CHAPTER - 2
SINGLE-TERM WALSH SERIES SOLUTION OF
FIRST-ORDER STATE-SPACE SYSTEMS

2.1 INTRODUCTION

Two orthogonal sets of functions, namely, the Walsh Functions (WF) and the Block Pulse Functions (BPF), have recently attracted the attention of researchers in various fields of Engineering and Science. Single-Term Walsh Series (STWS) was developed in 1980 to remove certain inconveniences in WF and BPF. In this chapter the Walsh Series and Block Pulse Functions methods of analysis are discussed briefly. The STWS approach in the first-order state-space formulation is explained with examples.

2.2 WALSH FUNCTIONS


2.2.1 Walsh Series

A function $f(t)$ integrable in $[0, 1)$ can be expanded using Walsh Series as

$$f(t) = \sum_{i=0}^{\infty} f_i \psi_i(t)$$  \hspace{1cm} (2.1)

where $\psi_i$ is the $i$th WF and $f_i$ is the corresponding co-efficient. Figure 2.1 shows a set of Walsh Functions. In practice, as an approximation a finite series is considered, say, with $m$ terms. If we write the coefficients of WF concisely as $m$ vectors, with the following notations,

$$F = [f_0, f_1, \ldots, f_{m-1}]^T$$  \hspace{1cm} (2.2)

and $\Psi = [\psi_0(t), \psi_1(t), \ldots, \psi_{m-1}(t)]^T$  \hspace{1cm} (2.3)
where \( m = 2^k \), \( k \) is an integer and \( T \) denotes transpose, then equation (2.1) becomes
\[
f(t) \cong F^T \Psi(t)
\] (2.4)

The coefficients \( f_j \) are so chosen as to minimize the mean square error given by
\[
\varepsilon = \frac{1}{2} \int [f(t) - F^T \Psi(t)]^2 \, dt
\] (2.5)

The coefficients are given by
\[
f_j = \frac{1}{2} \int f(t) \psi(t) \, d(t)
\] (2.6)

Rao et al. [21] and Chen et al. [8] have shown that
\[
\frac{1}{2} \int f(t) \, dt = F^T E \Psi(t)
\] (2.7a)

where \( E \) is called the operational matrix for integration in WF and it is given by
\[
E(m \times m) =
\begin{bmatrix}
E(m/2 \times m/2) & - \frac{1}{2m} I(m/2 \times m/2) \\
\frac{1}{2m} I(m/2 \times m/2) & O(m/2 \times m/2)
\end{bmatrix}
\] (2.7b)
\[ E_{2 \times 2} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix} \quad \text{and} \quad E_{1 \times 1} = \frac{1}{2} \quad (2.7c) \]

The integrals of the first four WF are shown in Figure (2.2).

### 2.3 BLOCK PULSE FUNCTIONS

BPF were effectively used by several authors for the analysis of dynamic systems. Sannuti [17] presented a method for numerically integrating a system of differential equations based on BPF. Shih [18] used BPF for the solution of state-space equations and Rao et al. [20] for synthesis of systems with time delays. Palanisamy et al. applied BPF for system identification [23], for analysis of stiff systems [25] and for solution of variational problems [35].

#### 2.3.1 Operational Matrix for Block Pulse Functions

A set of BPF required to get piecewise constant solutions of a differential equation is defined on an unit interval \((0, 1)\) as follows:
For each integer \( i, 1 \leq i \leq m \), the function \( \phi_i(t) \) is given by
\[
\phi_i(t) = \begin{cases} 
1 & \text{for } \frac{i - 1}{m} < t \leq \frac{i}{m} \\
0 & \text{otherwise}
\end{cases}
\] (2.8)

A function which is integrable in \([0, 1)\) can be approximately represented as shown below by an \( m \) vector \( \Phi(t) \) with \( \phi_i(t) \) as its \( i \)th component:
\[
f(t) = \sum_{i=1}^{m} f_i \phi_i(t)
\] (2.9)

where \( f_i \) are the coefficients to be determined such that the integral square error
\[
E = \sum_{i=1}^{m} \left[ f(t) - \sum_{i=1}^{m} f_i \phi_i(t) \right]^2 dt
\] (2.10)

is minimized, \( f_i \) is the average value of \( f(t) \) in the interval
\[
\frac{i - 1}{m} < t \leq \frac{i}{m}
\]
and is given by
\[
f_i = m \int_{\frac{i - 1}{m}}^{\frac{i}{m}} f(t) dt
\] (2.11)

Expanding the integral \( \int \phi_i(t) dt \) into the set of basic functions \( \Phi(t) \), the co-efficients can be determined and written in matrix
form as follows:

\[
\int_0^t \Phi(t) \, dt = P \Phi(t)
\]

(2.12)

where \( P \) is the operational matrix for integration of BPF and is given by

\[
P = \begin{bmatrix}
\frac{1}{2} & 1 & \cdots & \cdots & \cdots \\
0 & \frac{1}{2} & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \frac{1}{2}
\end{bmatrix}
\]

(2.13)

2.4 SOLUTION OF STATE EQUATIONS BY KRONECKER PRODUCT FORMULA IN WALSH SERIES

Chen and Hsiao [8] developed a simple procedure for solving state equations by Kronecker product formula eliminating the laborious use of look-up table by Corrington [7], as explained below.
Consider the following state equation

\[
\dot{X}(t) = A \, X(t) + B \, u(t) \tag{2.14}
\]

\[
X(0) = X_0
\]

where \(X\) is a state vector of \(n\) components, \(u\) is an input vector of \(1\) component, \(A\) and \(B\) are \(n \times n\) and \(n \times 1\) matrices, respectively.

Making \(m\)-term approximation, \(X\) can be written in terms of Walsh Series as below:

\[
\begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
c_{10} & c_{11} & \cdots & c_{1(m-1)} \\
c_{20} & c_{21} & \cdots & c_{2(m-1)} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n0} & c_{n1} & \cdots & c_{n(m-1)}
\end{bmatrix}
\begin{bmatrix}
\psi_0 \\
\psi_1 \\
\vdots \\
\psi_{m-1}
\end{bmatrix}
\]

Writing each column as a vector, equation (2.15) becomes:

\[
\dot{X} = [c_{0, \, c_1, \, \ldots, \, c_{m-1}}] \psi \overset{\Delta}{=} C \, \psi \tag{2.15a}
\]
The state variable \( X \) may be obtained by integration as
\[
X(t) = C \int_{0}^{t} Y(s) \, ds + X_0
\]  
(2.16)

The integration can be performed approximately by using the \( E \) matrix as below.
\[
\int_{0}^{t} Y(s) \, ds = E \psi(t)
\]  
(2.17)

The input vector can also be expressed by Walsh Series as
\[
u = H \psi
\]  
(2.18)

where \( H \) is a \( 1 \times m \) matrix. In terms of equations (2.15) - (2.18), equation (2.14) gets changed to
\[
\begin{align*}
CY &= A (CE\psi + X_0) + BH \psi \\
CY &= A CE\psi + AX_0 + BH \psi
\end{align*}
\]  
(2.19) (2.19a)

\( AX_0 \) can be written as the product of matrix \( G \) of \( m \) columns and the vector \( \psi \).
\[
AX_0 = [AX_0, 0, 0, \ldots] \psi = G \psi
\]  
(2.20)

Finally we have
\[
C = ACE + k
\]  
(2.21)

where \( G + BH \triangleq k \)
If we rearrange the \( n \times m \) matrix \( C \) as an \( nm \)-vector \( c \) by changing its first column into the first \( n \) components of the vector, the second column into second \( n \) components of the vector, etc., and rearrange \( k \) in the same manner, we finally arrive at a form in terms of Kronecker product for equation (2.21).

\[
c = [A \otimes E^T] c + k
\]  

(2.22)

where \( A \otimes E^T \) is a Kronecker product, defined as

\[
A \otimes E^T = \begin{bmatrix}
e_{11} A & e_{21} A & \cdots & e_{m1} A \\
e_{12} A & e_{22} A & \cdots & e_{m2} A \\
\cdots & \cdots & \cdots & \cdots \\
e_{1m} A & e_{2m} A & \cdots & e_{mm} A
\end{bmatrix}
\]

(2.23)

and \( E^T \) is the transpose of \( E \)

The solution of \( c \) comes from equation (2.22) directly,

\[
C = [I - A \otimes E^T]^{-1} k
\]  

(2.24)

Once \( c \) has been decided, the Walsh Series representation for the state variable is determined. The state variable vector is then found by substitution.

\[
X = CE^\psi + X_0
\]  

(2.25)
2.5 SINGLE-TERM WALSH SERIES APPROACH

Rao et al. [21] introduced a method of computation beyond the limit of initial normal interval in Walsh Series analysis which was later referred to as STWS approach. BPF method used simpler operational matrix than the original Walsh Series for the analysis of dynamic systems. But STWS approach has eliminated the necessity of using operational matrix. Palanisamy applied STWS in optimal control [22] and nonlinear systems [24] and Palanisamy et al. in time varying and nonlinear networks [33], smoothing circuits [34] and singular systems [38]. Subbayyan et al. [30] applied STWS in electronic circuit design.

In this chapter STWS approach for the analysis of dynamic systems represented in first-order state-space formulation is illustrated through two examples. The results are listed out for comparison with the newly developed second-order state-space formulation (to be discussed in Chapter 3).

In the STWS approach, a function $x(t)$ may be approximated using a single WF $\psi_0$ alone in the normalized interval $\tau \in [0,1)$ for $t \in [0, 1/m)$ when $t = \tau/m$ as

$$x(\tau) = B_1 \psi_0 (\tau)$$  \hspace{1cm} (2.26)
$B_1$ is the block pulse value of $x(t)$ in $t \in [0, 1/m)$. Let us consider the integral

$$
\int_{0}^{1} x(t) \, dt
$$

(2.27)

Expanding $x(t)$ in STWS in $t \in [0, 1)$, it can be written as

$$
x(t) = C_1 \psi_0(t)
$$

(2.28)

Integrating equation (2.28) with the operational matrix $E = \frac{1}{2}$

$$
B_1 \psi_0(t) = \frac{1}{2} C_1 \psi_0(t) + x(0)
$$

(2.29)

where $C_1$ is the block pulse value of the rate vector given by

$$
\frac{1}{m} \int_{0}^{1} x(t) \, dt, \quad t \in [0, \frac{1}{m})
$$

(2.30)

Normalizing equation (2.30) with $t = \frac{\tau}{m}$

$$
C_1 = \int_{0}^{1} x(t) \, dt = x(1) - x(0), \quad \tau \in [0, 1)
$$

(2.31)

$$
x(1) = C_1 + x(0)
$$

(2.32)

and for any interval $i$

$$
x(i) = C_i + x(i - 1), \quad i = 1, 2, 3, \ldots
$$

(2.33)
Equation (2.33) may generally be used to extend the solutions to other intervals by using block pulse values of the rate vector. \( x(i) \) are the discrete values.

### 2.5.1 Single-Term Walsh Series Solution of First-Order State-Space Systems

A dynamic system represented by the state-space equation

\[
\dot{X}(t) = A X(t) + B u(t) \quad (2.14)
\]

can be analysed using STWS. Normalizing the above equation with \( t = \frac{T}{m} \), the following equation is obtained.

\[
\dot{X}(\tau) = \frac{A}{m} X(\tau) + \frac{B}{m} u(\tau) \quad (2.34)
\]

Expressing equation (2.34) in STWS with

\[
\begin{align*}
\dot{X}(\tau) & = C(i) \psi_0(\tau) \\
X(\tau) & = B(i) \psi_0(\tau) \\
u(\tau) & = H(i) \psi_0(\tau)
\end{align*}
\quad (2.35)
\]

the recursive relationships for both block pulse and discrete values are
\[ C(i) = \left[ I - \frac{A}{2m} \right]^{-1} G(i) \]
\[ B(i) = \frac{1}{2} C(i) + X(i-1) \]
\[ X(i) = C(i) + X(i-1) \]

where \( G(i) = \frac{A}{m} X(i-1) + \frac{B}{m} H(i) \) and \( i = 1, 2, 3, \ldots \).

If for the representation of the system dynamics the second-order state-space formulation is adopted in place of the first-order formulation (2.14), as shown below,

\[ X(t) = A X(t) + B X(t) + c u(t) \]

then the sizes of the system matrices \( A, B \) and \( c \) get considerably reduced. For the STWS approach the possibility that the resulting algorithm assumes more effectiveness, leading to lesser computational efforts, deserves further investigations. This is what exactly is pursued and developed in detail in the chapter to follow (Chapter 3).

2.6 ILLUSTRATIVE EXAMPLES

Example 2.1

Consider the example 1 of Sivaramakrishnan and Srisailam [41].
\[ x + 3x + 2x = 0 \]  

(2.38)

with \( x(0) = 2 \) and \( x(0) = -3 \).

The first-order formulation is obtained as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

(2.39)

with

\[
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}
\]

The recursive relationships (2.36) are applied and the results are listed out in table 2.1 for comparison with second-order state-space formulation (to be developed in Chapter 3).

**Example 2.2**

Consider the example 2 of Sivaramakrishnan and Srisailam [41].

\[ \ddots \ddots \ddots x + 6x + 11x + 6x = 12 u(t) \]  

(2.40)

with \( x(0) = 5 \), \( x(0) = -6 \) and \( x(0) = 14 \), excited by an unit step input \( u(t) \).
The first-order formulation is obtained as:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 & -11 & -6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
-12
\end{bmatrix} u(t)
\]

with

\[
\begin{bmatrix}
x_1(0) \\
x_2(0) \\
x_3(0)
\end{bmatrix} =
\begin{bmatrix}
5 \\
-6 \\
14
\end{bmatrix}
\]

The STWS solution is listed out in Table 2.2 for comparison with second-order state-space formulation (to be developed in Chapter 3).

2.7 CONCLUSION

A general introduction to WF and BPF has been given. STWS method of analysing dynamic systems in first-order state-space formulation has been introduced. STWS method gives solution for both block pulse and discrete values; it does not require operational matrix for integration and eliminates the laborious use of Kronecker product of matrices.
Availing of the above advantageous features of STWS, a new method is proposed in Chapter 3. The results of illustrative examples are listed out for future comparison [with the results in Chapter 3 (Table 3.1 and Table 3.2)].
TABLE 2.1
Solution of Example 2.1 with m = 4

<table>
<thead>
<tr>
<th>Time sec.</th>
<th>X1 Block Pulse Value</th>
<th>X2 Discrete Value</th>
<th>X1 Block Pulse Value</th>
<th>X2 Discrete Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td></td>
<td>- 3.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>1.688888</td>
<td>- 2.488889</td>
<td>1.377778</td>
<td>- 1.977778</td>
</tr>
<tr>
<td>0.375</td>
<td>1.171358</td>
<td>- 1.377778</td>
<td>0.964938</td>
<td>- 1.324938</td>
</tr>
<tr>
<td>0.625</td>
<td>0.825723</td>
<td>- 1.113723</td>
<td>0.686508</td>
<td>- 0.902508</td>
</tr>
<tr>
<td>0.875</td>
<td>0.591029</td>
<td>- 0.763829</td>
<td>0.495550</td>
<td>- 0.625150</td>
</tr>
</tbody>
</table>
Table 2.2
Solution of Example 2.2 with \( m = 4 \)

<table>
<thead>
<tr>
<th>Time Sec.</th>
<th>( x_1 ) Block Pulse Value</th>
<th>( x_1 ) Discrete Value</th>
<th>( x_2 ) Block Pulse Value</th>
<th>( x_2 ) Discrete Value</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>- 2.643094</td>
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<td></td>
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<tr>
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<td></td>
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<tr>
<td>0.875</td>
<td>2.659330</td>
<td>0.968733</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 2.1 A Set of Walsh Functions

(a) Zeroth Walsh Function

(b) First Walsh Function

(c) Second Walsh Function

(d) Third Walsh Function
FIGURE 2.2 INTEGRALS OF A SET OF WALSH FUNCTIONS

(a) INTEGRAL OF ZEROETH WALSH FUNCTION

(b) INTEGRAL OF FIRST WALSH FUNCTION

(c) INTEGRAL OF SECOND WALSH FUNCTION

(d) INTEGRAL OF THIRD WALSH FUNCTION