CHAPTER 6

FAILURE LINE THEORY

6.1 Introduction

In an axially loaded thin-walled member, beyond overall buckling (Fig. 6.1a), at the vicinity of collapse, localised elastic buckling results (Fig. 6.1b) and failure lines get initiated. These on further loading develop into a failure pattern (Fig. 6.1c). These failure mechanisms are found to be in a definite shape as observed in the experimental test specimens. Each component plate fails in a pattern termed as basic mechanism which when fitted together, so that their deflections are compatible, lead to a structural failure mechanism. Collapse, accompanied by decrease in axial load with increasing lateral deflections, when represented graphically is known as a collapse curve. This part of structural response, the load shedding, gives an indication of the nature of failure, whether sudden (brittle) or gradual (ductile) [18, 31]. The collapse curves can also be used to predict the failure load corresponding to a limiting deflection defined by the material strengths.
In this chapter, failure patterns for thin-walled RC columns that are identified for different sections under study are presented. The characteristic load-lateral deflection interaction ($P$ versus lateral displacement) of basic mechanisms are derived for reinforced concrete on the basis of the rigid plastic theory. Both isotropic and orthotropic reinforcement patterns are covered. The structural failure mechanisms are derived from the basic plate mechanisms such that they satisfy equilibrium and compatibility criteria. Collapse curves (load shedding curves) are drawn for different geometric sections under study. From these collapse curves, the ultimate load is predicted, corresponding to characteristic deflection defined by a limiting curvature. Axially loaded and eccentrically loaded columns with positive eccentricity are considered. However, the study is limited to columns with hinged end supports only.

6.2 Assumptions

1. Steel yields along failure lines.
2. Maximum strain in concrete is 0.0035.
3. Elastic deformations when compared to plastic deformations are negligible, hence neglected. Therefore plastic deformation occurs along failure lines and the regions between failure lines remain flat.
6.3 Basic Mechanisms

Fig. 6.2 indicates some of the observed structural mechanisms in columns with various cross sections and Fig. 6.3 shows the basic mechanisms required to form the structural mechanisms. As the structural mechanisms are formed of basic mechanisms it becomes necessary to derive the load-deflection interaction equations for basic mechanisms in order to derive the characteristic equations for structural mechanisms. It is seen from the basic mechanisms that all the failure lines do not lie at right angles to the direction of thrust. So it also becomes necessary to derive an expression for the moment capacity of a plastic hinge not perpendicular to the line of thrust which will facilitate the formation of governing equations for basic mechanisms.

6.3.1 Failure line perpendicular to line of thrust

The axial load P and moment capacity $M_p'$ of the cross section of a rectangular plate of width $b$, thickness $t$, area of reinforcement $A_s$ at an effective depth $t'$ and effective cover $t_c$ (Fig. 6.4) are

Compressive force $C = 0.54 f_{ck} x_{ul} b$ .... (6.1)

Tensile force $T = f_{y} t_s b$ .... (6.2)
where

\[ f_{ck} = \text{ultimate 28 days cube compressive strength of concrete} \]

\[ f_y = \text{yield strength of steel} \]

\[ t_s = \text{equivalent thickness of steel} = \frac{A_s}{b} \]

For equilibrium of axial forces

\[ P = C - T \quad \ldots \quad (6.3) \]

\[ P = (0.54f_{ck}x_u - f_yt_s)b \quad \ldots \quad (6.4) \]

Where

\[ x_u = \text{depth of neutral axis} \]

For moment equilibrium about centroidal axis

\[ M_p' = C \left( \frac{t}{2} - 0.42x_u \right) + T \left( \frac{t}{2} - t_c \right) \ldots \quad (6.5) \]

where

\[ t_c = \text{effective cover} \]

Substituting Eq. (6.1) and Eq. (6.2) in Eq. (6.5)

\[ M_p' = 0.54f_{ck}x_u b \left( \frac{t}{2} - 0.42x_u \right) \]

\[ + f_yt_s b \left( \frac{t}{2} - t_c \right) \ldots \quad (6.6) \]
6.3.2 Failure line inclined to line of thrust

If a failure line is oriented at an angle $\theta$ to the cross section (Fig. 6.5), the moment capacity of the cross section $M_p''$ is determined from the vector diagram shown in Fig. 6.5b

$$M_p'' = M_p''' \sec \theta \quad \ldots \quad (6.7)$$

where

$$M_p''' = \text{moment capacity of failure line}$$

From Johansen's stepped yield criterion [23] $M_p'''$ is determined by taking into account reinforcement orthotropy as

$$M_p''' = M_p' (\cos^2 \theta + \mu \sin^2 \theta) \sec \theta \quad \ldots \quad (6.8)$$

where $\mu$ is approximated as the ratio of moment capacity in the transverse direction per unit length to the moment capacity in the loading direction per unit length.

Substituting Eq. (6.8) in Eq. (6.7)

$$M_p'' = M_p' (\cos^2 \theta + \mu \sin^2 \theta) \sec^2 \theta \quad \ldots \quad (6.9)$$

For reinforcement isotropy

$$\mu = 1$$
and
\[ M_p'' = M_p' \sec^2 \theta \quad \ldots (6.10) \]

Eq. (6.4), Eq. (6.6) and Eq. (6.9) are required to derive characteristic equations for basic mechanisms.

6.3.3 Basic mechanism B1 (Fig. 6.3a)

Considering the equilibrium of one half of mechanism
\[ P y_c = M_p' \quad \ldots (6.11) \]

where
\[ y_c = \text{lateral deflection of failure line} \]

Substituting Eq. (6.6) in Eq. (6.11)
\[ P y_c = 0.54 f_{ck} x_u b \left( \frac{t}{2} - 0.42 x_u \right) \]
\[ + f_{y} t_s b \left( \frac{t}{2} - t_c \right) \quad \ldots (6.12) \]

From Eq. (6.4)
\[ x_u = \frac{P + f_{y} t_s b}{0.54 f_{ck} b} \quad \ldots (6.13) \]

Substituting Eq. (6.13) in Eq. (6.12) and simplifying
\[ P = \left[ \frac{b}{2B} (C-y_c) \pm (C-y_c)^2 + 4BD \right] \quad \ldots (6.14) \]
where

\[ A = fyts \]  
\[ B = \frac{0.42}{0.54f_{ck}} \]  
\[ C = \frac{t}{2} - 2AB \]  
\[ D = A(t-AB-t_c) \]

In Eq. (6.14) positive sign is considered as it leads to positive minimum value of $P$.

6.3.4 Basic mechanism B2 (Fig. 6.3b)

Considering the equilibrium of a longitudinal strip ABC of width $dh$ and with central deflection $y_c$

\[ dP y_c = M_p'' + M_p' \]  
\[ dP y_c = k_1 M_p' \]

where $k_1$ accounting for reinforcement orthotropy is

\[ k_1 = 2 + \mu \tan^2 \theta \]

and for reinforcement isotropy

\[ k_1 = 1 + \sec^2 \theta \]
Replacing $y_C$ as $\frac{Y_c}{k_1}$ and $b$ as $dh$ in Eq. (6.14)

\[
dP = \frac{1}{2B} \left[ \left( C - \frac{Y_C}{k_1} \right) + \sqrt{(C-y_C)^2 + 4BD} \right] dh
\]

Substituting

\[
y_C = \frac{y}{b} h
\]

where

$y$ = lateral deflection at $D$

$h$ = distance of strip from $O$

and integrating

\[
P = \frac{1}{2B} \left[ b \left( C - \frac{F_1 b}{2} \right) + I_1 \right]
\]

where

\[
I_1 = -\frac{H^2}{2F_1} \left[ \frac{C-F_1 b}{H} \sqrt{1 + \left( \frac{(C-F_1 b)^2}{H^2} \right)} 
+ \log \left( \frac{(C-F_1 b)}{H} + \sqrt{1 + \left( \frac{(C-F_1 b)^2}{H^2} \right)} \right) 
- \frac{C}{H} \sqrt{1 + \frac{C^2}{H^2}} 
- \log \left( \frac{C}{H} + \sqrt{1 + \frac{C^2}{H^2}} \right) \right]
\]

\[\cdots (6.23)\]

\[\cdots (6.24)\]

\[\cdots (6.25)\]

\[\cdots (6.26)\]
where

\[ F_1 = \frac{Y}{k_1 b} \]  \hspace{0.5cm} \ldots (6.27) \\

\[ H = 4BD \]  \hspace{0.5cm} \ldots (6.28) \\

The moment of \( P \) about an axis through \( O \) is found by integrating the expression \( hdp \) whence

\[
(\text{Pe}) = \frac{1}{2B} \left[ b^2 \left( \frac{C}{2} - \frac{F_1 b}{3} \right) + \frac{1}{3F_1^2} \left[ \left( C - F_1 b \right)^2 + H^2 \right]^{3/2} - \left( C^2 + H^2 \right)^{3/2} + 3 CF_1 I_1 \right] \]  \hspace{0.5cm} \ldots (6.29)

6.3.5 Basic mechanism B3 (Fig. 6.3c)

Considering the equilibrium of a longitudinal strip ABC of width \( dh \) and with relative deflection between A and B as \( 2y_C \)

\[
dP(2y_C) = 2M_p'' \]  \hspace{0.5cm} \ldots (6.30) \\

\[
dPy_C = k_2 M_p' \]  \hspace{0.5cm} \ldots (6.31) \\

where \( k_2 \) accounting for reinforcement orthotropy is

\[
k_2 = 1 + \mu \tan^2 \theta \]  \hspace{0.5cm} \ldots (6.32)

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and for reinforcement isotropy

\[ k_2 = \sec^2 \theta \quad \ldots (6.33) \]

The derivation of the characteristic equation for this mechanism is similar to that of type B2, the only difference being in the value of \( k_1 \). The characteristic equations are derived by replacing \( k_1 \) by \( k_2 \) in Eq. (6.25) and Eq. (6.29).

The characteristic equations are

\[ P = \frac{1}{2B} \left[ b \left( C - \frac{F_2 b}{2} \right) + I_2 \right] \quad \ldots (6.34) \]

where

\[ I_2 = - \frac{H^2}{2F} \left[ \frac{C-F_2b}{H} \sqrt{1 + \frac{(C-F_2b)^2}{H^2}} \right. \]

\[ + \log \left( \frac{C-F_2b}{H} + \sqrt{1 + \frac{(C-F_2b)^2}{H^2}} \right) \]

\[ - \frac{C}{H} \sqrt{1 + \frac{c^2}{H^2}} - \log \left( \frac{C}{H} + \sqrt{1 + \frac{c^2}{H^2}} \right) \] \quad \ldots (6.35)

\[ F_2 = \frac{Y}{k_2b} \quad \ldots (6.36) \]
and

\[
(\text{Pe}) = \frac{1}{2B} \left[ b^2 \left( \frac{C}{2} - \frac{F_2b}{3} \right) \right. \\
+ \frac{1}{3F_2^2} \left[ \left( \frac{(C - F_2b)^2 + H^2}{3/2} \right) - \left( \frac{C^2 + H^2}{3CF_2I_2} \right)^{3/2} \right] \right] \quad \ldots \ldots (6.37)
\]

### 6.3.6 Basic mechanism B4 (Fig. 6.3d)

Considering the equilibrium of a longitudinal strip ABC of width \(dh\) and with central deflection \(y_C\)

\[
dP \text{y}_C = M_p'' + M_p' = k_1 M_p' \quad \ldots \ldots (6.38)
\]

Replacing \(y_C\) as \(\frac{y_C}{k_1}\) and \(b\) as \(dh\) in Eq. (6.14)

\[
dP = \frac{1}{2B} \left[ \left( \frac{C - \frac{y_C}{k_1}}{\sqrt{\left( \frac{C - \frac{y_C}{k_1} \right)^2 + 4BD}} \right) \right] \text{dh} \quad \ldots \ldots (6.39)
\]

Substituting

\[
y_C = \frac{y_8}{\alpha} \quad \ldots \ldots (6.40)
\]

and \(dh = \text{kdis} \quad \ldots \ldots (6.41)\)
where

\[ y = \text{lateral deflection of D} \]
\[ \beta = \text{angle subtended by arc OC} \]
\[ \alpha = \text{semi-central angle} \]
\[ R = \text{radius of the section} \]

Integrating Eq. (6.39)

\[ P = \frac{R}{2B} \left[ \alpha \left( C - \frac{F_3 \alpha}{2} \right) + I_3 \right] \quad \ldots (6.42) \]

where

\[ I_3 = -\frac{H^2}{2F} \left[ \frac{C-F_3 \alpha}{H} \sqrt{1 + \frac{(C-F_3 \alpha)^2}{H^2}} \right. \]
\[ + \log \left( \frac{C-F_3 \alpha}{H} + \sqrt{1 + \frac{(C-F_3 \alpha)^2}{H^2}} \right) \]
\[ - \frac{C}{H} \sqrt{1 + \frac{C^2}{H^2}} - \log \left( \frac{C}{H} + \sqrt{1 + \frac{C^2}{H^2}} \right) \]
\[ \ldots (6.43) \]

\[ F_3 = \frac{y}{k_1 \alpha} \quad \ldots (6.44) \]

The moment of \( P \) about an axis through \( O \) is found by integrating the expression \( hdp \), whence

\[ (Pe) = \frac{R^2}{2B} \left[ I_4 - I_5 \right] \quad \ldots (6.45) \]
where

\[
I_4 = \alpha \left( C - \frac{F_3 \alpha}{2} \right) + I_3 \quad \ldots \ldots (6.46)
\]

\[
I_5 = I_{51} + I_{52} \quad \ldots \ldots (6.47)
\]

where

\[
I_{51} = (C - F_3 \alpha) \sin \alpha - F_3 \cos \alpha + F_3 \quad \ldots \ldots (6.47a)
\]

\[
I_{52} = \alpha/2 \left[ 0.55 \cos(0.88\alpha) \sqrt{(C - F_3 0.88\alpha)^2 + H^2} 
+ 0.88\cos(0.5\alpha) \sqrt{(C - 0.05\alpha F_3)^2 + H^2} 
+ 0.55\cos(0.11\alpha) \sqrt{(C - 0.11\alpha F_3)^2 + H^2} \right] \quad \ldots \ldots (6.47b)
\]

6.3.7 Basic mechanism B5 (Fig. 6.3e)

Considering the equilibrium of longitudinal strip ABC of width \(dh\) and with the relative deflection between A and B as \(2y_c\)

\[
dP y_c = k_2 M_p' \quad \ldots \ldots (6.48)
\]

The derivation of the characteristic equation for this mechanism is similar to that for type B4, the only difference being in the value of \(k_1\). Therefore replacing \(k_1\) by \(k_2\) in Eq. (6.42) and Eq. (6.45), the characteristic equations are derived as

\[
P = \frac{R}{2B} \left[ \alpha \left( C - \frac{F_4 \alpha}{2} \right) + I_6 \right] \quad \ldots \ldots (6.49)
\]
where

\[ I_6 = - \frac{H^2}{2F} \left[ \frac{C-F_4\alpha}{H} \sqrt{1 + \frac{(C-F_4\alpha)^2}{H^2}} \right. \]

\[ + \log \left( \frac{C-F_4\alpha}{H} + \sqrt{1 + \frac{(C-F_4\alpha)^2}{H^2}} \right) \]

\[ - \frac{C}{H} \sqrt{1 + \frac{C^2}{H^2}} - \log \left( \frac{C}{H} + \sqrt{1 + \frac{C^2}{H^2}} \right) \right] \]

\[ \cdots \cdots (6.50) \]

\[ F_4 = \frac{y}{k_2\alpha} \]

\[ \cdots \cdots (6.51) \]

and

\[ (Pe) = \frac{R^2}{2B} [I_7 - I_8] \]

\[ \cdots \cdots (6.52) \]

where

\[ I_7 = \alpha \left( C - \frac{F_4\alpha}{2} \right) + I_6 \]

\[ \cdots \cdots (6.53) \]

\[ I_8 = I_{81} + I_{82} \]

\[ \cdots \cdots (6.54) \]

where

\[ I_{81} = (C - F_4\alpha)\sin\alpha - F_4\cos\alpha + F_4 \]

\[ \cdots \cdots (6.54a) \]
6.3.8 Basic mechanism B6 (Fig. 6.3f)

This mechanism is nicknamed as flip disc mechanism by Murray [29, 31] and is normally found to occur in webs of channel and Z sections when the web fails in compression. This mechanism takes the form of curve ABEDGF as shown in Fig. 6.3f. Because of the skew symmetric nature of Z sections, point A lies away from mid depth of the plate. Its location is defined by coordinates a_4 and a_3. For simplicity of analysis this mechanism is idealised into straight lines FA, AE, EG and GF.

For failure lines AE and EG, considering the equilibrium of a longitudinal strip BCD of width dh, and taking the deflections at B and D with respect to C as \( y_C \)

\[
\frac{dP}{d\gamma} = 2M_\gamma'' \quad \ldots \ldots (6.55)
\]

\[
\frac{dP}{y_\gamma} = k_2 M_\gamma' \quad \ldots \ldots (6.56)
\]

Similar to mechanism B3 the characteristic equation is derived as

\[
P_1 = \frac{1}{2B} \left[ a_1 \left( C - \frac{F_2 a_1}{2} \right) + I_9 \right] \quad \ldots \ldots (6.57)
\]
where

\[ I_9 = - \frac{H^2}{2F} \left\{ \frac{C-F_2a_1}{H} \sqrt{1 + \frac{(C-F_2a_1)^2}{H^2}} + \log \left( \frac{C-F_2a_1}{H} + \sqrt{1 + \frac{(C-F_2a_1)^2}{H^2}} \right) \right\} + \frac{C}{H} \sqrt{1 + \frac{C^2}{H^2}} - \log \left( \frac{C}{H} + \sqrt{1 + \frac{C^2}{H^2}} \right) \]

\[ \ldots \ldots (6.58) \]

Similarly for failure lines AF and FG the characteristic equation is

\[ P_2 = \frac{1}{2F} \left[ a_2 \left( C - \frac{F_2a_2}{2} \right) + I_{10} \right] \]

\[ \ldots \ldots (6.59) \]

where

\[ I_{10} = - \frac{H^2}{2F} \left\{ \frac{C-F_2a_2}{H} \sqrt{1 + \frac{(C-F_2a_2)^2}{H^2}} + \log \left( \frac{C-F_2a_2}{H} + \sqrt{1 + \frac{(C-F_2a_2)^2}{H^2}} \right) \right\} + \frac{C}{H} \sqrt{1 + \frac{C^2}{H^2}} - \log \left( \frac{C}{H} + \sqrt{1 + \frac{C^2}{H^2}} \right) \]

\[ \ldots \ldots (6.60) \]

The characteristic equation for the entire mechanism is derived by adding Eq. (6.57) and Eq. (6.59)

\[ P = P_1 + P_2 \]

\[ \ldots \ldots (6.61) \]
A study is undertaken with the use of a computer to determine the effect of the position of $A$ on the failure load $P$ by numerically varying the values of $a_1$ and $a_3$. This study revealed that the effect is very small and therefore convenient values of $a_1$ and $a_3$ are chosen as $b/2$ and zero respectively.

6.4 Structural Mechanisms

A structural mechanism is an assembly of the basic mechanisms which fit together and deform in a manner compatible with the deformation of one another and with the deformation of the failure mechanisms as a whole. Some of the structural mechanisms observed in the laboratory tests are shown in Fig. 6.2. Mechanisms CF, TF, AF, CAF and ZF are formed in column members when they fail in flexural mode with the flanges subjected to compression. The flanges fail in compression by formation of basic mechanism B2 and a portion of web is subjected to tension. The members failing in torsional - flexural mode collapse with the formation of mechanisms CT, AT, TT and CAT. The flanges fail in basic mechanism B3 permitting the panels to twist freely and the web bends inducing tension in web. The derivation of the characteristic equations for the structural mechanisms shown in Fig. 6.2, by combining the basic mechanisms, are presented.
6.4.1 Mechanism CF in column - channel (Fig. 6.2a)

The flanges fail by formation of basic plate mechanism B2. The web is subjected to compression and tension with neutral axis lying at a depth $x_u$ from the junction of flange and web. The value of $x_u$ depends on the magnitude of the central deflection $\delta$ of the column. The free body of one half of the column in its deformed position (Fig. 6.6a) is analysed.

Because of the lateral movement $y$ of the edge E of the column, the axial shortening of one half of the strut is $y^2/(2btan\theta)$ and for compatibility

$$\alpha_1 = \frac{y^2}{2b^2tan\theta} \quad \ldots \ldots (6.62)$$

and

$$\delta = \frac{y^2L}{4b^2tan\theta} \quad \ldots \ldots (6.63)$$

where

$L = $ Length of the column

$b = $ breadth of the flange

$\theta = $ inclination of failure line to the cross section

$\alpha_1 = $ angle of rotation
For equilibrium

\[ P = 2(P_f)_1 + P_w \]  \hspace{1cm} ....(6.64)

where

\[ P = \text{longitudinal load} \]
\[ P_f = \text{longitudinal load in flange, given by Eq. (6.25)} \]
\[ P_w = \text{longitudinal load in web} \]

Moment equilibrium about the junction of flange and web leads to

\[ P(e_m + \delta) = 2(P_fe_f)_1 + M_w - P_w \left(\frac{t_w}{2}\right) \]  \hspace{1cm} ....(6.65)

where

\[ M_w = \text{moment in web} \]
\[ t_w = \text{thickness of the web} \]
\[ e_m = \text{distance from the edge of the plate element to line of } P \]
\[ (P_fe_f)_1 = \text{moment due to axial load in flange about the junction of flange and web given by Eq. (6.29)} \]
\[ P_w = (0.54f_{ck}x_u - f_yt_s)a \]  \hspace{1cm} ....(6.66)

where

\[ a = \text{width of the web} \]
Considering uniform thickness, i.e., thickness of the flange \( t_f \) = thickness of the web \( t_w \) = \( t \) and substituting Eq. (6.66) in Eq. (6.64)

\[
x_u = (P + A_1) \frac{1}{B_1}
\]

where

\[
A_1 = -2(P_f)_1 + f_y t_s a \quad \ldots \ldots (6.69)
\]

\[
B_1 = 0.54 f_{ck} a \quad \ldots \ldots (6.70)
\]

Substituting Eq. (6.66), Eq. (6.67) and Eq. (6.68) in Eq. (6.65) and simplifying

\[
P = - \frac{(2A_1D_1+e_m+\delta) + \sqrt{(2A_1D_1+e_m+\delta)^2 + 4C_1D_1}}{2D_1}
\]

\[\ldots \ldots (6.71)\]

where

\[
C_1 = 2(P_f e_f)_1 - D_1 A_1^2 + f_y t_s a(t-t_c) \quad \ldots \ldots (6.72)
\]

\[
D_1 = \frac{0.42}{B_1} \quad \ldots \ldots (6.73)
\]
6.4.2 Mechanism CT in column - channel (Fig. 6.2b)

Basic plate mechanism B3 forms in the flange and the web is subjected to compression and tension with the depth of neutral axis \( x_u \) measured from the junction of flange and web. The derivation of the characteristic equation for this mechanism is similar to that of type CF, with the only difference of \( k_1 \) being replaced by \( k_2 \). The free body diagram of one half of the column in its deformed position is considered (Fig. 6.6b) and the central deflection \( \delta \) is

\[
\delta = \frac{y^2L}{4b^2\tan \theta}
\]

The characteristic equation is

\[
P = \frac{(2A_2D_1 + e_m + \delta) + \sqrt{(2A_2D_1 + e_m + \delta)^2 + 4C_2D_1}}{2D_1}
\]

where

\[
A_2 = -2(P_f)^2 + f_yt_s a 
\]

\[
C_2 = 2(P_f e_f) - D_1A_2^2 + f_yt_s a(t-t_c)
\]

6.4.3 Mechanism TF in column - trough (Fig. 6.2c)

This mechanism is similar to the mechanism CF, the difference being in the member where the flange of the trough is inclined at an angle \( \phi \). The free body diagram of
one half of the column in its deformed position (Fig. 6.6c) is considered and the central deflection

\[ \delta = \frac{y^2L}{2b^2 \tan \theta} \]  

For equilibrium

\[ P = 2(P_f)_{1} + P_w \]  

\[ P(e_m + \delta) = 2(P_f e_f)_{1} \cos \phi + M_w - P_w \left( \frac{t_w}{2} \right) \]  

Choosing uniform thickness \( t \) and \( \phi = 45^\circ \), the characteristic equation is derived similar to the mechanism CF as

\[ p = \frac{-A_1 D_1 + e_m + \delta) + \sqrt{(2A_1 D_1 + e_m + \delta)^2 + 4C_3 D_1}}{2D_1} \]  

where

\[ C_3 = \sqrt{2(P_f e_f)_{1} - A_1^2 D_1 + f_y t_y a(t - t_c)} \]  

6.4.4 Mechanism TT in column - trough: (Fig. 6.2d)

The derivation of the characteristic equation is similar to the mechanism TF. Considering the free body of one half of column in its deformed position (Fig. 6.6d) the
characteristic equation is derived as

\[ p = \frac{-(2A_2D_1 + \text{em} + \delta) + \sqrt{(2A_2D_1 + \text{em} + \delta)^2 + 4C_4D_1}}{2D_1} \]  

\[ \ldots (6.83) \]

where

\[ C_4 = \sqrt{2(P_f e_f)^2 - A_2^2D_1 + f_yt_s a(t-t_c)} \]

and the central deflection \( \delta = \frac{y^2L}{2b^2\tan\theta} \)

6.4.5 Mechanism AF in column - angle (Fig. 6.2e)

Both the legs of the column fail by basic mechanism B2. A small region at the junction of the legs is subjected to tension. This tensile force is neglected as the area over which it acts is small compared to the area of legs.

The central deflection of the strut for a lateral displacement \( y \) at E is (Fig. 6.6e)

\[ \delta = \frac{y^2L}{2a^2\tan\theta} \]  

\[ \ldots ..(6.84) \]

For moment equilibrium about the junction of legs

\[ p(e_m + \delta) = \sqrt{2(P_f e_f)} \]  

\[ \ldots (6.85) \]

6.4.6 Mechanism AT in column - angle (Fig. 6.2f)

The legs of the column fail by basic mechanism B3. Neglecting the tensile force occurring at the junction of
legs, and considering free body diagram (Fig. 6.6f)

the central deflection

\[ \delta = \frac{y^2L}{2a^2\tan \theta} \] ...... (6.86)

and

\[ P(e_m + \delta) = \sqrt{2}(P_e)_2 \] ...... (6.87)

6.4.7 Mechanism CAF in column - circular arc (Fig. 6.2g)

The column fails by formation of two basic mechanisms B4. Neglecting the tension as in mechanism AF, the governing equation is derived as follows.

The central deflection of the strut for a lateral displacement of \( y \) at \( E \) is (Fig. 6.6g)

\[ \delta = \frac{y^2L}{4R^2(1-\cos \alpha)^2\tan \theta} \] ...... (6.89)

For moment equilibrium of the free body of one half of strut about an axis passing through \( O \) and lying in the plane of the cross section

\[ P(e_m + \delta) = 2(P_e)_1 \] ...... (6.90)

where \( (P_e)_1 \) is given by Eq. (6.45)
6.4.8 Mechanism CAT in column - circular arc (Fig. 6.2h)

Referring to Fig. 6.6h the central deflection

\[ \delta = \frac{y^2L}{4R^2(1 - \cos^2\alpha)^2 \tan \theta} \]

\[ \text{.....(6.91)} \]

and for moment equilibrium about an axis passing through O and lying in the plane of cross section

\[ P(e_m + \delta) = 2(Pe)_2 \]

\[ \text{.....(6.92)} \]

where \((Pe)_2\) is given by Eq. (6.52)

6.4.9 Mechanism ZF in column - Z-section (Fig. 6.2i)

Axially loaded columns with point symmetric Z section generally fail in flexural mode because of high torsional rigidity. In this mode the failure is initiated in the compression flange with basic mechanism B2 and the web fails in flip disc mechanism B6 and the other flange is subjected to tension. The characteristic equation is derived by combining these basic mechanisms and satisfying compatibility in deflection.

Analysing the free body of one half of the column in its deformed position (Fig. 6.6i), the central deflection for a lateral deflection \(y\) of point E

\[ \delta = \frac{y^2L}{4b^2 \tan \theta} \]

\[ \text{.....(6.93)} \]
For moment equilibrium about the junction of compression flange with web

\[ p(m + \delta) = (P_{fe})_1 - P_w \left( \frac{t_w}{2} \right) + P_{ft} \left( \frac{b}{2} \right) \ldots (6.94) \]

Where

\( (P_{fe})_1 \) = moment of axial force in compression flange (Eq. (6.29))
\( P_w \) = axial load in web (Eq. (6.66))
\( t_w \) = thickness of the web, for uniform thickness \( t_w = t \)
\( P_{ft} \) = load in tension flange

\( P_{ft} \) is determined by assuming the entire steel in the tension region to have yielded. Therefore

\[ P_{ft} = f_y b t_s \ldots \ldots (6.95) \]

6.4.10 Mechanism ZT in column - Z-section (Fig. 6.2j)

Eccentrically loaded column with point-symmetric Z section may also fail in torsional - flexural mode. When a member fails in this mode, the mechanism formed consists of basic mechanism B3 in compression flange, flip disc mechanism B6 in web and tension in the other flange. Analysing the free body diagram of one half of the column in its deformed position (Fig. 6.6j), the central deflection
for a lateral deflection $y$ of point $E$

$$\delta = \frac{y^2L}{4b^2\tan\theta} \quad \ldots \ldots (6.96)$$

For moment equilibrium about the junction of compression flange with web

$$P(e_m + \delta) = (P_{ef})^2 - P_w\left(\frac{l_w}{2}\right) + P_{ft}\left(\frac{b}{2}\right) \quad \ldots \ldots (6.97)$$

6.5 Orientation of Failure Lines

In the characteristic equation derived for various structural mechanisms, it is necessary to determine the value of $\theta$ which defines the shape of the mechanism and predicts a minimum value of $P$. A numerical study undertaken using a computer revealed that the change in $P$ is insensitive to change in $\theta$ between values $15^\circ$ and $45^\circ$ for all mechanisms. In the laboratory tests, $\theta$ is found to be closer to $15^\circ$. Therefore choosing a value of $\theta$ equal to $15^\circ$, the collapse curves are drawn and presented in Fig. 6.7 to Fig. 6.15.

6.6 Critical Deflection

The collapse curve apart from indicating the nature of failure, can also be used to predict the load carrying capacity approximately, since this is the starting point of the draw down curve. In the vicinity of collapse, local
elastic buckling results (Fig. 6.1b) with a lateral deflection \( y \). With the increase in \( y \) the strains in the materials reach their limiting values and the failure lines get initiated. The curvature and hence the lateral deflection \( y \) corresponding to the limiting strains are found from which the central deflection when the failure lines begin to form is determined. The load corresponding to this central deflection, termed as critical deflection, in a collapse curve determines approximately the failure load.

A failure line may be initiated with the section subjected to both tensile and compressive strains or the entire section subjected to compressive strains only. When the failure is initiated by tension, the strains in concrete and steel at critical section are assumed as (Fig. 6.17c)

Strain in concrete = 0.0035

Strain in steel = 0.002 + \( \frac{f_y}{E_s} \)

where

\( f_y \) = yield strength of steel

\( E_s \) = modulus of elasticity of steel = 200GPa

\[
\text{curvature } (\Phi) = \frac{1}{r} = \frac{0.007575}{(t-t_c)} \quad \ldots \ldots (6.98)
\]

where

\( r \) = radius of curvature
Assuming the profile of the local buckling (Fig. 6.17a and Fig. 6.17b) to be circular

\[ y = \frac{(btan\theta)^2}{2} \times \frac{1}{r} \ldots \ldots (6.99) \]

Substituting Eq. (6.98) in Eq. (6.99)

\[ y_{cr} = \frac{(btan\theta)^2}{(t - t_c)} \times 0.00378 \ldots \ldots (6.100) \]

Substituting Eq. (6.100) in the respective equations for central deflections the critical central deflections of columns are arrived at as:

for channel and Z sections

\[ \delta_{cr} = \frac{b^2Ltan^3\theta}{(t - t_c)} \times 3.585 \times 10^{-6} \ldots \ldots (6.101) \]

for trough section

\[ \delta_{cr} = \frac{b^2Ltan^3\theta}{(t - t_c)} \times 7.17 \times 10^{-6} \ldots \ldots (6.102) \]

for angle section

\[ \delta_{cr} = \frac{a^2Ltan^3\theta}{(t - t_c)} \times 7.17 \times 10^{-6} \ldots \ldots (6.103) \]
for circular arc section

\[ \delta_{cr} = \frac{R^2(1-\cos \alpha)^2L\tan^3 \theta}{t - t_c} \times 3.585 \times 10^{-6} \]  

\[ \cdots (6.104) \]

A criterion to determine whether tension or compression initiates the failure is presented. When tension initiates the failure lines, the interaction of load and moment is in the region AB and shown in Fig. 6.18.

When the failure line is initiated by compression, the compressive strain in concrete is assumed to vary from 0.0035 at highly compressed fibre to 0.002 in the least stressed fibre [39].

Similar to tension failure the following equations are arrived at.

Curvature \((\Phi) = \frac{0.0015}{(t-t_c)} \cdots (6.105)\)

\[ Y_{cr} = \frac{(b\tan \theta)^2}{(t - t_c)} \times 0.00075 \cdots (6.106) \]

and critical central deflections for columns with channel and Z sections

\[ \delta_{cr} = \frac{b^2L\tan^3 \theta}{(t - t_c)} \times 0.14 \times 10^{-6} \cdots (6.107) \]
For trough section

\[ \delta_{cr} = \frac{b^2L \tan^3 \theta}{(t - t_c)} \times 0.28 \times 10^{-6} \quad \ldots (6.108) \]

For angle section

\[ \delta_{cr} = \frac{a^2L \tan^3 \theta}{(t - t_c)} \times 0.28 \times 10^{-6} \quad \ldots (6.109) \]

For circular arc section

\[ \delta_{cr} = \frac{R^2(1 - \cos \alpha)^2L \tan^3 \theta}{(t - t_c)} \times 0.14 \times 10^{-6} \quad \ldots (6.110) \]

The position of the neutral axis when P-M interaction is at point A is,

\[ x_{uA} = \frac{f_{ys}t_s}{0.54f_{ck}} \quad \ldots (6.111) \]

The position of neutral axis for balanced case (point B) is

\[ x_{uB} = 0.462(t - t_c) \quad \ldots (6.112) \]

For the section to fail in tension

\[ x_{uA} \leq x_{uB} \quad \ldots (6.113) \]

which leads to a condition

\[ t = \frac{f_{ys}t_s}{0.249 f_{ck}} + t_c \quad \ldots (6.114) \]
6.7 Concluding Remarks

Based on the analysis incorporating considerations of an appropriate failure mechanism, characteristic equations are derived for columns under study for flexure mode and torsional-flexural buckling mode. This approach provides information regarding the post failure load shedding, the failure pattern and the ultimate deflection apart from enabling the determination of failure load.
FIGURE 6.1 FORMATION OF STRUCTURAL MECHANISM

GLOBAL BUCKLING (a)
LOCAL BUCKLING (b)
STRUCTURAL MECHANISM (c)
FIGURE 6.2-1 FAILURE MECHANISMS

(a) MECHANISM: 'CF'

(b) MECHANISM: 'CT'

(c) MECHANISM: 'TF'

(d) MECHANISM: 'TT'

(e) MECHANISM: 'AF'

(f) MECHANISM: 'AT'
FIGURE 6.2-2 FAILURE MECHANISMS

MECHANISM: 'CAF'

MECHANISM: 'CAT'

MECHANISM: 'ZF'

MECHANISM: 'ZT'

FIGURE 6.2-2 FAILURE MECHANISMS
FIGURE 6.3 BASIC PLATE MECHANISMS
\[
t_s = \frac{A_s}{b}
\]

\[e_c = \text{COMPRESSIVE STRAIN IN CONCRETE} = 0.0035\]

\[e_s = \text{TENSILE STRAIN IN STEEL}\]

**FIGURE 6.4 FAILURE LINE PERPENDICULAR TO LINE OF THRUST**

\[
f = b t
\]

**FIGURE 6.5 FAILURE LINE INCLINED TO LINE OF THRUST**
MECHANISM: CF
(a)

MECHANISM: CT
(b)

MECHANISM: TF
(c)

MECHANISM: TT
(d)

MECHANISM: AF
(e)

MECHANISM: AT
(f)

FIGURE 6.6-1 FREE BODY DIAGRAM OF ONE HALF OF COLUMN DURING COLLAPSE
MECHANISM: CAF
(g)

MECHANISM: CAT
(h)

MECHANISM: ZF
(i)

MECHANISM: ZT
(j)

FL = FAILURE LINE
FD = FLIP DISC

FIGURE 6.6-2 FREE BODY DIAGRAM OF ONE HALF OF COLUMN DURING COLLAPSE
FIGURE 6.7 DRAWS DOWN CURVE (AXIAL LOAD)

MECHANISM: CT

CHANNEL SECTION

\[ a = 200 \text{ mm} \]
\[ b = 200 \text{ mm} \]
\[ t = 10 \text{ mm} \]
\[ L = 1400 \text{ mm} \]
\[ f_{ck} = 23.92 \text{ N/mm}^2 \]
\[ \rho = 0.0076 \]
FIGURE 6.8 DRAW DOWN CURVE (AXIAL LOAD)

MECHANISM: TT

- \( a = 200 \text{ mm} \)
- \( b = 200 \text{ mm} \)
- \( t = 14 \text{ mm} \)
- \( L = 1810 \text{ mm} \)
- \( f_{ck} = 16.2 \text{ N/mm}^2 \)
- \( p = 0.0036 \)
- \( \phi = 45^\circ \)
FIGURE 6.9 DRAW DOWN CURVE (AXIAL LOAD)

MECHANISM: AT

\[ P = 0.0046 \]

\[ \sigma = 200 \text{ mm} \]
\[ t = 16.5 \text{ mm} \]
\[ L = 1400 \text{ mm} \]
\[ f_{ck} = 12.75 \text{ N/mm}^2 \]
FIGURE 6.10 DRAW DOWN CURVE (AXIAL LOAD)

- $R = 750 \text{ mm}$
- $t = 20 \text{ mm}$
- $L = 900 \text{ mm}$
- $\alpha = 30.2^\circ$
- $f_{ck} = 19.4 \text{ N/mm}^2$
- $P = 0.0053$

DEFLECTION (mm)

- $P$ (kN)
MECHANISM: CT

FIGURE 6:12 DRAW DOWN CURVE (ECCENTRIC LOAD)

Channel section

\[ P = \frac{f_{ck} \cdot e_x}{t} \]

where

- \( f_{ck} = 25 \text{ kN/mm}^2 \)
- \( e_x = 0.0069 \)

Dimensions:
- \( a = 200 \text{ mm} \)
- \( b = 200 \text{ mm} \)
- \( t = 11 \text{ mm} \)
- \( L = 920 \text{ mm} \)
- \( e_x = 50 \text{ mm} \)

Deflection (mm)

Channel section \( P \) (kN)
Figure 6.13 Draw Down Curve (Eccentric Load)

DEFLECTION (mm) 151

MECHANISM: TT

TRough SECTION

\( \theta_1 = 45^\circ \)

\( f_{ck} = 4.4 \text{ N/mm}^2 \)

\( \rho = 0.0035 \)

\( b = 200 \text{ mm} \)

\( t = 14.5 \text{ mm} \)

\( h_x = 35 \text{ mm} \)

\( L = 1400 \text{ mm} \)

\( a = 200 \text{ mm} \)
MECHANISM: AT

ANGLE SECTION

\[ \begin{align*}
    \alpha &= 200 \text{ mm} \\
    t &= 15.5 \text{ mm} \\
    L &= 1400 \text{ mm} \\
    e_x &= 10 \text{ mm} \\
    f_{ck} &= 28 \text{ N/mm}^2 \\
    \rho &= 0.0049
\end{align*} \]

**FIGURE 6.14** DRAW DOWN CURVE (ECCENTRIC LOAD)
FIGURE 6.15 DRAW DOWN CURVE (ECCENTRIC LOAD)

R = 750 mm
L = 900 mm
α = 36.8°
x = 85 mm
f_{ck} = 275 N/mm²
p = 0.0086

MECHANISM: CAT

CIRCULAR ARC SECTION

DEFLECTION (mm)

P (kN)

0 0.4 0.8 1.2 1.6 2.0 2.4 2.8 3.2 3.6 4.0 4.4 4.8 5.2

200 180 160 140 120 100 80 60

DRAW DOWN CURVE (ECCENTRIC LOAD)
FIGURE 6.16 DRAW DOWN CURVE (ECCENTRIC LOAD)
(a) FOR MECHANISMS - FLEXURE

(b) FOR MECHANISMS - TORSION

(C) STRAIN DISTRIBUTION

\[ \varepsilon_C = \text{STRAIN IN CONCRETE} = 0.0035 \]

\[ \varepsilon_S = \text{STRAIN IN STEEL} = 0.002 + \frac{f_y}{E_s} \]

\[ t' = t - t_c \]

\[ \phi = \text{CURVATURE} \]

FIGURE 6.17 CRITICAL DEFLECTION

A - PURE BENDING
B - BALANCED CASE

FIGURE 6.18 P-M INTERACTION DIAGRAM