BILINEAR DEGENERATED SHELL ELEMENT

A simple and efficient four noded shell element was presented by Kanok-Nukulchai [3] and its application to a number of practical cases was illustrated. The following two assumptions are made:

1. Normals to the midsurface remain straight after deformation. Thus, the formulation include transverse shear deformation and Kirchhoff—Love hypothesis is not assumed.
2. Stresses normal to the midsurface are zero.

**B.1 Shape Functions for Geometry and Displacement**

The four noded element is evolved from an eight noded solid element. The midsurface enclosed by four straight sides forms a hyperbolic paraboloid. The shell element is shown in Fig. B.1 The shape function to describe the midsurface in terms of natural coordinates is the same as given by Eq. B.0b) for two dimensional isoparametric element. Thus,

\[
N_i = \frac{1}{4} (1 + r_i) (1 + s_i) \quad i = 1, \ldots, 4
\]

where \(r_i\) and \(s_i\) are the natural coordinates of node \(i\)

The thickness of the shell element in the direction normal to the midsurface \(t\) is required and is specified as input. Using the shape functions, the thickness at point \(i\) is

\[
\begin{align*}
N_1 &= \frac{(1-r)(1-s)}{4} \\
N_2 &= \frac{(1+r)(1-s)}{4} \\
N_3 &= \frac{(1-r)(1+s)}{4} \\
N_4 &= \frac{(1+r)(1+s)}{4}
\end{align*}
\]
functions, the coordinates of any point in the element can be uniquely given in terms of nodal coordinates and thicknesses as,

$$\begin{align*}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
x_i \\
y_i \\
z_i \\
\end{bmatrix} + \frac{1}{2} h_i \begin{bmatrix}
l_{3i} \\
m_{3i} \\
n_{3i} \\
\end{bmatrix}
\end{align*}
$$

(B.2)

where \( x_i, y_i, z_i \) are the global coordinates of the midsurface node \( i \).

\( h_i \) is the thickness at node \( i \) and

\( l_{3i}, m_{3i} \) and \( n_{3i} \) are the normal unit vector at node \( i \).

Fig. B.1: Four nodded shell element

At any point \((r, s)\) on the midsurface \((t=0)\) an orthogonal set of local coordinate axes \( x', y', z' \) are constructed, \( e_3' \) is the normal unit vector and \( e_1' \) and \( e_2' \) are tangent to the midsurface. It is well known from vector algebra that the cross product of two vectors gives a vector oriented normally to the plane given by the two vectors and unit vector is obtained by dividing it by its scalar length. For details of vector products reference \([13]\) may be consulted. Thus,
The partial derivatives such as \( \frac{\partial x}{\partial r} \), etc. can be obtained from Eq. (B.1). Now the direction cosines of the new axes \( x', y', z' \) with respect to \( x, y, z \) are defined by \([D]\) matrix, as

\[
[D] = \begin{bmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
  n_1 & n_2 & n_3
\end{bmatrix}
\]
The displacement variation in the element can be expressed as

$$\begin{bmatrix}
  u_i \\
  v_i \\
  w_i
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
  u_i \\
  v_i \\
  w_i \\
  u_i^* \\
  v_i^* \\
  w_i^*
\end{bmatrix}$$  \hspace{1cm} (B.5)$$

where \( u_i, v_i, w_i \) are the displacements of the node \( i \) on the midsurface along the global \( x, y, z \) directions; \( u_i^*, v_i^*, w_i^* \) are the relative nodal displacements along \( x, y, z \) directions produced by the rotation of the normal at the node \( i \).

The displacements \( u_i^*, v_i^*, w_i^* \) are to be expressed explicitly in terms of the rotations \( \theta_{xi}, \theta_{yi}, \theta_{zi} \) at each node \( i \) about the global axes. Using the shell assumption that straight normals to the midsurface remain straight after deformation, the displacements produced by the normal rotations \( \alpha_{xi}^i \) and \( \alpha_{yi}^i \) can be calculated as (Fig. B.2)

$$\begin{bmatrix}
  u_i' \\
  v_i' \\
  w_i'
\end{bmatrix} = \frac{1}{2} \theta h_i \begin{bmatrix}
  \alpha_{xi}^i \\
  \alpha_{yi}^i \\
  0
\end{bmatrix}$$  \hspace{1cm} (B.6)$$

where \( u_i', v_i', w_i' \) are displacement components along \( x', y', z' \) at node \( i \) and \( \alpha_{xi}^i, \alpha_{yi}^i \) are rotations about \( x' \) and \( y' \) respectively.

The components of these displacements along the global directions, \( u_i^*, v_i^*, w_i^* \), can now be got by knowing the direction cosines of \( x', y', z' \), with respect to \( x, y, z \) (Eq. B.4)

$$\begin{align*}
  u_i^* &= l_{1i} u_i' + l_{2i} v_i' \\
  v_i^* &= m_{1i} u_i' + m_{2i} v_i' \\
  w_i^* &= n_{1i} u_i' + n_{2i} v_i'
\end{align*}$$  \hspace{1cm} (B.7)$$

Substituting from Eq. (B.6) into Eq. (B.7) and arranging the terms in matrix form we get

$$\begin{bmatrix}
  u_i^* \\
  v_i^* \\
  w_i^*
\end{bmatrix} = \frac{1}{2} \theta h_i \begin{bmatrix}
  l_{1i} & -l_{2i} \\
  m_{1i} & -m_{2i} \\
  n_{1i} & -n_{2i}
\end{bmatrix} \begin{bmatrix}
  \alpha_{xi}^i \\
  \alpha_{yi}^i
\end{bmatrix}$$  \hspace{1cm} (B.8)$$

We can now express \( \alpha_{xi}^i \) and \( \alpha_{yi}^i \) in terms of global rotations \( \theta_{xi}, \theta_{yi}, \theta_{zi} \) as,

$$\begin{align*}
  \alpha_{xi}^i &= l_{1i} \theta_{xi} + m_{1i} \theta_{yi} + n_{1i} \theta_{zi} \\
  \alpha_{yi}^i &= l_{2i} \theta_{xi} + m_{2i} \theta_{yi} + n_{2i} \theta_{zi}
\end{align*}$$  \hspace{1cm} (B.9)$$
Arranging the terms in matrix form we get

\[
\begin{pmatrix}
\alpha_{21} \\
\alpha_{11}
\end{pmatrix} =
\begin{bmatrix}
I_{21} & m_{21} \\
I_{11} & m_{11}
\end{bmatrix}
\begin{pmatrix}
\theta_{x1} \\
\theta_{z1}
\end{pmatrix}
\]

\[
\text{(B.10)}
\]

Substituting Eq. (B.10) into Eq. (B.8) we get

\[
\begin{pmatrix}
u_i^* \\
w_i^* \\
w_i^*
\end{pmatrix} = \frac{1}{2} h_l \left[ D_l \right] \begin{pmatrix}
\theta_{x1} \\
\theta_{z1}
\end{pmatrix}
\]

\[
\text{(B.11)}
\]
where

\[
[D_i] = \begin{bmatrix}
I_{11} & -l_{21} & m_{11} & -m_{21} & n_{11} & -n_{21} \\
-m_{21} & I_{21} & n_{21} & -n_{11} & l_{11} & -l_{21} \\
-I_{11} & I_{21} & n_{11} & -n_{21} & l_{21} & -l_{11}
\end{bmatrix}
\]

which results in

\[
[D_i] = \begin{bmatrix}
0 & n_{31} & -m_{31} \\
-n_{31} & 0 & l_{31} \\
m_{31} & -l_{31} & 0
\end{bmatrix}
\]

(B.12)

The terms \(l_{31}, m_{31}, \) and \(n_{31}\) are direction cosines of unit vector \(e_3\) as defined in equation (B.3a) and are to be evaluated at node \(i\).

Substituting Eq. (B.12) into Eq. (B.5) the displacement variation is expressed in terms of nodal values, as,

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \frac{1}{2} th_i [D_i] \begin{bmatrix}
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1}
\end{bmatrix}
\]

Substituting from Eq. (11.12) we get,

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \frac{1}{2} th_i \begin{bmatrix}
0 & 0 & 0 \\
l_{31} & \theta_{z1} & -n_{31} \\
m_{31} & \theta_{z1} & -l_{31}
\end{bmatrix} \begin{bmatrix}
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1}
\end{bmatrix}
\]

(B.13(a))

B.3.2 Strain—Displacement Matrix

Assuming \(\varepsilon_z = 0\), the strain components along the local axes of the shell element are given by,

\[
\{\varepsilon'\} = \begin{bmatrix}
\varepsilon_{x'} \\
\varepsilon_{y'} \\
\gamma_{x'y'} \\
\gamma_{x'z'} \\
\gamma_{y'z'}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial x'} \\
\frac{\partial v'}{\partial x'} \\
\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} \\
\frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} \\
\frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}
\end{bmatrix}
\]

(B.14)
The strain components in the local axes system can be obtained through the Eq. B14.a and the [D] matrix is given by Eq. (B.4). The derivative of \( u, v, w \) with respect to \( x, y, z \) to be used in Eq. B14.a are computed from Eq. B14.b. For this purpose the derivatives of \( u, v, w \) with respect to \( r, s, t \) are required and they are obtained by differentiating Eq. (B.13). Thus,

\[
\begin{bmatrix}
\frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\
\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial w}{\partial s} \\
\frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} & \frac{\partial w}{\partial t}
\end{bmatrix}
= \sum_{i=1}^{4}
\begin{bmatrix}
\frac{\partial N_i}{\partial r} u_i \\
\frac{\partial N_i}{\partial s} v_i \\
\frac{\partial N_i}{\partial t} w_i
\end{bmatrix}
\]

Thus,

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial r} (n_3 \theta_{y_i} - m_3 \theta_{z_i}) \\
\frac{\partial N_i}{\partial s} (l_3 \theta_{x_i} - n_3 \theta_{z_i}) \\
\frac{\partial N_i}{\partial t} (m_3 \theta_{x_i} - l_3 \theta_{y_i})
\end{bmatrix}
= \sum_{i=1}^{4} \frac{h_i}{2} \begin{bmatrix}
\frac{\partial N_i}{\partial r} (n_3 \theta_{y_i} - m_3 \theta_{z_i}) \\
\frac{\partial N_i}{\partial s} (l_3 \theta_{x_i} - n_3 \theta_{z_i}) \\
\frac{\partial N_i}{\partial t} (m_3 \theta_{x_i} - l_3 \theta_{y_i})
\end{bmatrix}
\]

Making use of the procedure and equations indicated above, all the derivatives necessary to compute \( \{\epsilon'\} \) of Eq. B.14 can be obtained. The strain displacement matrix \([B]\) may be split up conveniently into two matrices \([B_m]\) and \([B_s]\) such that

\[
\{\epsilon'_m\} = \{\epsilon'_s\} = \sum_{i=1}^{4} [B_m] \{d_i\} \quad \text{(B.16a)}
\]

\[
\{\gamma'_{s'}\} = \sum_{i=1}^{4} [B_s] \{d_i\} \quad \text{(B.16b)}
\]

where \(\{d_i\}\) represents the global displacements and rotations at each node.

The strain-displacement matrices \([B_{mi}]\) is further split as

(i) \([B_{1mi}]\), (ii) \([B_{2mi}]\), and (iii) \([B_{3mi}]\).

\([B_{1mi}]\) is formed considering only in plane displacements \(u_i, v_i, \text{ and } w_i\).

\([B_{2mi}]\) and \([B_{3mi}]\) are formed considering rotations \(\theta_{x_i}, \theta_{y_i}, \theta_{z_i}\).
Similarly the strain-displacements matrix \([B_{11}]\) is split into (i) \([B_{11l}]\), (ii) \([B_{21l}]\), and (iii) \([B_{31l}]\).

\([B_{11l}]\) is formed considering only in plane displacements and
\([B_{21l}]\) and \([B_{31l}]\) are formed considering rotations only.

(a) **Formulation of \([B_{1ml}]\)** Making use of Eq. B .15, the derivatives of \(u'\) and \(v'\) with respect to \(x'\) and \(y'\) are computed. Arranging the terms with respect to inplane displacements \(u_i, v_i\) and \(w_i\), \([B_{1ml}]\) matrix is constructed. A typical term is given below:

\[
\frac{\partial u'}{\partial x'} = \sum_{l=1}^{4} \left[ l \left( \frac{\partial N_i}{\partial x} l_i u_i + \frac{\partial N_i}{\partial x} m_i v_i + \frac{\partial N_i}{\partial x} n_i w_i \right) \right] + m_i \left( \frac{\partial N_i}{\partial y} l_i u_i + \frac{\partial N_i}{\partial y} m_i v_i + \frac{\partial N_i}{\partial y} n_i w_i \right) + n_i \left( \frac{\partial N_i}{\partial z} l_i u_i + \frac{\partial N_i}{\partial z} m_i v_i + \frac{\partial N_i}{\partial z} n_i w_i \right)
\]

(B.17)

Similarly \(\frac{\partial v'}{\partial x'}\) and \(\left( \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} \right)\) can be computed. Now the inplane strains due to \(u_i, v_i\) and \(w_i\) can be expressed as,

\[
\begin{align*}
\begin{bmatrix}
\in_x' \\
\in_y' \\
\gamma_{x'y'}
\end{bmatrix}
&= \begin{bmatrix}
\frac{\partial u'}{\partial x'} \\
\frac{\partial v'}{\partial y'} \\
\frac{\partial u' + \partial v'}{\partial x'}
\end{bmatrix} = \sum_{l=1}^{4} \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} \begin{bmatrix}
l_1B'(1, i) & m_1B'(1, i) & n_1B'(1, i) \\
l_2B'(1, i) & m_2B'(1, i) & n_2B'(1, i) \\
l_1B'(2, i) & m_1B'(2, i) & n_1B'(2, i) \\
l_2B'(2, i) & m_2B'(2, i) & n_2B'(2, i)
\end{bmatrix}
\end{align*}
\]

(B.18)

where

\[
[B_{1ml}] = \begin{bmatrix}
l_1B'(1, i) & m_1B'(1, i) & n_1B'(1, i) \\
l_2B'(2, i) & m_2B'(2, i) & n_2B'(2, i) \\
l_1B'(2, i) & m_1B'(2, i) & n_1B'(2, i) \\
l_2B'(1, i) & m_2B'(1, i) & n_2B'(1, i)
\end{bmatrix}
\]

(B.19)

and

\[
B'(1, i) = \frac{\partial N_i}{\partial x} l_1 + \frac{\partial N_i}{\partial y} m_1 + \frac{\partial N_i}{\partial z} n_1
\]

\[
B'(2, i) = \frac{\partial N_i}{\partial x} l_2 + \frac{\partial N_i}{\partial y} m_2 + \frac{\partial N_i}{\partial z} n_2
\]

(B.20)
(b) Formulation of $[B_{2ml}]$ and $[B_{3ml}]$. The same procedure as in the case of $[B_{1ml}]$ is followed and here the contributions due to rotations are considered. A typical term is given below

$$
\frac{\partial u'}{\partial x'} = \sum_{i=1}^{4} \frac{h_i}{2} \left[ l_i \left( \frac{\partial N_i}{\partial x} + J_{1i}^{\ast} N_i \right) + m_i \left( \frac{\partial N_i}{\partial y} + J_{2i}^{\ast} N_i \right) + n_i \left( \frac{\partial N_i}{\partial z} + J_{3i}^{\ast} N_i \right) \right] \times [(n_3, \theta_y - m_3, \theta_z) N_i]
$$

$$
+ (l_3, \theta_{z2l} - n_3, \theta_{zl}) m_i + (m_3, \theta_{zl} - l_3, \theta_{z2l}) n_i
$$

(B.21)

Arranging the relation between the inplane strains and rotations in matrix form we get,

$$
\begin{bmatrix}
\epsilon_x' \\
\epsilon_y' \\
\gamma_{x'y}'
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial x'} \\
\frac{\partial u'}{\partial y'} \\
\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}
\end{bmatrix} = \sum_{i=1}^{4} \left[ [B_{2ml}] + i[B_{3ml}] \right] \begin{bmatrix}
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1}
\end{bmatrix}
$$

(B.22)

In $[B_{2ml}]$ we have terms such as $l_i J_{1i}^{\ast} + m_i J_{2i}^{\ast} + n_i J_{3i}^{\ast}$ and $l_2 J_{2l}^{\ast} + m_2 J_{2l}^{\ast} + n_2 J_{3l}^{\ast}$.

It can be shown that $J_{1i}^{\ast} = l_i$, $J_{2l}^{\ast} = m_3$, and $J_{3l}^{\ast} = n_3$.

And by orthogonality condition we have,

$$
l_1 l_3 + m_1 m_3 + n_1 n_3 = 0
$$

$$
l_2 l_3 + m_2 m_3 + n_2 n_3 = 0
$$

(B.23)

(B.24)

Hence, it can be observed that $[B_{2ml}] = [0]$.

Therefore, Eq. (B.22) reduces to,

$$
\begin{bmatrix}
\epsilon_x' \\
\epsilon_y' \\
\gamma_{x'y}'
\end{bmatrix} = \sum_{i=1}^{4} t \left[ B_{3ml} \right] \begin{bmatrix}
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1}
\end{bmatrix}
$$

(B.25)

where

$$
[B_{3ml}] = \frac{1}{2} \begin{bmatrix}
B'(1, i)(m_3, n_1 - n_3, m_1) & B'(1, i)(n_3, l_1 - l_3, n_1) & B'(1, i)(l_3, m_1 - m_3, l_1) \\
B'(2, i)(m_3, n_1 - n_3, m_1) & B'(2, i)(m_3, n_2 - n_3, m_2) & B'(2, i)(n_3, l_2 - l_3, n_2) \\
B'(2, i)(m_3, n_1 - n_3, m_1) & B'(2, i)(n_3, l_1 - l_3, n_1) & B'(2, i)(l_3, m_1 - m_3, l_1) \\
B'(1, i)(m_3, n_1 - n_3, m_1) & B'(2, i)(m_3, n_2 - n_3, m_2) & B'(1, i)(m_3, n_2 - n_3, m_2)
\end{bmatrix}.
$$

(B.26)
(c) Formulation of \([B_{1il}]\)  Similarly the derivatives are worked out for computing \(\gamma_{x',z'}\) and \(\gamma_{y',z'}\), due to displacements \(u', v', w'\). Thus,

\[
\begin{bmatrix}
\gamma_{x',z'} \\
\gamma_{y',z'}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \\
\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}
\end{bmatrix} = \sum_{i=1}^{4} [B_{1il}] \begin{bmatrix}
u_l \\
w_l
\end{bmatrix} \tag{B.27}
\]

where

\[
[B_{1il}] = \begin{bmatrix}
l_1B'(3, i) + m_1B'(1, i) & n_1B'(3, i) + n_2B'(1, i) \\
l_1B'(3, i) & m_2B'(3, i) + n_2B'(1, i)
\end{bmatrix}
\]

and \(B'(3, i) = \frac{\partial N_i}{\partial x} l_1 + \frac{\partial N_i}{\partial y} m_1 + \frac{\partial N_i}{\partial z} n_3 \tag{B.29}\)

(d) Formulation of \([B_{2il}]\) and \([B_{3il}]\)  The strains \(\gamma_{x',z'}\) and \(\gamma_{y',z'}\) due to rotations \(\theta, \phi, \psi\) are expressed as,

\[
\begin{bmatrix}
\gamma_{x',z'} \\
\gamma_{y',z'}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \\
\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}
\end{bmatrix} = \sum_{i=1}^{4} [[B_{2il}] + t[B_{3il}]] \begin{bmatrix}
\theta_{x'i} \\
\theta_{y'i}
\end{bmatrix} \tag{B.30}
\]

The matrices \([B_{2il}]\) and \([B_{3il}]\) are constructed in a similar manner as before and are given as follows:

\[
[B_{2il}] = \frac{l_i}{2} N_i B^* \begin{bmatrix}
(m_3n_1-n_3m_1) & (n_3l_1-l_3n_1) & (l_3m_1-m_3l_1) \\
(m_3n_2-n_3m_2) & (n_3l_2-l_3n_2) & (l_3m_2-m_3l_2)
\end{bmatrix} \tag{B.31}
\]

where \(B^* = m_3J^*_3 + n_3J^*_3\) \tag{B.32}\)

and,

\[
[B_{3il}] = \frac{l_i}{2} N_i B^* \begin{bmatrix}
B'(3, i)(m_3n_1-n_3m_1) & B'(3, i)(n_3l_1-l_3n_1) & B'(3, i)(l_3m_1-m_3l_1) \\
B'(3, i)(m_3n_2-n_3m_2) & B'(3, i)(n_3l_2-l_3n_2) & B'(3, i)(l_3m_2-m_3l_2)
\end{bmatrix} \tag{B.33}
\]

We can now arrange the strain-displacement matrix in the following form:

\[
\begin{bmatrix}
\{\varepsilon_m\} \\
\{\varepsilon_i\}
\end{bmatrix} = \sum_{i=1}^{4} \begin{bmatrix}
[B_{1il}] & t[B_{3il}]
\end{bmatrix} \begin{bmatrix}
d'' \\
d'
\end{bmatrix} \tag{B.34}
\]
\[ d' = [v, w] \]
\[ \{d\} = [\theta_x, \theta_y, \theta_z] \]  
(B.34)

### B.3.3 Stress-Displacement Matrix

The element stresses and nodal displacements are related as

\[
\{\sigma\} = [C] [B] \{d\} = [CB] \{d\} \quad \text{where } [C] \text{ is the constitutive matrix}
\]

For the case of isotropic material, the constitutive relation is given by Eq. B.34. Now imposing the condition that \(\sigma_x = 0\), the following relation is obtained for the stress-strain relation in \(x', y', z'\) coordinates;

\[
\{\sigma'\} = [C] \{\varepsilon'\}
\]

i.e.,

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{x'y'} \\
\tau_{x'z'} \\
\tau_{y'z'} \\
\end{bmatrix} = \begin{bmatrix}
1 & \mu & 0 & 0 & 0 \\
\mu & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1-\mu}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha(1-\mu)}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\gamma(1-\mu)}{2} \\
\end{bmatrix} \begin{bmatrix}
\varepsilon_x' \\
\varepsilon_y' \\
\gamma_{x'y'} \\
\gamma_{x'z'} \\
\gamma_{y'z'} \\
\end{bmatrix}
\]

(B.35)

Where \(\alpha\) is a numerical correction factor used to account for a better representation of shear deformation and is explained in Chapters [7 and 10]. A value of 5/6 has been suggested.

To facilitate adoption of different numerical integration schemes for bending and shear contributions to the stiffness matrix, the constitutive matrix is split into \([C_m]\) and \([C]\) as,

\[
[C] = \begin{bmatrix}
[C_m] & [0] \\
\frac{E}{1-\mu^2} & \frac{\mu}{1-\mu^2} \\
[0] & [C'] \\
\end{bmatrix}
\]

(B.36)

where

\[
[C_m] = \frac{E}{1-\mu^2} \begin{bmatrix}
1 & \mu & 0 \\
\mu & 1 & 0 \\
0 & 0 & \frac{1-\mu}{2} \\
\end{bmatrix}
\]

(B.37)
It may be noted that $E^*. B .37$ is the same as for the plane stress conditions given by $E^*. B .38.a$ and $E^*. B .38.b$ is similar to $E^*. B .38.a$ except that the terms are multiplied by $h$ to get the corresponding shear forces in the case of plate bending problems.

**B .3.4 Element Stiffness Matrix**

It is convenient to split the stiffness matrix into two parts; the bending and membrane effects and transverse shear effects. This will allow the use of appropriate order of numerical integration of each part.

Thus,

$$ [k] = [k]_m + [k]_t $$

i.e.,

$$ [k] = \sum_{i=1}^{4} \sum_{j=1}^{4} ([k]_{ij})_m + ([k]_{ij})_t $$

$$ ([k]_{ij})_m = \int_{\nu} [B_{ml}]^T [C_m] [B_{mj}] dV $$

$$ ([k]_{ij})_t = \int_{\nu} [B_{sl}]^T [C_s] [B_{sj}] dV $$

Where $E^*. B .40(a)$ gives the contribution due to bending and membrane effects and $E^*. B .40(b)$ gives transverse shear contribution to stiffness matrix.

Substituting from $E^*. B .34$ for bending and membrane contribution into $E^*. B .40(a)$ we get,

$$ ([k]_{ij})_m = \int_{\nu} \left[ \frac{[B_{1ml}]^T}{[l[B_{3ml}]^T]} \right] [C_m] \left[ \frac{[B_{1mj}]}{[l[B_{3mj}]]} \right] dV $$

Simplifying the above equation and expressing it in natural coordinates we get,

$$ ([k]_{ij})_m = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \left[ \frac{[B_{1ml}]^T [C_m] [B_{1mj}]}{[l[B_{3ml}]^T [C_m] [B_{3mj}]]} \right] dV $$

$$ |J(r,s,t)| dr ds dt $$

where $|J|$ is the determinant of the Jacobian matrix defined in $E^*. B .42.a$.**

To be consistent with the shell assumption $|J(r,s,t)|$ can be approximated by $|J(r,s,t)|$.**
Since \([B_{1m}]\) and \([B_{3m}]\) are functions of \(r\) and \(s\) only, the integral of Eq. B.42 can be analytically integrated with respect to \(t\) and thus we get,

\[
[k]_{lm} = \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix}
\frac{2[B_{1ml}]^T [C_m]}{J_{r, s, o}} & \begin{bmatrix} [B_{1ml}] & [B_{3ml}] \end{bmatrix} \\
0 & \frac{3[B_{3ml}]^T [C_m]}{J_{r, s, o}}
\end{bmatrix} \begin{bmatrix} [B_{1ml}] & [B_{3ml}] \end{bmatrix} \, dr \, ds \quad (B.43)
\]

Similarly substituting from Eq. B.34 for shear contribution into Eq. B.40(b) we get,

\[
[k]_{yj} = \int_{r} \begin{bmatrix}
[B_{1y}]^T & \begin{bmatrix} [B_{1y}] & [B_{3y}] \end{bmatrix} \\
2[B_{3y}]^T & 2[B_{3y}] + t[B_{3y}]
\end{bmatrix} \begin{bmatrix} [B_{1y}] & [B_{3y}] \end{bmatrix} \, dV \quad (B.44)
\]

Integrating across the thickness as before, we get

\[
[k]_{yj} = \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix}
\frac{2[B_{1y}]^T [C_j]}{J_{r, s, o}} & \begin{bmatrix} [B_{1y}] & [B_{3y}] \end{bmatrix} \\
2[B_{3y}] & 2[B_{3y}] + \frac{3[B_{3y}]^T [C_j]}{J_{r, s, o}}
\end{bmatrix} \begin{bmatrix} [B_{1y}] & [B_{3y}] \end{bmatrix} \, dr \, ds \quad (11.45)
\]

The size of each submatrix in Eqs. B.43 and B.45 is \(6 \times 6\). Thus the bending and membrane, and shear stiffness contributions to the element stiffness matrix can be computed as,

\[
[k]_m = \begin{bmatrix}
[k_{11}] & [k_{12}] & [k_{13}] & [k_{14}] \\
[k_{21}] & [k_{22}] & [k_{23}] & [k_{24}] \\
[k_{31}] & [k_{32}] & [k_{33}] & [k_{34}] \\
[k_{41}] & [k_{42}] & [k_{43}] & [k_{44}]
\end{bmatrix}
\]

The Eq. B.46 is similar to Eq. B.46 for the computation of plate bending stiffness. The submatrix \([k_{11}], [k_{12}], [k_{13}]\) and \([k_{14}]\) are evaluated by letting \(i = 1, j = 1, 2, 3, 4\). The other submatrices are evaluated in a similar manner.

Numerical integration procedure is used to evaluate the stiffness matrix of Eqs. B.43 and B.45. A \(2 \times 2\) Gauss quadrature is used to evaluate the integral B.43, bending and membrane contribution. To avoid shear locking effect as explained in chapter 10 one point Gauss quadrature is used to evaluate the integral B.45, shear contribution to the stiffness matrix.

**B.3.5 Torsional Stiffness**

The four noded degenerated shell element described earlier employs six degrees of freedom at each node. However, no stiffness corresponding to the torsional rotation degree of freedom exists in the local axes system in the formulation. But when non-planar elements join at a node \(i\), the resistance to this rotation is got due to transformation of stiffness coefficients paral-
rotation of the midsurface is \( \theta_\text{el} \) to global direction, i.e. along \( \theta_\text{el} \). When the finite element mesh is refined, the angles between elements may become close to \( 2\pi \) and this will result in a very small amount of stiffness for the torsional rotation. Therefore, any slight disturbance in the load corresponding to this degree of freedom will amplify the torsional rotation degree of freedom to an unrealistic value and thus will affect the global solution.

This problem is common to all shell elements which use six global degrees of freedom at each node. But the difficulty is got over by providing a fictitious torsional spring along the local normal direction at each node of the element. This technique increases the amount of work in data preparation and is also unsatisfactory since for a flexible system it produces an unrealistic amount of strain energy in the spring by a rigid body motion.

For the four noded shell element, the rotation of the normal and the mid-surface displacement field are independent. As shown in Fig. B.3(c) the rotation of the midsurface is \( \frac{1}{2} \left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) \).

The deviation of the torsional rotation of the normal from that of the mid-surface is assumed to have the governing strain energy,

\[
U_i = \alpha_i G h \int_{\Omega} \left[ \alpha_3 \frac{1}{2} \left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) \right]^2 dA \tag{B.47}
\]

where \( \alpha_i \) is known as torsional coefficient.

If \( \alpha_i, G, h \) is chosen to be large relative to the factor \( E h^3 \) used in bending energy calculations, Eq. B.47 will play the role of penalty function and results in the desired constraint at the Gauss points [3] as

\[
\alpha_i' \approx \frac{1}{2} \left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) \tag{B.48}
\]

The use of penalty functions in the potential energy functional and illustration of constraints for beam and plate are described in reference [12].

Now the torsional stiffness coefficient is derived from Eq. B.47. Following the procedure indicated in the earlier section we get,

\[
\left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) = \sum_{j=1}^{5} \left[ l_1 \left( \frac{\partial N_j}{\partial x} \frac{\partial N_j}{\partial x} m_1 \frac{\partial N_j}{\partial x} n_2 \frac{\partial N_j}{\partial x} w_1 \right) + m_1 \left( \frac{\partial N_j}{\partial y} \frac{\partial N_j}{\partial y} m_2 \frac{\partial N_j}{\partial y} n_2 \frac{\partial N_j}{\partial y} w_1 \right) + n_1 \left( \frac{\partial N_j}{\partial z} \frac{\partial N_j}{\partial z} m_2 \frac{\partial N_j}{\partial z} n_2 \frac{\partial N_j}{\partial z} w_1 \right) - l_2 \left( \frac{\partial N_j}{\partial x} \frac{\partial N_j}{\partial x} m_1 \frac{\partial N_j}{\partial x} n_1 \frac{\partial N_j}{\partial x} w_1 \right) + m_2 \left( \frac{\partial N_j}{\partial y} \frac{\partial N_j}{\partial y} m_1 \frac{\partial N_j}{\partial y} n_1 \frac{\partial N_j}{\partial y} w_1 \right) + n_2 \left( \frac{\partial N_j}{\partial z} \frac{\partial N_j}{\partial z} m_1 \frac{\partial N_j}{\partial z} n_1 \frac{\partial N_j}{\partial z} w_1 \right) \right] \tag{B.49}
\]
If $\alpha_{zz}$ is the rotation about the local $z'$ axis at node $i$, then it can be expressed in terms of the global rotations $\theta_{zl}$ and $\theta_{z'l}$ as,

$$\alpha_{zz}' = l_{3l} \theta_{zl} + m_{3l} \theta_{z'l} + n_{3l} \theta_{zl} \quad (B.50)$$

If $\alpha_{z}$ is the rotation at any point $(r, s)$ on the midsurface, then the variation can be expressed through the same function $N_i$, as

$$\alpha_{z} = N_i (l_3 \theta_{zl} + m_3 \theta_{z'l} + n_3 \theta_{zl}) \quad (B.51)$$

Arranging Eq. B.49 and B.50 in matrix form and substituting in Eq. B.47 we can express $U_t$ in terms of the torsional stiffness matrix as,

$$U_t = \{d\}^T [k], \{d\} \quad (B.52)$$

where, the submatrix $[k_{ij}]$, for torsional stiffness, is given by

$$[k_{ij}] = \alpha, G h \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} [R_{ml}]^T [R_{ml}] & [R_{ml}]^T [R_{nj}] \\ [R_{ml}]^T [R_{ml}] & [R_{ml}]^T [R_{nj}] \end{bmatrix} |d| dr ds \quad (B.53)$$

where $\alpha_i =$ torsional coefficient

$G =$ shear modulus

$h =$ thickness

$$[R_{ml}] = \frac{1}{2} \left[ B' (2, i) - B' (1, i) 0 \right] [D]^T \quad (B.54a)$$

$$[R_{nj}] = N_i [l_3, m_3, n_3] \quad (B.54b)$$

$$d A = |e' \times e'| = |d| dr ds$$

$$|d| = \sqrt{\left(\frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial x}{\partial s} \frac{\partial z}{\partial r} - \frac{\partial x}{\partial r} \frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial r} \frac{\partial z}{\partial s} - \frac{\partial y}{\partial s} \frac{\partial z}{\partial r}\right)^2} \quad (B.54c)$$

It can also be shown that

$$|d| = |J| \times \sqrt{(J_{11})^2 + (J_{22})^2 + (J_{33})^2} \quad (B.54d)$$

Now the torsional stiffness matrix $[k]$, is evaluated from $[k_{ij}]$, following the procedure described for evaluating bending and membrane contribution, and shear contribution by letting $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$. (Refer description under Eq. (B.46))

A $1 \times 1$ Gauss quadrature is used in evaluating $[k]$, at the centre of the element to avoid an overconstrained situation similar to shear locking behaviour explained earlier.

It was demonstrated in reference [3] that the converged solution is insensitive to $\alpha$, as long as $\alpha$, is large enough ($>0.1$) to sufficiently restrain the torsional modes. This experiment also indicates that the addition of torsional stiffness will not degrade the behaviour of the system.