CHAPTER 7

ALTERNATE FORMULATION USING DEGENERATED LAYERED PLATE AND SHELL ELEMENTS

7.1 INTRODUCTION

Composite thin-walled non-prismatic beams, plates and shells may also be analysed using degenerated plate / shell element. The general purpose program FEAST-C [2] is developed based on well established finite element method for obtaining solutions of various structural analysis problems using the degenerated element originally developed by Irons and Ahmed. In this chapter, the general finite element formulation for static, free vibration and stability analysis using the degeneration concept is discussed. For completeness, the element stiffness matrix, mass matrix and geometric stiffness matrices are also given even though they are found in many text books and proceeding volumes.

7.2 CONCEPTS

Basically there are two concepts that have been pursued to derive thin shell finite elements as shown in Fig. 7.1.

7.2.1 Classical Concept

The finite element discretization is introduced into the shell surface model. This 2D idealization is described by a linear or non-linear shell theory usually derived by introducing thin shell assumptions into the field equations of three-dimensional
continuum. In most cases the Kirchhoff-Love hypothesis is presumed so that C\(^1\) continuity requirement is to be satisfied.

### 7.2.2 Degeneration Concept

The formulation directly discretizes the 3D field equations in terms of mid surface variables applying simultaneously corresponding to shell assumptions. In general, it is assumed that normal to the shell surface remains straight after deformation allowing for shear deformations. These displacements and rotations are independent variables so that C\(^0\) continuity is required.

### 7.3 STATIC ANALYSIS

The beam, plate and shell structures are discretized into a number of plate and shell elements connected at a set of discrete points, known as nodes as shown in Fig. 7.2. The displacement at any point \(\{r\}^T = <u, v, w>\) within an element is given by

\[
r = [N]\{r\}
\]

(7.1)

where \([N]\) are the shape functions and \(\{r\}\) are the nodal variables denoting displacements in \(x, y\) and \(z\) directions and rotations about three axes. The strain vector is defined as

\[
<\varepsilon> = <\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}>
\]

(7.2)

or

\[
\{\varepsilon\} = \{\varepsilon\}_L + \{\varepsilon\}_NL
\]

(7.3)

\(\{\varepsilon\}_L\) is linear in displacements and \(\{\varepsilon\}_NL\) is quadratic in displacements given by

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The linear and nonlinear strain components may be written in terms of nodal displacements as

$$
\{\varepsilon\} = [B]\{\delta\} + [B_{NL}]\{\nu\}
$$

(7.6)

where \(\{\varepsilon\}\) is the strain vector, \([B]\) linear strain displacement matrix and \([B_{NL}]\) is the nonlinear strain displacement matrix. Let the initial strains arising due to temperature change, moisture (hygrothermal) can be denoted by \(\{\varepsilon_0\}\) so that constitutive relations are written as

$$
\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\})
$$

(7.7)
The strain energy of the element is given by

\[ U = \frac{1}{2} \int \varepsilon^T \sigma \, dv \] \hspace{1cm} (7.8)

\[ = \frac{1}{2} \int \{r\}^T (\{B\} + [B_{NL}]) \{D\} \{(\{B\} + [B_{NL}])\{r\} - \{\varepsilon_0\}\} \, dv \] \hspace{1cm} (7.9)

The potential of applied loads \( V \) is given by

\[ V = -\{r\}^T \{f_c\} - \int r^T \{b\} \, dv - \int r^T \{d\} \, dA \] \hspace{1cm} (7.10)

where \( \{f_c\} \) is the concentrated loads, \( \{b\} \) body forces / unit volume, \( \{d\} \) denote the surface forces.

The total potential \( \Pi \) is given by

\[ \Pi = U + V \] \hspace{1cm} (7.11)

Using the stationary property of the total potential for equilibrium, we get

\[ [k] \{r\} + [k_G] \{\varepsilon\} + \{f_b\} + \{f_d\} + \{f_{e_0}\} - \{f_c\} = \{0\} \] \hspace{1cm} (7.12)

where

\[ [k] = \int [b] \,^T \{D\} \{B\} \, dv = \text{Elastic stiffness matrix} \]

\[ [k_G] = \int [B_{NL}] \,^T \{\sigma\} \, dv = \text{Geometric stiffness matrix} \]

\[ \{f_b\} = -\int [N] \,^T \{b\} \, dv = \text{Load vector due to body force} \]

\[ \{f_d\} = -\int [N] \,^T \{d\} \, dA = \text{Load vector due to surface forces} \]

\[ \{f_{e_0}\} = -\int [B] \,^T \{D\} \{\varepsilon_0\} \, dv = \text{Load vector due to initial strains} \]
Eqn. 7.12 is the equilibrium equation for an element. Obtaining similar equation for all the elements of the structure and using standard procedures, the equilibrium equation for a complete structure can be formed as

\[
\{K\} \{\ddot{r}\} + \{K_{ri}\} \{r\} + \{F_p\} + \{F_d\} + \{F_{vb}\} - \{F_c\} = \{0\}
\]  

(7.13)

In Eqn. 7.13 all the matrices and vectors are assembled global matrices and vectors for the complete structure. Solution of Eqn. 7.13 is possible through any standard algorithms to obtain the nodal variables \{r\} after proper constraints are imposed. The strains and stresses can then be evaluated using Eqn. 7.6 and Eqn. 7.7.

### 7.4 FREE VIBRATION ANALYSIS

The kinetic energy \( T \) of an element is written as

\[
T = \frac{1}{2} m \dot{r}^2 \, dv
\]  

(7.14)

Assuming that the structure is executing harmonic oscillations, which means that \( r(x,y,z,t) \) is of the form \( r(x,y,z) \, e^{i \omega t} \) and Eqn. 7.14 may be rewritten as

\[
T = \frac{1}{2} \omega^2 \int \dot{r}^T \, m \, r \, dv
\]  

(7.15)

Using Eqn. 7.1 for the field variables in terms of nodal variables Eqn. 7.15 becomes

\[
T = \frac{\omega^2}{2} \int \{\xi\}^T [N]^T \, [m] \, [N] \, dv \, \{\xi\}
\]  

(7.16)

Defining Lagrangian \( L \) as

\[
L = U - T
\]  

(7.17)
Now Lagrangian $L$ is written as

$$L = \frac{1}{2} \int \varepsilon^T \sigma \, dv - \frac{\omega^2}{2} \int \varepsilon^T \bar{m} \, \dot{r} \, dv \tag{7.18}$$

For stationary value of $L$, $\frac{\partial L}{\partial \{r\}} = 0$, we get

$$[k] \{\ddot{r}\} - \omega^2 [m] \{r\} = 0 \tag{7.19}$$

where the element mass matrix $[m]$ is given by

$$[m] = \int [N]^T \bar{m} [N] \, dv \tag{7.20}$$

The assembly of elemental equilibrium equations leads to global equilibrium equation as

$$([K] - \omega^2 [M]) \{\ddot{r}\} = \{0\} \tag{7.21}$$

The Eigen value problem can be solved using standard algorithms to compute Eigen values and Eigen vectors.

### 7.5 STABILITY ANALYSIS

The stability analysis is carried out in two steps. In the first step, the stress analysis is carried out for the given loads. In the second step, the geometric stiffness matrices (will be presented later) for all the elements are evaluated and the assembled governing equation is

$$([K] - P_{cr} [K_G]) \{\ddot{r}\} = \{0\} \tag{7.22}$$

Similar to Eqn.7.21, the eigen values and the eigen vectors (buckled shapes) are computed using any of the standard eigen value extraction schemes. The analysis of
layered curved / straight panels and shells is generally carried out by employing doubly
curved shell elements. The coordinates $\bar{X}, \bar{Y}, \bar{Z}$ of a point anywhere in a shell element
is expressed as

$$
\bar{X} = X + \frac{t}{2} \zeta V_{nx} \\
\bar{Y} = Y + \frac{t}{2} \zeta V_{ny} \\
\bar{Z} = Z + \frac{t}{2} \zeta V_{nz}
$$

(7.23)

where $X$, $Y$ and $Z$ are the Cartesian coordinates of a point on the reference mid plane
and $t$ denotes the thickness and $\zeta$ is the natural coordinate in the normal direction and
$V_{nx}, V_{ny}, V_{nz}$ represent the components of mid surface normal at the point in $X$, $Y$ and
$Z$ directions respectively. The Cartesian coordinate of a point in the element are
related to the nodal coordinates as

$$
\bar{X} = \sum N_i X_i + \frac{\zeta}{2} \sum t_i N_i V'_{nx} \\
\bar{Y} = \sum N_i Y_i + \frac{\zeta}{2} \sum t_i N_i V'_{ny} \\
\bar{Z} = \sum N_i Z_i + \frac{\zeta}{2} \sum t_i N_i V'_{nz}
$$

(7.24)

where $N_i$ are the shape functions, $t_i$ nodal thickness and the summation is over a
number of nodes per element. The normal vector $V_n$ at a point on the mid surface
$\zeta = 0$ is computed by taking the cross product of the first two rows $J_1, J_2$ of Jacobian
matrix given by
\[
[J] = \begin{bmatrix}
\frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} & \frac{\partial Z}{\partial \xi} \\
\frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} & \frac{\partial Z}{\partial \eta} \\
\frac{\partial X}{\partial \zeta} & \frac{\partial Y}{\partial \zeta} & \frac{\partial Z}{\partial \zeta}
\end{bmatrix}
\]

(7.25)

and \( \vec{V}_n \) is given by

\[
\vec{V}_n = \frac{J_1 \times J_2}{|J_1 \times J_2|}
\]

(7.26)

where \( \xi, \eta, \zeta \) are the natural coordinates. The components of \( V_n^k \) can be expressed in terms of rotations at the nodal point \( k \). However, there is no unique way of proceeding. An efficient way is to define two unit vectors \( ^0V_1^k \) and \( ^0V_2^k \) that are orthogonal to \( ^0V_n^k \) initially. In the undeformed configuration

\[
^0V_1^k = \frac{e_y \times ^0V_n^k}{|e_y \times ^0V_n^k|}
\]

(7.27)

and

\[
^0V_2^k = ^0V_n^k \times ^0V_1^k
\]

(7.28)

Considering the total Lagrangian description, wherein all the variables are referred to the undeformed configuration and defining \( \alpha_k \) and \( \beta_k \) as the rotation of the direction vector \( V_n^k \) after deformation about the vectors \( V_1^k \) and \( V_2^k \) for small angles, we can get

\[
V_n^k = -V_2^k \alpha_k + V_1^k \beta_k
\]

(7.29)
Subtracting the $X$, $Y$, $Z$ coordinates of any point in the undeformed configuration from the deformed configuration the displacements are obtained. (Bathe [4])

$$
\bar{u} = \sum N_i u_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{2x} \alpha_i + V^i_{1x} \beta_i)
$$

$$
\bar{v} = \sum N_i v_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{2y} \alpha_i + V^i_{1y} \beta_i)
$$

$$
\bar{w} = \sum N_i w_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{2z} \alpha_i + V^i_{1z} \beta_i)
$$

(7.30)

$$
\bar{u} = \sum N_i u_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{ny} \theta_{zi} + V^i_{nz} \theta_{yi})
$$

$$
\bar{v} = \sum N_i v_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{nz} \theta_{xi} + V^i_{nx} \theta_{zi})
$$

$$
\bar{w} = \sum N_i w_i + \frac{\xi}{2} \sum t_i N_i (-V^i_{nx} \theta_{yi} + V^i_{ny} \theta_{xi})
$$

(7.31)

The local coordinate system $(X', Y', Z')$ is shown in Fig. 7.3 and it is constructed as follows. The cross product of $V_n$ and the vector connecting nodes 1 and 2 is taken as $V_d$. The cross product of $V_n$ and $V_d$ gives orthogonal vector $V_2$ and the cross product of $V_2$ and $V_n$ gives $V_1$. Defining the three vectors in terms of direction cosines as

$$
\bar{V}_1 = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}
$$

$$
\bar{V}_2 = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}
$$

$$
\bar{V}_n = l_3 \hat{i} + m_3 \hat{j} + n_3 \hat{k}
$$

(7.32)

The local coordinate system $(X', Y', Z')$ is related to the global coordinate system $(X, Y, Z)$ through these vectors as

$$
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} =
\begin{bmatrix}
l_1 & m_1 & n_1 \\
l_2 & m_2 & n_2 \\
l_3 & m_3 & n_3
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
$$

(7.33)

Defining local strains and global strains as
In the local coordinate system, $\varepsilon_Z$ is neglected as only first order shear deformation theory is employed. The strain in local system is related to strain in global system as

$$\{\varepsilon\} = [T] \{\varepsilon\}$$

(7.35)

Where $[T]$ is given by

$$[T] = \begin{bmatrix}
E_1 & E_2 \\
E_3 & E_4
\end{bmatrix}$$

(7.36)

where

$$[E_1] = \begin{bmatrix}
l_1^2 & m_1^2 & n_1^2 \\
l_2^2 & m_2^2 & n_2^2
\end{bmatrix}$$

(7.37a)

$$[E_2] = \begin{bmatrix}
l_1m_1 & l_1n_1 & m_1n_1 \\
l_2m_2 & l_2n_2 & m_2n_2
\end{bmatrix}$$

(7.37b)

$$[E_3] = 2 \begin{bmatrix}
l_1l_2 & m_1m_2 & n_1n_2 \\
l_1l_3 & m_1m_3 & n_1n_3
\end{bmatrix}$$

(7.37c)

$$[E_4] = \begin{bmatrix}
l_1m_2 + l_2m_3 & n_1l_2 + n_2l_1 & m_1n_2 + m_2n_1 \\
l_1m_3 + l_2m_3 & n_1l_3 + n_2l_2 & m_1n_3 + m_2n_1
\end{bmatrix}$$

(7.37d)

The global strain vector may be written as

$$<\varepsilon> = \begin{bmatrix}
d\varepsilon_x \\
d\varepsilon_y \\
d\varepsilon_z \\
d\gamma_{XY} \\
d\gamma_{XZ} \\
d\gamma_{YZ}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial X} \\
\frac{\partial v}{\partial Y} \\
\frac{\partial w}{\partial Z} \\
\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} \\
\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X} \\
\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}
\end{bmatrix}$$

(7.38)
The global strain vector may be written in another form in terms of $< \varepsilon >$ as

$$< \varepsilon >= \left[ \begin{array}{cccccc}
\frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} & \frac{\partial \varepsilon}{\partial X} & \frac{\partial \varepsilon}{\partial Y} & \frac{\partial \varepsilon}{\partial Z}
\end{array} \right]$$

(7.39)

Hence $< \varepsilon >$ may be written in terms of $< \bar{\varepsilon} >$ as

$$\{\varepsilon\} = [L] \{\bar{\varepsilon}\}$$

(7.40)

where $[L]$ is given by

$$[L] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}$$

(7.41)

The gradients of the displacements with respect to normal coordinates may be written in terms of the gradients of displacements with respect to global coordinates as

$$\left[ \begin{array}{c}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta}
\end{array} \right] = [J] \left[ \begin{array}{c}
\frac{\partial u}{\partial X} \\
\frac{\partial u}{\partial Y} \\
\frac{\partial u}{\partial Z}
\end{array} \right]$$

(7.42)

Hence the gradients of the displacements with respect to global coordinates are now written in terms of the gradients of displacements with respect to normal coordinates as

$$\left[ \begin{array}{c}
\frac{\partial u}{\partial X} \\
\frac{\partial u}{\partial Y} \\
\frac{\partial u}{\partial Z}
\end{array} \right] = [J]^{-1} \left[ \begin{array}{c}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta}
\end{array} \right]$$

(7.43)
Denoting the displacement gradients with respect to normal coordinates as \( \{ \varepsilon_n \} \) as

\[
\langle \varepsilon_n \rangle = \left< \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial \xi} \frac{\partial \eta}{\partial \eta} \frac{\partial \xi}{\partial \xi} \frac{\partial \xi}{\partial \eta} \right>
\] (7.44)

Hence \( \{ \bar{\varepsilon} \} \) is written in terms of \( \{ \varepsilon_n \} \) as

\[
\{ \bar{\varepsilon} \} = [J^*] \{ \varepsilon_n \}
\] (7.45)

where

\[
[J^*] = \begin{bmatrix}
J^{-1} & [0] & [0] \\
[0] & J^{-1} & [0] \\
[0] & [0] & J^{-1}
\end{bmatrix}
\] (7.46)

Substituting for displacements in \( \{ \varepsilon_n \} \) we can write in terms of nodal displacements as

\[
\{ \varepsilon_n \} = \{ [S_0] + \xi[S_1] \} \{ r \}
\] (7.47)

Combining Equations 7.35, 7.40 and 7.45 the strains in local system can be written in terms of nodal displacements of global system as

\[
\{ \varepsilon' \} = [T][L][J^*] \{ [S_0] + \xi[S_1] \} \{ r \}
\] (7.48)

or

\[
\{ \varepsilon' \} = \{ [B_0] + \xi[B_1] \} \{ r \}
\] (7.49a)

where

\[
[B_0] = [T][L][J^*][S_0]; \quad [B_1] = [T][L][J^*][S_1]
\]

In Eqn. 7.48 and 7.49.b \([S_0]\) and \([S_1]\) are explicitly given by
\[ [S_0] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix} ; \quad [S_1] = \frac{t_i}{2} \begin{bmatrix} 0 & G_{12} \\ 0 & G_{22} \end{bmatrix} \]

(7.50)

where sub matrices are given by

\[ [F_{11}] = \begin{bmatrix} N_{i,\dot{t}} & 0 & 0 \\ N_{i,\eta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad [F_{21}] = \begin{bmatrix} 0 & N_{i,\dot{\eta}} & 0 \\ 0 & N_{i,\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad [F_{31}] = \begin{bmatrix} 0 & 0 & N_{i,\dot{\eta}} \\ 0 & 0 & N_{i,\eta} \\ 0 & 0 & 0 \end{bmatrix} \]

(7.51)

\[ [F_{12}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} V_{nz} N_i & -\frac{1}{2} t_i V_{n\eta} N_i \end{bmatrix} \]

(7.52a)

\[ [F_{22}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} t_i V_{nz} N_i & 0 & \frac{1}{2} t_i V_{n\eta} N_i \end{bmatrix} \]

(7.52b)

\[ [F_{32}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} t_i V_{n\eta} N_i & -\frac{1}{2} t_i V_{nz} N_i & 0 \end{bmatrix} \]

(7.52c)

\[ [G_{12}] = \begin{bmatrix} 0 & V_{nz} N_{i,\dot{t}} & -V_{n\eta} N_{i,\dot{\eta}} \\ 0 & V_{nz} N_{i,\eta} & -V_{n\eta} N_{i,\eta} \\ 0 & 0 & 0 \end{bmatrix} \]

(7.52d)

\[ [G_{22}] = \begin{bmatrix} -V_{nz} N_{i,\dot{\eta}} & 0 & -V_{n\eta} N_{i,\dot{t}} \\ -V_{nz} N_{i,\eta} & 0 & -V_{n\eta} N_{i,\eta} \\ 0 & 0 & 0 \end{bmatrix} \]

(7.52e)
The nodal displacements are given by
\[
< r > = \begin{pmatrix} u_i \ v_i \ w_i \ \theta_{xi} \ \theta_{yi} \ \theta_{zi} \end{pmatrix}
\]  
(7.53)

### 7.6 STIFFNESS MATRIX FOR COMPOSITE LAMINA

Consider a composite lamina as shown in Fig. 7.4. The fibres are oriented at an angle of \( \theta \) with respect to \( X' \) axis. The constitutive properties of the composite lamina with respect to LT system is given by
\[
\{\sigma\}_{LT} = [Q] \{\epsilon\}_{LT}
\]  
(7.54)
where \([Q]\) is the constitutive matrix with respect to LT system. The stresses with respect to \( X' \ Y' \) system can be written as
\[
\{\sigma\}' = [R] \{\sigma\}_{LT}
\]  
(7.55)
and the strain with respect to LT system can be obtained with respect to strain in \( X' \ Y' \) system as
\[
\{\epsilon\}_{LT} = [R]^T \{\epsilon\}'.
\]  
(7.56)
Substituting Eqn. 7.54 and 7.56 in Eqn. 7.55 we get
\[
\{\sigma\}' = [R][Q][R]^T \{\epsilon\}' = [\overline{Q}] \{\epsilon\}'.
\]  
(7.57)
where \([\overline{Q}]\) represents the constitutive matrix for a composite lamina wherein the properties have been transformed from material principal to the local coordinate system.
The strain energy for a composite plate is written as

\[
U = \frac{1}{2} \sum_{i}^{n_{\text{layer}}} \int \{e\}^T \{\mathbf{Q}\} \{e\}' \, dv
\]  

(7.58)

Substituting Eqn. 7.49.a in Eqn. 7.58 we get

\[
U = \frac{1}{2} \{\mathbf{r}\}^T \sum \int ((B_o) + \xi(B_i))^T \{\mathbf{Q}\}((B_o) + \xi(B_i)) \, dv \{\mathbf{r}\}
\]  

(7.59)

dv = dt \times dA and carrying out the integration in thickness direction we get

\[
U = \frac{1}{2} \{\mathbf{r}\}^T [k] \{\mathbf{r}\}
\]  

(7.60)

Considering the variation of strain energy with respect to nodal displacements we get

\[
[k] \text{ in Eqn. 7.12 as}
\]

\[
[k] = \int\{[B_o]^T [A] [B_o]^T [B] [B_o]^T [B] [B_o] + [B_o]^T [B] [B_o] + [B_o]^T [D] [B_o] \} \, dA
\]  

(7.61)

where

\[
[A] = \sum_{h_k}^{h_{k+1}} \int \{\mathbf{Q}\} \, dt = \sum_{k=1}^{n_{\text{layer}}} \{\mathbf{Q}\}(h_{k+1} - h_k)
\]

(7.62a)

\[
[B] = \sum_{h_k}^{h_{k+1}} \int \{\mathbf{Q}\} \xi \, dt = \frac{1}{2} \sum_{k=1}^{n_{\text{layer}}} \{\mathbf{Q}\}(h_{k+1}^2 - h_k^2)
\]

(7.62b)

\[
[D] = \sum_{h_k}^{h_{k+1}} \int \{\mathbf{Q}\} \xi^2 \, dt = \frac{1}{3} \sum_{k=1}^{n_{\text{layer}}} \{\mathbf{Q}\}(h_{k+1}^3 - h_k^3)
\]

(7.62c)

where \(h_k\) is the normalized height of \(k\) th layer from the bottom surface. \([A]\), \([B]\) and \([D]\) are called axial, coupled and bending stiffness matrices of the composite plate respectively.
7.7 NUMERICAL INTEGRATION

To arrive at the stiffness matrix, numerical integration is carried out using Gaussian quadrature. The recommendations given by FEAST-C [2] are shown in Table 7.1. In the case of coplanar elements, the assembled stiffness matrix will contain a large number of zero diagonals corresponding to $\theta_x, \theta_y, \theta_z$ depending on the plane of the plate. This makes global stiffness matrix singular or ill conditioned. To avoid this problem, a small stiffness of $1/1000^{th}$ smallest diagonal is placed at the corresponding locations.

7.8 LOAD VECTOR

7.8.1 Pressure Load

The element surface pressure load is computed as follows

$$\{F\}^e = \int [N]^T p(X', Y') V_n \, dA \quad (7.63)$$

where

$$p(X', Y') = \sum N_i p_i \quad (7.64)$$

$p_i$ is the pressure intensity at node 'i'.

7.8.2 Mass Matrix

The kinetic energy is written as

$$Y = \frac{1}{2} \int \rho (\ddot{u},_x + \ddot{v},_x + \ddot{w},_x) \, dv \quad (7.65)$$

where, $t$ denotes differentiation with respect to time 't'.

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Using kinematics Eqn. 7.31 for a layered plate/shell

\[ T = -\frac{\omega^2}{2} \int \{r\}^T \{ [R_0]^T + \varepsilon[R_1]^T \} \rho \{ [R_0] + \varepsilon[R_1] \} dv \{r\} \]  

(7.66)

where

\[ [R_0] = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 & 0 \end{bmatrix} \]  

(7.67)

and

\[ [R_1] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & t_i V_{n2} N_i & -t_i V_{n2} N_i \\ 0 & 0 & 0 & t_i V_{n2} N_i & 0 & t_i V_{n2} N_i \\ 0 & 0 & 0 & t_i V_{n2} N_i & -t_i V_{n2} N_i & 0 \end{bmatrix} \]  

(7.68)

The variation of kinetic energy leads to

\[-\omega^2[M] \{\dot{r}\} = -\omega^2 \{ \int \rho [R_0]^T [R_0] + \rho \varepsilon ([R_0]^T [R_1] + [R_1]^T [R_0]) + \rho \varepsilon^2 [R_1]^T [R_1] \} dv \{r\} \]  

(7.69)

Once again integration is carried out in the thickness direction as shown below.

Denoting

\[ R = \int_{h_k}^{h_{k+1}} \rho_i (h_{k+1} - h_k) \]  

(7.70a)

\[ S = \frac{1}{2} \int_{h_k}^{h_{k+1}} \rho_i (h_{k+1}^2 - h_k^2) \]  

(7.70b)

\[ I = \frac{1}{3} \int_{h_k}^{h_{k+1}} \rho_i (h_{k+1}^3 - h_k^3) \]  

(7.70c)

Substituting Eqn. 7.70 in Eqn. 7.69, we get
\[
[M] = \int \left\{ [R[R_0] + S([R_0][R_1] + [R_1][R_0]) + I[R_1][R_1] \} \right\} \, dA \quad (7.70d)
\]

### 7.9 GEOMETRIC STIFFNESS MATRIX FOR STABILITY ANALYSIS

In Eqn. 7.5 the nonlinear contribution for strain vector is given. To derive the stiffness matrix let us follow the steps given below.

\[
\{e\}_{NL} = \frac{1}{2} [A][G]\{r\} \quad (7.71)
\]

where

\[
[A] = \begin{bmatrix}
\frac{\partial u}{\partial X} & 0 & 0 & \frac{\partial v}{\partial X} & 0 & 0 & \frac{\partial w}{\partial X} & 0 & 0 \\
0 & \frac{\partial u}{\partial Y} & 0 & 0 & \frac{\partial v}{\partial Y} & 0 & 0 & \frac{\partial w}{\partial Y} & 0 \\
0 & 0 & \frac{\partial u}{\partial Z} & 0 & 0 & \frac{\partial v}{\partial Z} & 0 & 0 & \frac{\partial w}{\partial Z} \\
\frac{\partial u}{\partial Y} & \frac{\partial u}{\partial X} & 0 & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial X} & 0 & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial X} & 0 \\
\frac{\partial u}{\partial Z} & \frac{\partial u}{\partial X} & 0 & \frac{\partial v}{\partial Z} & \frac{\partial v}{\partial X} & 0 & \frac{\partial w}{\partial Z} & \frac{\partial w}{\partial X} & 0 \\
0 & \frac{\partial u}{\partial Y} & 0 & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} & 0 & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial Z} & 0 \\
\end{bmatrix}
\quad (7.72)
\]

\[
[G]\{r\} = \{\varepsilon\} \quad (7.73)
\]

and \([G]\) is given by

\[
[G] = [J^*](\{S_0\} + \zeta\{S_1\}) \quad (7.74)
\]

and hence nonlinear strain displacement matrix \([B]_{NL}\) is given by

\[
[B]_{NL} = [A][G] \quad (7.75)
\]
and geometric stiffness matrix \([K]_g\) is given as

\[
[K]_g = \int [B]_{NL} \{\sigma\} \, dv \tag{7.76}
\]

This can be simplified as

\[
[K]_g \{r\} = \int [G]^T [\bar{\sigma}] [G] \, dv \{r\} \tag{7.77}
\]

where

\[
[\bar{\sigma}] = \begin{bmatrix}
[\sigma] & [0] & [0] \\
[0] & [\sigma] & [0] \\
[0] & [0] & [\sigma]
\end{bmatrix} \tag{7.78}
\]

where

\[
[\sigma] = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{bmatrix} \tag{7.79}
\]

\([\bar{\sigma}]\) is the global stress state which has been computed during pre-buckling analysis and can be expressed as (Fig. 7.5).

\[
\bar{\sigma} = \sigma_1 + \sigma_2 \zeta \tag{7.80}
\]

Neglecting higher order terms involving \(\zeta^2, \zeta^3\) and carrying out thickness integral we get

\[
[k_G] = \int [S_o J^* N_i J^* S_o + S_o J^* M_2 J^* S_o + S_o J^* M_1 J^* S_i + S_i J^* M_1 J^* S_o] \, dA \tag{7.81}
\]

where \(N_i, M_1, M_2\) are the stress resultants and stress couples respectively.
7.10 SHAPE FUNCTIONS

The shape functions for the family of quadrilateral plate / shell elements in terms of natural coordinates are given below.

7.10.1 Four Node Quadrilateral Element

For the four node quadrilateral element shown in Fig. 7.6 the shape functions are given by

\[ N_i = \frac{1}{4} (1 + \xi_i \xi_j)(1 + \eta_i \eta_j) \]  

(7.82)

7.10.2 Nine Node Quadrilateral Element [87]

Other than four corner nodes there are five more nodes as shown in Fig. 7.7. For the nine node quadrilateral element, the shape functions are given by

\[ N_1 = \frac{1}{4} (\xi - \xi^2)(\eta - \eta^2) \]

\[ N_2 = -\frac{1}{4} (\xi + \xi^2)(\eta - \eta^2) \]

\[ N_3 = \frac{1}{4} (\xi + \xi^2)(\eta + \eta^2) \]

\[ N_4 = -\frac{1}{4} (\xi - \xi^2)(\eta + \eta^2) \]

\[ N_5 = -\frac{1}{2} (1 - \xi^2)(\eta - \eta^2) \]
\[ N_0 = \frac{1}{2} (\xi + \xi^2)(1 - \eta^2) \]
\[ N_7 = \frac{1}{2} (\xi^2 - \xi^3)(\eta + \eta^2) \]
\[ N_8 = -\frac{1}{2} (\xi^2 - \xi^3)(1 - \eta^2) \]
\[ N_9 = (1 - \xi^2)(1 - \eta^2) \]  
(7.83)

7.11 SUMMARY

In this Chapter, the stiffness, mass and geometric stiffness matrices are given for layered degenerated plate / shell element for static, dynamic and buckling analysis. The package FEAST-C [2] is used in Chapters 9, 10 and 11 for obtaining deflection, buckling load and natural frequency for thin-walled beam, plate and shell elements. These values are subsequently considered as constraints for finding the optimal lay-up of composite structures using Evolution Strategies and Genetic Algorithms.
**TABLE 7.1 ORDER OF GAUSSIAN QUADRATURE**

<table>
<thead>
<tr>
<th>ELEMENT TYPE</th>
<th>GAUSSIAN QUADRATURE ORDER</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BENDING</td>
</tr>
<tr>
<td>4 node</td>
<td>2x2</td>
</tr>
<tr>
<td>8 node</td>
<td>3x3</td>
</tr>
<tr>
<td>9 node</td>
<td>3x3</td>
</tr>
<tr>
<td>5 node</td>
<td>3x2</td>
</tr>
<tr>
<td>6 node</td>
<td>3x3</td>
</tr>
<tr>
<td>7 node</td>
<td>3x3</td>
</tr>
</tbody>
</table>

**Fig. 7.1 CLASSICAL AND DEGENERATION CONCEPT**
Fig. 7.2 SHELL

Fig. 7.3 SHELL MID SURFACE

Fig. 7.4 COMPOSITE LAMINA
Fig. 7.5 STRESS STATE

Fig. 7.6 a REAL ELEMENT  Fig. 7.6 b PARENT ELEMENT

Fig. 7.7 NINE NODE QUADRILATERAL