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Chapter 5

Existence and Uniqueness of Nonlinear Fourth Order Ordinary Differential Equations by using Banach Fixed Point Theorem

5.1 Introduction:
In this chapter, we discussed Picard’s Existence and Uniqueness Theorem for nonlinear third order ordinary differential equations by using Lipschitz condition and Banach Fixed Point Theorem is used. Hence Picard-Lipschitz Fundamental Theorem is proved.

5.2 Statement of problem (Picard’s Theorem) : [33]
Consider the Ordinary Differential Equation,
\[ z^{iv} = f(y,z,z',z'',z''') \text{with} \]
\[ \begin{align*}
\emptyset(y_0) &= z_0 \\
\emptyset'(x_0) &= y_0' \\
\emptyset''(x_0) &= y_0'' \\
\emptyset'''(x_0) &= y_0'''
\end{align*} \] (5.1)

has a solution in the rectangle \( R : |x - a| \leq h, |y - b| \leq k \) such that:

(i) \( f \) is bounded by \( M \) i.e. \( |f(x,y,y',y'',y''')| \leq M \) and \( Mh \leq k \), since \( f \) is continuous on \( R \).

(ii) Lipschitz condition is satisfied by \( f \) on \( R \).

Also, the solution is unique.

5.2.1 Lipschitz Condition : [16]
A function \( f(y,z) \) on a rectangle \( S \) is continuous and Lipschitz condition is satisfied by this function with constant \( A \) if there exists a real \( A > 0 \) s. t.

\[ |f(y,i) - f(y,j)| < A|i-j| \text{where } (y,i) \in S, (y,j) \in S. \] (5.2)

This condition is between continuity and differentiability, is the right condition to prove Picard’s Existence Theorem.
If \( f \) is continuously differentiable w. r. t. on \([z_1, z_2]\) then by using Mean Value Theorem, we get

\[
f(y, z_2) - f(y, z_1) = f_z(y, z_3) (z_2 - z_1)
\]

for \( z_3 \in [z_1, z_2] \), hence \( f \) satisfies Lipschitz condition on this interval.

### 5.2.2 Example:

Verify that whether the function \( f(y, z) = y^2 z^{1/5} \) satisfies Lipschitz condition on the rectangle \( \{ (y, z) : |y| \leq i, |z| \leq j \} \), where \( i > 0 \) and \( j > 0 \) or not.

We take the real root of \( z^{1/5} \) i.e. if \( y < 0 \), we take \( -|y|^{1/5} \).

If \( f \) is locally derivable w.r.t. on bounded interval, then obviously it is Lipschitz on this interval. When the derivative of \( f \) is not bounded, the problem arises with \( y = 0 \). If we consider \( f \) is Lipschitz on the interval which contains zero, then for \( y \) belongs to such interval i.e. \( |y| < i \), we get a constant \( L \) s. t.

\[
|y^2||z^{1/5}| \leq L|z|
\]

It is not true for constant \( L \) and the continuous function which is not Lipschitz at \( z = 0 \). Hence at initial condition \( z(0) = 0 \), Picard’s theorem will be failed.

But on a rectangle \( S \), Picard’s theorem will be satisfied, the rectangle is given by,

\[
|y| < i, \ |z-b| < j \text{ when } j < b, \ s. t. \ y \neq 0 \text{ in this region.}
\]

Consider this, let \( M \) be a bound of \( f \) in \( S \) which is given in Picard’s theorem:

Suppose,

\[
\max |y^2 z^{1/5}| \leq M = i^2 (b + j)^{1/5}.
\]

Now apply Picard’s theorem on a rectangle with \( i > 0 \) satisfying,

\[
i < \frac{j}{i^2(b+j)^{1/5}}.
\]
i.e.

\[ i^3 < \frac{i}{(b+j)^{1/5}}. \]

By solving directly, we have

\[ z = \left(4y^{3/15} + b^{4/5}\right)^{5/4}. \]

This solution is true for \( b = 0 \) but the trivial solution is also true.

### 5.2.3 Contraction Mapping Principle (Complete Metric Space):

**Theorem:**

Let \( T: X \to X \) be a contraction mapping on a complete metric space \( X \). The \( T \) has a unique fixed point \( x \in X \).

**Proof:** For \( y_0 \in X \), we define \( \{y_k\} \) by putting,

\[ y_{k+1} = T(y_k) \]

\( \forall k \geq 0, \text{ let } d_0 = d(y_0, y_1), \text{ then,} \]

\[ d(y_k, y_{k+1}) = d(T(y_{k-1}), T(y_0)) \leq \alpha d(y_{k-1}, y_k) \]

For \( \geq 1 \), then by induction \( d(y_k, y_{k+1}) \leq \alpha^k d_0 \). For given \( \epsilon > 0 \) take \( N \) s.t. \( \alpha^N d_0 < \epsilon \), for \( \alpha < 1 \). For \( m, n > N \), put \( d(n, m) \leq d(n, m) \leq \sum_{k=m}^{n-1} \alpha^k d_0 \leq \sum_{k=m}^{\infty} \alpha^k d_0 = \frac{\alpha^m d_0}{1-\alpha} \leq \frac{\alpha^N d_0}{1-\alpha} < \epsilon \)

Since \( X \) is a complete metric space, therefore \( \{y_k\} \) is Cauchy sequence which converges to \( y \in X \).

Now \( d(T(y_k), T(y)) \leq \alpha d(y_k - y) \to 0 \), i.e. \( T(y_k) \to T(y) \).

Since \( T(y_k) = y_{k+1} \) converges to \( y \), hence \( T(y) = y \).

For proving the uniqueness, let \( T(z'') = z'' \), then

\[ d(y, z'') = d(T(y), T(z'')) \leq \alpha d(y, z'') \]

This is true only if \( d(y, z'') = 0 \), i.e. \( y = z'' \).
5.2.4 Theorem:

Suppose $f(t,z)$ is continuous in $t$ and Lipschitz with respect to $z$ on the domain $S = [a,b] \times [c,d]$. Then, given any point $(t_0, z_0)$ in $S$, $\exists \epsilon > 0$ and a unique solution $z(t)$ of the Initial Value Problem

$$\frac{d^4f}{dy^4} = f(t, y, y', y'', y''') , y'''(t_0) = y'''_0$$

on the interval $(t_0 - \epsilon, t_0 + \epsilon)$.

**Proof of the existence and uniqueness theorem:**

Consider the Initial Value Problem

$$\frac{d^4f}{dy^4} = f(t, y, y', y'', y''') , y'''(t_0) = y'''_0 \quad (5.3)$$

The solution is as fixed point of a mapping on a complete metric space, then we have to prove that, if the time interval is sufficiently small, this mapping is a contraction mapping. First of all we define the mapping. Consider some interval $(t_0 - \epsilon, x_0 + \epsilon), f$ is continuous, and the solution is $y(t)$ of (5.3).

After taking integration between $t_0$ to $t$, we get

$$y(t) = y'''_0 + \int_{t_0}^{t} f(s, y(s), y'(s), y''(s), y'''(s)) ds \quad (5.4)$$

The above equation is known as integral equation.

Hence solution of (5.3) is also a solution of the integral equation.

In converse part, consider that, continuous function $y$ satisfies (5.4), then $f(s, y(s), y'(s), y''(s), y'''(s))$ is continuous, hence by the Fundamental Theorem of Calculus gives that $y$ is derivable and $\frac{dy}{dt} = f(t, y(t), y'(t), y''(t), y'''(t))$. Again, at $t = t_0$, we get $y'(t_0) = y'''_0$. Thus any continuous solution of (5.4) is a solution of (5.3). Therefore we solve (5.3).

For a function $y$ which is continuous, we define $T(y)$ is,

$$T(y)(t) = y'''_0 + \int_{t_0}^{t} f(s, y(s), y'(s), y''(s), y'''(s)) ds$$

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then \( y \) is solution of (3) iff \( T(y) = y \) i.e. \( y \) be the fixed point of the mapping \( T \). Now we define a complete metric space with \( T \) is a contraction mapping. For \( \epsilon > 0 \) and \( \eta > 0 \), we defines the space

\[ X = C([t_0 - \epsilon, t_0 + \epsilon], [y_0 - \eta, y_0 + \eta]). \]

Then by theorem, “The space \( C([a, b], [c, d]) \) is complete.”, \( X \) is complete.

5.3 Picard’s Theorem for System via the CMP

Application of the contraction mapping principle is to prove the existence and uniqueness of solutions of nonlinear differential equations. Here we will discuss this in metric space.

Suppose \( Y \) be a Banach space, defined with a norm \( \|\cdot\|_Y \).

By using definition, \( Y \) is a complete metric space with a metric \( d \) s.t.

\[ d(i, j) = \|i - j\|_Y \quad \forall \ i, j \in Y \]

5.3.1 Contraction Mapping Principle (Banach Space):

Theorem (Contraction Mapping Principle):

Let us suppose \( R \) is a closed subset of Banach space \( Y \) and consider the mapping \( T: R \rightarrow R \) on \( R \) s.t.

\[ \|T_i - T_j\|_Y < \alpha \|i - j\|_Y \quad \forall \ i, j \in R \]

for \( \alpha < 1 \), then \( T \) has a unique fixed point \( i \in R \) s.t.

\[ T_i = i. \]

5.3.2 Picard’s Existence Theorem for autonomous system:

Consider vector field \( f: X \rightarrow X \) which is Lipschitz on a closed ball \( B_S(i_0) \), of radius \( S > 0 \) for \( i_0 \in X \). Suppose

\[ M = \text{Sup}_{t \in B_S(i_0)} \|f(t)\| < \infty \]

then the Initial Value Problem
\[ i^{iv}(y) = f(i'''(y)), i'''(0) = i'''_0 \]

has a uniquelocal solution which is continuously derivable\(i'''(y)\), on the interval\(-\delta < y < \delta\), with

\[ \delta = S/M \]

**Proof:**

By writing the Initial Value Problem into an integral equation\(I = Ti\), where

\[ Ti(t) = i'''_0 + \int_0^y (i'''(s)) ds. \]

Suppose that \(Y\) be denoted by \(C([-\eta, \eta], X)\), the Banach space of continuous mappings from the interval \([-\eta, \eta]\) into \(X\), defined with sup norm, which is as follows,

\[ ||i||_Y = \sup_{|y| \leq \eta} ||i(y)||. \]

By applying Contraction Mapping Principle, we get

We defines \(R\) which is closed as follows:

\[ R = C([-\eta, \eta], B_S(i_0)), \text{ where } 0 < \eta < S/MT \]

The functions which are in \(Y\) whose values are within the ball \(B_S(i_0)\) i.e. within \(X\).

Now we prove that \(T: Y \rightarrow Y\) is a contraction mapping on \(R\) when \(\eta\) is sufficiently small.

for \(i \in \text{Rand each } y \in [-\eta, \eta]\),

\[ ||Ti(y) - i_0|| = ||\int_0^y f(i) ds|| \leq M\eta < S \]

Therefore \(Ti \in R\), hence \(T: R \rightarrow R\).

We have,
\[ \|T_i - T_j\|_Y = \sup_{|y| \leq \eta} \int_0^y [f(i(s)) - f(j(s))] ds \leq L \eta \|i - j\|_Y \]

On \( B_S(i_0).L \) is a Lipschitz constant for \( f \). Thus if we take

\[ \eta \leq \min \left\{ \frac{1}{2L}, \frac{1}{2\delta} \right\} \]

then \( T \) is a contraction mapping on \( Y \), it has a unique fixed point.

Since \( \eta \) depends on \( L \) and on the distance between \( S \) and \( i_0 \) of \( B_S(i_0) \), by making repetition of application of the above result, we get a unique local solution for \( |y| < S/M = \delta \) which we desire to prove.

For proving this, extend the interval of existence to the R.H.S. in the positive direction of \( y \) because we can reverse \( x \) to prove the result to L.H.S.

Consider \( \eta \neq \delta \) i.e., \( \delta > 1/2L \), thus we proved local existence only for \( \eta \) arbitrarily close to \( \eta = 1/2L \) where \( \tilde{i}(y_i) = i_1 \), say. By construction \( \|i_1 - i_0\| = S_1 \leq M\eta \). Thus \( -\eta \leq (S - S_1)/M \).

Select a new ball with radius \( S' = S_1 + S_1/2 \) about \( i_1 \), s. t. \( B_{S'}(i_1) \) is inside \( B_S(i_0) \), and \( M \) again applies. The above local existence argument applies to the initial value problem starting at \( i(y_1) = i_1 \) and hence we can take a further step so that the solution exists either for all \( y < 2\eta \) or for all \( y < y_1 + S'/M \). In the latter case we are done since \( R'/M \) is greater than the previous shortfall. In the former case we can iterate and take another step.

Now we can see that the interval of existence only depends on the norm of \( u \). Assuming that \( f \) is Lipschitz continuous on any ball of any size then the only way in which the solution of an ODE ceases to exist is if \( ||i(y)|| \) becomes unbounded. We shall pick up this idea to prove the global existence of solutions for well behaved some class of problems later on.

An examination of this proof, given above, reveals that we could allow a non-autonomous system, where the mapping \( f \) depends explicitly on \( y \), provided that the...
bounds $M$ and Lipshitz constant is valid for all $y$ with in the interval $(-\delta, \delta)$ where we seek existence. Hence we have the following.

### 5.3.3 Picard’s Existence Theorem for non-autonomous system:

Consider that: $[-\delta, \delta] \times X \to X$, $\forall y \in [-\delta, \delta]$, s.t., the field $f(y, i)$ is Lipschitz w.r.t. $i$ on a closed ball $B_\delta(i_0)$, of radius $S > 0$ for $i_0 \in X$.

Consider an upper bound for $||f||$:

$$M = \sup_{i \in B_\delta(i_0), |y| < \delta} ||f(y, i)|| < \infty$$

Then the Initial Value Problem

$$i'''(y) = f(y, i'''(y)), i'''(0) = i'''_0$$

has a uniquelocal solution which is continuously derivable $i'''(y)$, on the interval $-\delta < y < \delta$, with

$$\delta = S/M$$

**Note**: Consider that $f(y, z''')$ is continuous in $S$. Then integrating (5.1) between $a$ to $y$ variable $y$.

$$[z'''](y) = \int_a^y f(t, z'''(t))dt = z''(y) - z''(a)$$

After rearrangement,

$$z'''(y) = b + \int_a^y f(t, z'''(t))dt$$

We convert the Differential Equation into an Integral Equation, value of $z'''$ is given in an integral rather than a differential. Here we consider the particular case of (4.5). A solution is obtained by iteration or successive approximation.

### 5.4 Picard’s Method of Successive Approximation:

Successive approximations, are given by $z_n(y)$ initially with $z_0(y)$.

$$\begin{align*}
z_0(y) &= b \\
z_{n+1}(y) &= z_n(y) + \int_a^y f(t, z_n(t))dt
\end{align*}$$

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i.e., consider that \( y \) equals to its starting value, and after that put the current value in the R.H.S. of (5.7) to get the next value. Here we know that if this process converges and converges to solution of (5.7).

Let us consider the differences between successive iterations

\[
e_0(y) = b \\
e_{n+1}(y) = z_{n+1}(y) - z_n(y)
\]  

and

\[
z_n(y) = \sum_{k=0}^{n} e_k(y)
\]

We written \( z_n(y) \) as in the form of a series (5.8), we want to prove that this series converges, hence we want the differences \( e_n(y) \) to be small. To prove this, consider some assumptions and conditions on \( f \), but

\[
e_{n+1}(y) = z_{n+1}(y) - z_n(y) \\
= \int_{a}^{y} [f(t,z_n(t)) - f(t,z_{n-1}(t))] dt
\]

\[
\therefore |e_{n+1}(y)| < \left| \int_{a}^{y} [f(t,y_n) - f(t,z_{n-1})] dt \right|
\]  

5.5 Picard-Lipschitz Theorem : [3]

(Picard-Lipschitz) Fundamental Theorem of O.D.E.s

Given the Initial Value Problem

\[
z^{iv}(x) = F(y,z'''(y)), \quad z'''(a) = Y'''
\]

If \( F \) is continuous in \( x \) and Lipschitz in \( y \) in a neighbourhood of the initial point \( y \in (a - h, a + h), z \in (Y - l, Y + l) \), then the o.d.e. has a unique solution on some (smaller) interval, \( y \in (a - r, a + r) \), that depends continuously on \( Y \).

Proof : The Ordinary Differential Equation with initial condition converted into an integral equation :
\[ z(y) = Y''' + \int_a^y f(s, z'''(s)) \, ds \]

Taking Picard iteration on \( y \in [\alpha, \beta] \)

\[
\begin{align*}
z_0(y) & := Y''' \\
z_1(y) & := Y''' + \int_a^y F(s, Y''') \, ds \\
z_2(y) & := Y''' + \int_a^y F(s, z_1'''(s)) \, ds \\
& \vdots \\
z_{n+1}(y) & := Y''' + \int_a^y F(s, z_n'''(s)) \, ds
\end{align*}
\]

Here, we observe that each function is continuous in \( y \).

Suppose that all \( y \) and \( Y''' \), which are in the equation are in the rectangle \((a - r, a + r) \times (Y''' - l, Y''' + l)\), where \( r \leq h \) s.t. \( F \) is Lipschitz condition on them.

1. The Iteration Converges : for each \( y \in (a - r, a + r) \), the sequence \( z'''_{n}(y) \) converges. Here we say \( z'''_{n} \) is convergent to \( z''' \). Here we show that, by the induction on \( n \) is as follows :

\[
|z'''_{n+1}(y) - z'''_{n}(y)| \leq \frac{ck_{n}y - a|^{n+1}}{(n + 1)!}
\]

When \( n = 0 \),

\[
|z'''_{1}(y) - z'''_{0}(y)| = \left| \int_a^y F(s, Y''') \, ds \right| \\
\leq c| \int_a^y 1 \, ds | \\
\leq c|y - a|
\]

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Where \( c := \max_{R \in [a-r,a+r]} F(s, Y'') \) is exists when \( F \), continuous in \( s \).

Suppose that, we prove for \( n - 1 \),

\[
|z''_{n+1}(y) - z''_n(y)| = \left| \int_a^y F(s, z''_n(s) - F(s, z''_{n-1}(s)) \, ds \right|
\]

\[
\leq \int_a^y |F(s, z''_n(s) - F(s, z''_{n-1}(s))| \, ds
\]

\[
\leq k \int_a^y k^{n-1}|s - a|^n \frac{n!}{n!} ds
\]

\[
= \frac{ck^n|y - a|^{n+1}}{(n+1)!}
\]

\[
\sum_{n=0}^\infty \frac{ck^n|y - a|^{n+1}}{(n+1)!}
\]

is convergent.

\[
\therefore \sum_{n=0}^\infty |z''_{n+1}(y) - z''_n(y)| \text{ is absolutely convergent.}
\]

Therefore, \( \lim_{n \to \infty} z''_n(y) = Y'' + \sum_{n=0}^\infty z''_{n+1}(y) - z''_n(y) \)

Which converges uniformly to \( z''(y) \), because

\[
|z''_{n+1}(y) - z''_n(y)| \leq \frac{ck^{n+1}|y - a|^{n+1}}{k(n + 1)!}
\]

\[
\Rightarrow |z''(y) - z''_N(y)| \leq \sum_{n=0}^{N-1} |z''_{n+1}(d) - z''_n(y)|
\]

\[
\leq \sum_{n=N}^\infty \frac{ck^{n+1}|y - a|^{n+1}}{(n+1)!}
\]

\[
\leq \frac{ck^{N+1}|y - a|^{N+1}}{k(N+1)!} e^kh
\]
We know that the uniform convergence of continuous functions is continuous.

2. $z'''(y)$ is a solution: Take $z'''_n$ which is near to $z'''$. This is true because $z'''_n$ converges uniformly to $z'''$.

$$\forall \epsilon > 0, \exists N, \forall y \in [a - r, a + r], n > N \Rightarrow |z'''_n(y) - z'''(y)| < \epsilon$$

Thus,

$$\left| \int_a^y F(s, z'''_n(s)) ds - \int_a^y F(s, z'''(s)) ds \right| \leq \int_a^y |F(s, z'''_n(s)) - F(s, z'''(s))| ds$$

$$\leq | \int_a^y k|z'''_n(s) - z'''(s)| ds|$$

$$\leq k |y - a|$$

$$\leq \epsilon kh$$

By taking the limit $n \to \infty$ of the above,

$$z'''_{n+1}(y) = Y''' + \int_a^y F(s, z'''_n(s)) ds$$

Thus $z'''$ is a solution of the Ordinary Differential Equation

$$z'''(y) = Y''' + \int_a^y F(s, z'''(s)) ds$$

3. $z'''$ is unique: Consider that $i'(y)$ is also a solution,

$$i'''(y) = Y''' + \int_a^y F(s, i'''(s)) ds.$$ 

Since $z''$ and $i''$ are continuous, therefore they are bounded on $[a - r, a + r]$.

$$|z'''(y) - i'''(y)| < C \ \forall y \in [a - r, a + r]$$

Thus,

$$|z'''(y) - i'''(y)| = \left| \int_a^y F(s, z'''(s)) - F(s, i'''(s)) ds \right|$$

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\[ \leq \int_a^y k|z'''(s) - i'''(s)|ds \]
\[ \leq kC|y - a| \]
\[
|z'''(y) - i'''(y)| = \left| \int_a^y F(s, z''(s)) - F(s, i''(s))ds \right| 
\]
\[ \leq \int_a^y k|z''(s) - i''(s)|ds \]
\[ \leq \int k^2C|s - a|ds \]
\[ \leq Ck^2 \frac{|y-a|^2}{2} \]

After repetition of this procedure, we have,
\[ |z'''(y) - i'''(y)| \leq \frac{Ck^n|y-a|^n}{n!} \rightarrow 0 \text{ as } n \rightarrow \infty \]

Hence we get \( z'''(y) = i'''(y) \) on \([a - r, a + r]\).

4. The solution depends continuously on \( Y''' \): Suppose \( z''' \) be the unique solution of Ordinary Differential Equation, with initial condition \( z'''(a) = Y''' \); let \( i''' \) be the unique solution with initial condition \( i'''(a) = Y''' + \delta \). Both \( z'''(a) \) and \( i'''(a) \) satisfies the equations
\[ z'''(y) = Y''' + \int_a^y F(s, z''(s))ds, \quad i'''(y) = Y''' + \delta + \int_a^y F(s, i''(s))ds \]

Hence,
\[ |z'''(y) - i'''(y)| \leq |\delta| + \int_a^y F(s, z''(s)) - F(s, i''(s))ds \]
\[ \leq |\delta| + k \int_a^y |z''(s) - i''(s)|ds \quad \text{(5.10)} \]
Since \( z''' \) and \( t''' \) are continuous functions, therefore \( |z'''(s) - t'''(s)| \) must be bounded,
\[
|z'''(y) - t'''(y)| \leq \text{Con}(a - r, a + r); \text{by putting in (5.11) we have}
\]
\[
|z'''(y) - t'''(y)| \leq |\delta| + kC|y - a|
\]
By repetition of this process, by induction, we show that
\[
|z'''(y) - t'''(y)| \leq |\delta| + \sum_{n=1}^{\infty} \frac{k^n |y - a|^n}{n!} 
\]
Therefore, \( |z'''(y) - t'''(y)| \) \( \leq |\delta|e^{k|y-a|} \) this gives that \( t''(y) \to z''(y) \) as \( \delta \to 0 \).

5. Here we assume that all the \( z^iv \) remain in \( (Y''' - l, Y''' + l) \), hence we apply the Lipschitz condition.

Suppose \( r = \min(h, l/M) \) where \( M = \max_{y \in (a-h, a+h)} |F(y, z)| \).

For \( y \in (a - r, a + r) \),
\[
z'''_0(y) = Y''' \in (Y''' - l, Y''' + l)
\]
\[
|z'''_{n+1}(y) - Y'''| = |\int_a^y F(s, z'''_n(s)) \, ds| 
\]
\[
\leq M|y - a| \leq Mr \leq l
\]
\[
\therefore z'''_{n+1}(y) \in (Y''' - l, Y''' + l) \text{ by induction on } n.
\]

5.6 Alternative Proof of Fundamental Theorem of Ordinary Differential Equations using Banach’s fixed point theorem

Let the set of continuous functions on some bounded closed interval \( I \subset \mathbb{R} \), and define
\[
||f|| := \max_{y \in I} |f(y)|. \text{Here we prove that } ||f + g|| \leq ||f|| + ||g||, \text{and } ||f_n - f|| \to 0
\]
when $f_n$ converges to $f$ uniformly. When the vector function $f$ is interpreted $|f(y)|$ is the Euclidean modulus of $f$.

**Banach’s Fixed Point Theorem**: If $T$ is a contraction mapping on $I$, i.e., there exists a constant $c < 1$ s.t.

$$||T(z''_1) - T(z''_2)|| \leq c||z''_1 - z''_2||,$$

Then the iteration $z''_{n+1} := T(z''_n)$ initially from $z''_0$, converges to $z''$, is unique fixed point of $T$, i.e., $T(z'') = z''$.

**Proof**: 

$$||z''_{n+1} - z''_n|| = ||T(z''_n) - T(z''_{n-1})||$$

$$\leq c||z''_n - z''_{n-1}||$$

$$\leq c^n||z''_1 - z''_0||$$

by induction. So for $n > m$

$$||z''_n - z''_m|| \leq ||z''_n - z''_{n-1}|| + \cdots + ||z''_{m+1} - z''_m||$$

$$= ||T(z''_{n-1}) - T(z''_{n-2})|| + \cdots + ||T(z''_{m}) - T(z''_{m-1})||$$

$$\leq (c^{n-1} + \cdots + c^m)||z''_1 - z''_0||$$

$$\leq \frac{c^m}{1-c}||z''_1 - z''_0|| \to 0 \text{ as } n, m \to \infty$$

Therefore, $|z''_n(y) - z''_m(y)| \leq ||z''_n - z''_m|| \to 0$ for each point $y$, therefore the convergence $z''_n(y) \to z''(y)$. This is a uniform convergence in $y$, i.e.,

$||z''_n - z''|| \to 0 \text{ as } n \to \infty$.

then the follows that

$$||T(z''_n) - T(z'')|| \leq c||z''_n - z''|| \to 0$$

Therefore as $n \to \infty$, the equation $z''_{n+1} = T(z''_n)$ may be written as $z''' = T(z'')$.

If $i = T(i)$, this fixed point is unique. Then,
\[ \|z''' - i'''\| = \|T(z'') - T(i'')\| \leq c\|z''' - i'''\| \]

\[ \therefore 0 \leq (1 - c)\|z''' - i'''\| \leq 0 \]

\[ \therefore \max_{y \in I}|z'''(y) - i'''(y)| = \|z''' - i'''\| = 0 \]

and \(z''' = i'''\) on \(I\).

**Proof**: of the Fundamental Theorem of Ordinary Differential Equations

Suppose

\[ T(z'') := Y''' + \int_a^y F(s, z''(s)) \, ds \]

On \(y \in [a - h, a + h]\) value of \(h\) taking later on. Then

\[ |T(z''_1) - T(z''_2)| = |\int_a^y F(s, z''_1(s)) - F(s, z''_2(s))| \, ds| \]

\[ \leq \int_a^y |F(s, z''_1(s)) - F(s, z''_2)| \, ds \]

\[ \leq k|z''_1(s) - z''_2(s)| \, ds \]

\[ \leq k|y - a|\|z''_1 - z''_2\| \]

\[ \therefore \|T(z''_1) - T(z''_2)\| \leq kh\|z''_1 - z''_2\| \]

By taking \(h\) s.t. \(h < 1/k\), then \(T\) becomes contraction map. Consider the Picard iteration \(z''_{n+1} = T(z''_n)\). Here each iteration is a continuous function since \(F\) and integration are continuous operations. Then by using Banach fixed point theorem these iterations uniformly converges to \(z''(\|z''_n - z''\| \to 0 \text{ as } n \to \infty)\). This unique function is the fixed point of \(T\), i.e., \(z''' = T(z''') = Y''' + \int_a^y F(s, z''(s)) \, ds\). On differentiating we get \(z^{iv}(y) = F(y, z''(y))\) by the fundamental theorem of calculus : \(z''(a) = Y''\).

If \(F\) is Lipschitz only in a nhd. of the starting point \((a, Y''')\), then each iteration \(z'''_n\) remains within this nhd. It is done by induction on \(n\),

\[ |z''_{n+1}(y) - Y'''| \leq \int_a^y |F(s, z'''_n(s))| \, ds \]

\[ \leq hc \leq l \]

Suppose \(h \leq l/c\), where \(c\) is max. value of \(|F(y, z'')|\) on rectangular nhd.
For proving that \( z'' \) depends continuously on \( Y'' \), suppose \( i \) is the unique solution of
\[
i'' = F(y, i''') \quad \text{where} \quad i'''(a) = Y''' + \delta, \text{then} \quad i''' = Y''' + \delta + \int_a^y F(s, i'''(s)) \, ds.
\]

Therefore,
\[
\left| |z''' - i'''| - |T(z''' - i''') - \delta| \right|
\leq |\delta| + |T(z''') - T(i'''')|
\leq |\delta| + c|z'' - i'''|
\therefore \quad |z''' - i'''| \leq \frac{|\delta|}{1 - c}
\]
hence, \( i''' \to z'' \) uniformly as \( \delta \to 0 \).

5.7 Conclusion:

In this work, an efficient simple method is presented for solving different orders of nonlinear ordinary differential equations. Some examples are given to show the simplicity and easiest way for computation.

By using different conditions, formulas, and theorems, the computations associated with the examples are performed. The solutions of nonlinear ordinary differential equations are extended by using existence and uniqueness.