i(z_1, z_2, z_3) = (4 - z_2 + z_1 + z_1^2 z_2) z_3 + 1 \neq 0, \quad |z_1| = 1, \quad |z_3| \leq 1 \]

Following the procedure from chapter 5 substitute \( z_1 = \theta_1, \quad z_2 = \theta_2, \quad z_3 = \theta_3 \) for \( \theta_1, \theta_2, \theta_3 \), each varying in the limit 0 to \( 2\pi \) and the values of \( \theta_1 \) and \( \theta_2 \) are kept constant while \( \theta_3 \) alone is varied.

Thus,

\[
B_i(\theta_1, \theta_2, \theta_3) = (4 - |\theta_2|^2 + |\theta_1|^2 + 2|\theta_2|\theta_3^2) |\theta_3|^2 + 1
\]

\[
= R(\theta_1, \theta_2, \theta_3) + j I(\theta_1, \theta_2, \theta_3)
\]

Where the real part and imaginary part respectively are

\[
R(\theta_1, \theta_2, \theta_3) = 1 + 4 \cos \theta_3 - \cos(\theta_2 + \theta_3) + \cos(\theta_1 + \theta_3) + \cos(2\theta_1 + \theta_2 + \theta_3)
\]

and

\[
I(\theta_1, \theta_2, \theta_3) = 4 \sin \theta_3 - \sin(\theta_2 + \theta_3) + \sin(\theta_1 + \theta_3) + \sin(2\theta_1 + \theta_2 + \theta_3)
\]

without loss of generality and for simplicity the following typical cases are tabulated for sufficiency test.
Values of $\theta_1$ and $\theta_2$ | Expression $T(\theta_3)$ | Result for $0 \leq \theta_3 \leq 2\pi$
---|---|---
$\theta_1 = \theta_2 = 0$ | $25 + 5 \cos \theta_3$ | $T(\theta_3) > 0$
$\theta_1 = \theta_2 = \pi/2$ | $[(4 \cos \theta_3 + \sin \theta_3)^2 + (4 \cos \theta_3 + \sin \theta_3) + (4 \sin \theta_3 - \cos \theta_3)^2]$ | $T(\theta_3) > 0$
$\theta_1 = \theta_2 = \pi$ | $9 + 3 \cos \theta_3$ | $T(\theta_3) > 0$

Similarly for other constant values of $\theta_1$ and $\theta_2$, $\theta_3$ is varied and tested using Pontryagin's inequality. It is found that for all these values, the inequality is satisfied.

Since the necessary and sufficiency conditions are satisfied, the 3-D system with the chosen $B(z_1, z_2, z_3)$ is stable.

**Example : 2**

Let $B(z_1, z_2, z_3) = 2 + z_3 - z_2 + z_1 + 6 z_1^2 z_2$

Then

(i) $B(z_1) = 2 + z_1 \neq 0, \ |z_1| \leq 1$

(ii) $B(z_1, z_2) = 2 - z_2 + z_1 + 6 z_1^2 z_2 \neq 0, \ |z_1| = 1$ and $|z_2| \leq 1$

(iii) $B(z_1, z_2, z_3) = 2 + z_3 - z_2 + z_1 + 6z_1^2 z_2 \neq 0, \ |z_1| = |z_2| = 1$ and $|z_3| \leq 1$
Necessary conditions tests are:

(i) \( T(\theta_1) = 4 + 2 \cos \theta_1 \)

(ii) \( T(\theta_1, \theta_2) = R(\ldots) \dot{I}(\ldots) - I(\ldots) \dot{R}(\ldots) \)

Where

\[
R(\ldots) = 2 \cos \theta_2 + \cos(\theta_1 + \theta_2) + 6 \cos 2\theta_1 - 1 \\
I(\ldots) = 2 \sin \theta_2 + \sin(\theta_1 + \theta_2) + 6 \sin 2\theta_1 \\
\dot{R}(\ldots) = -2 \sin \theta_2 - \sin(\theta_1 + \theta_2) \\
\dot{I}(\ldots) = 2 \cos \theta_2 + \cos(\theta_1 + \theta_2)
\]

For \( 0 \leq \theta_1 \leq 2\pi \), \( T(\theta_1) > 0 \)

With \( \theta_1 = 0^\circ \), \( T(\theta_1, \theta_2) = 9 + 15 \cos \theta_2 \)

When \( \theta_2 = \pi \), \( T(\theta_1, \theta_2) < 0 \) indicating unstable situation of the system. Further computation can be terminated.

8.4 SUMMARY

A new method of testing the stability of generalised multi-dimensional digital system extending the Pontryagin's stability criterion was presented in this chapter. For this purpose, unique representation of real and imaginary parts of characteristic polynomial of the system was employed. The newly proposed method was illustrated through examples.