CHAPTER 5

ELASTIC STABILITY OF COLUMNS WITH SMALL ECCENTRICITY

5.1 Introduction

Very rarely in practice, axially loaded member is encountered. Real construction practice presents walls and columns with some implicit eccentricity - due to erection, small permissible eccentricity in placing of loads, member imperfections and nonhomogeneous material dispersion etc. Hence it is necessary to study the stability response of columns when subjected to loads at small eccentricity. The term 'small eccentricity' implies that the whole cross section is under compression and no tension is developed i.e., the neutral axis lies outside the cross section. This chapter deals with thin-walled reinforced concrete columns subjected to loads at small eccentricity. The investigation is limited to positive eccentricity, eccentricity being measured with respect to minor flexural bending axis (Fig. 5.1 and Fig. 5.6)

5.2 Assumptions

The assumptions made in the proposed analysis of eccentrically loaded reinforced concrete columns are
1. Small deflection theory prevails and imperfections are neglected.

2. Plane sections before application of load remain plane after application of load also.

3. The eccentricity is small and measured along positive major principal axis - the eccentricity being considered small when tension is not developed in the cross section of the member.

4. The concrete remains uncracked at the onset of buckling.

5. Short term buckling alone is considered and the effect of creep is ignored.

6. The reinforcing steel does not yield at the onset of buckling.

7. The uniaxial nonlinear stress-strain relationship of concrete in compression [27] is given by

\[ f_c = \frac{f_o}{e_o^2} (2ee_o - e_2) \quad \ldots \quad (5.1) \]

where

\[ f_o = \text{ultimate 28 days compressive strength of concrete} \]

\[ e_o = \text{concrete strain corresponding to stress } f_o \]
5.3 Theory

The differential equations of equilibrium for a column loaded by a longitudinal load \( P \), with biaxial eccentricities, \( e_x \) and \( e_y \) which are constant along the length of the member are [49]

\[
E I_y \, u'' + P u'' + P(y_0 - e_y) \phi'' = 0 \quad \ldots \quad (5.2)
\]

\[
E I_x \, v'' + P v'' - P(x_0 - e_x) \phi'' = 0 \quad \ldots \quad (5.3)
\]

\[
E I_\Omega \, \phi'' - (GJ - P e_y \, E_1 - P e_x \, E_2 - P r_0^2) \phi'' + P(y_0 - e_y) u''
- P(x_0 - e_x) v'' = 0 \quad \ldots \quad (5.4)
\]

where

- \( u, v \) = displacements in \( x \) and \( y \) directions
- \( \phi \) = rotation about shear centre
- \( E \) = modulus of elasticity of the material
- \( G \) = shear modulus of the material
- \( J \) = torsional constant
- \( P \) = longitudinal load
- \( r_0 \) = polar radius of gyration of the cross section about shear centre
- \( I_x', I_y \) = principal moments of inertia
- \( I_\Omega \) = sectorial moment of inertia
- \( x_0, y_0 \) = distance between the shear centre and the centroid in the two principal directions
\[
\beta_1 = \frac{U_x}{I_x} - 2y_o \quad \ldots (5.5)
\]

\[
U_x = \int y^3 \, dA + \int yx^2 \, dA \quad \ldots (5.6)
\]

\[
\beta_2 = \frac{U_y}{I_y} - 2x_o \quad \ldots (5.7)
\]

\[
U_y = \int x^3 \, dA + \int xy^2 \, dA \quad \ldots (5.8)
\]

All derivatives are with respect to Z axis, the axial direction of the member. The solution of Eq. (5.2) to Eq. (5.4) using Galerkin's method [5, 10] leads to the following equations applicable to hinged-hinged end conditions [35].

\[
\begin{pmatrix}
(P_y - P) & 0 & -P_y \\
0 & (P_x - P) & P_x \\
-P_y & P_x & r^2(P\phi' - P)
\end{pmatrix}
\begin{cases}
u_o \\
v_o \\
\phi_o
\end{cases}
= \begin{pmatrix}
\begin{pmatrix}
\frac{p^2}{P_y}e_x(\frac{4}{\pi}) \\
-\frac{p^2}{P_x}e_y(\frac{4}{\pi}) \\
-p^2(\frac{a_ye_x}{P_y} - \frac{a_xe_y}{P_x})\frac{4}{\pi}
\end{pmatrix}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{pmatrix}
\frac{p^2}{P_y}e_x(\frac{4}{\pi}) \\
-\frac{p^2}{P_x}e_y(\frac{4}{\pi}) \\
-p^2(\frac{a_ye_x}{P_y} - \frac{a_xe_y}{P_x})\frac{4}{\pi}
\end{pmatrix}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{pmatrix}
\frac{p^2}{P_y}e_x(\frac{4}{\pi}) \\
-\frac{p^2}{P_x}e_y(\frac{4}{\pi}) \\
-p^2(\frac{a_ye_x}{P_y} - \frac{a_xe_y}{P_x})\frac{4}{\pi}
\end{pmatrix}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{pmatrix}
\frac{p^2}{P_y}e_x(\frac{4}{\pi}) \\
-\frac{p^2}{P_x}e_y(\frac{4}{\pi}) \\
-p^2(\frac{a_ye_x}{P_y} - \frac{a_xe_y}{P_x})\frac{4}{\pi}
\end{pmatrix}
\end{pmatrix}
\]

\[
where \quad P_x = \frac{\pi^2EI_x}{l^2} \quad \ldots (5.10)
\]
where

\[ l = \text{length of the column} \]

\[ a_x = x_0 - e_x \quad \ldots \quad (5.13) \]

\[ a_y = y_0 - e_y \quad \ldots \quad (5.14) \]

\[ r_1^2 = \frac{\xi_1 e_y + \xi_2 e_x + r_o^2}{r_2} \quad \ldots \quad (5.15) \]

The solution for Eq. (5.9) is obtained using the following functions for the deflection components satisfying static and kinematic boundary conditions

\[ u = \frac{p e_x}{2 E I_y} (z^2 - l z) + u_o \sin \frac{\pi z}{l} \quad \ldots \quad (5.16) \]

\[ v = \frac{p e_y}{2 E I_x} (z^2 - l z) + v_o \sin \frac{\pi z}{l} \quad \ldots \quad (5.17) \]

\[ \phi = \varphi_o \sin \frac{\pi z}{l} \quad \ldots \quad (5.18) \]

where \( u_o, v_o \) and \( \varphi_o \) are the coefficients to be determined.
However for singly symmetric sections and load acting in the plane of symmetry as shown in Fig. 5.1 to Fig. 5.6, \( e_y = y_0 = 0 \) that is \( a_y = 0 \), then Eq. (5.9) yields

\[
\begin{bmatrix}
(P_y - P) & 0 & 0 \\
0 & (P_x - P) & \rho_{x} \\
0 & \rho_{x} & r_1^2(P_{\phi} - P)
\end{bmatrix}
\begin{bmatrix}
u_o \\ v_o \\ \phi_o
\end{bmatrix} = \begin{bmatrix}\frac{2}{(P_y - P)}e_x \left(\frac{4}{\pi}\right)
\frac{2}{(P_y - P)}e_x \left(\frac{4}{\pi}\right) \\
0
\end{bmatrix}
\]

\[
\ldots(5.9a)
\]

Partitioning the matrix and uncoupling

\[
(P_y - P) u_o = \left(\frac{\nu^2}{P_y}\right)e_x \left(\frac{4}{\pi}\right)
\]

\[
\ldots(5.19)
\]

\[
\begin{bmatrix}
P_x - P & \rho_{x} \\
\rho_{x} & r_1^2(P_{\phi} - P)
\end{bmatrix}
\begin{bmatrix}
v_o \\ \phi_o
\end{bmatrix} = 0
\]

\[
\ldots(5.20)
\]

Eq. (5.19) gives \( u_o \) for a beam-column deforming in flexure. The small deflection theory predicts \( u_o \) to become infinite when \( P = P_y \).

On the other hand, the determination of torsional-flexural buckling load is an eigenvalue problem. Nontrivial solution of Eq. (5.20) is the bifurcation load \( P_{cr} = P_{\phi x} \) which is the critical load for eccentric torsional - flexural buckling.
5.4 Buckling of Thin-Walled RC Column Members

The buckling solutions for elastic material are applied to reinforced concrete using the constitutive material equations. The expression for stress in concrete $f_c$ at any strain $e$ is

$$f_c = \frac{0.85f_{cc}}{e_o^2} (2e_{o} - e^2) \quad \cdots (5.21)$$

Neglecting the contribution of steel to the modulus of elasticity of reinforced concrete, as it is small as seen in axially loaded columns, the tangent modulus of reinforced concrete $E_t$ is,

$$E_t = \frac{1.7f_{cc}}{e_o^2} (e_o - e) \quad \cdots (5.22)$$

and shear modulus is [20]

$$G_t = \frac{E_t}{2(1 + \mu)} \quad \cdots (5.23)$$

In thin-walled columns subjected to a load at small eccentricity, the neutral axis lies outside the cross section and hence no tension is developed in the section.

When the load is transferred to the member through an end block, it gets distributed uniformly. Due to such
distribution the bimoment caused is zero [53]. Therefore plane sections may be assumed to remain plane before and after application of the load and the strain variation is linear.

For a linear strain variation across the section, the stress distribution is according to Eq. (5.21) (Fig. 5.1 to Fig. 5.6). The derived equation relating the eccentricity and the position of neutral axis and the expression for longitudinal load in nondimensional form for the cross sections under consideration are given below. Tangent modulus corresponding to maximum strain in the section is used in the derivations. The equations are determined from equilibrium considerations.

5.4.1 Channel

From Fig. 5.1

\[ C_f = \frac{2bt}{3} (2f_{c1} + f_{c2}) \] ....(5.24)

\[ C_w = f_{c2} \text{ at} \] ....(5.25)

where \( f_{c1}, f_{c2} \) = extreme fibre stresses corresponding to strains \( e_1 \) and \( e_2 \)

\[ f_{c1} = \frac{0.85f_{cc}}{e_o^2} (2e_o e_1 - e_1^2) \] ....(5.26)
\[ f_{c2} = \frac{0.85f_{cc}}{e_0^2} (2e_0e_2 - e_2^2) \quad \ldots (5.27) \]

\[ e_2 = e_1 X' \quad \ldots (5.28) \]

\[ X' = \frac{1}{x_n} (x_n - b) \quad \ldots (5.29) \]

\[ C_f = \text{total force in flanges} \]
\[ C_w = \text{force in web} \]

Longitudinal load is sum of Eq. (5.24) and Eq. (5.25)

\[ P = \frac{4}{3} f_{c1}bt + f_{c2} \left( \frac{2bt}{3} + at \right) \quad \ldots (5.30) \]

Moment equilibrium about centroidal axis yy is

\[ P_{ex} = C_f(x_1 - \bar{x}) - C_w(x_2) \quad \ldots (5.31) \]

where

\[ x_1, x_2 = \text{position of the centroid} \]
\[ \bar{x} = \text{position of } C_f \text{ from the maximum stressed fibre} \]

\[ \bar{x} = \frac{3b(f_{c1} + f_{c2})}{4(2f_{c1} + f_{c2})} \quad \ldots (5.32) \]
Substituting Eq. (5.30) in Eq. (5.31) and rewriting

\[ \frac{e_x}{a} = \frac{C_f}{f_{cc}bt} \left[ k B_1 - k \left( \frac{x}{a} \right) \right] - \frac{C_w}{f_{cc}at} B_2 \]

\[ k \left( \frac{C_f}{f_{cc}bt} \right) + \frac{C_w}{f_{cc}at} \]

\[ ......(5.33) \]

where \( B_1 \) and \( B_2 \) are functions of \( b/a \) provided in Appendix A.

The limiting value of small eccentricity is determined from Eq. (5.33) when \( x_n = b \)

The longitudinal load (Eq. (5.30)) in a nondimensional form is represented as

\[ \frac{P}{P_x} = N_{el} \left( \frac{1}{a} \right)^2 \]

\[ ......(5.34) \]

where \( P_x \) and \( E_t \) are given by Eq. (5.10) and Eq. (5.22) and

\[ I_x = C a^3 t \]

\[ ......(5.35) \]

\[ N_{el} = \frac{1}{2 \pi^2 C \left( e_0 - e_1 \right)} \left[ \frac{4k}{3} \left( 2e_0 e_1 - e_1^2 \right) \right. \]

\[ + \left( \frac{2k}{3} + 1 \right) \left\{ 2e_0 \left( e_1 X' \right) - \left( e_1 X' \right)^2 \right\} \]

\[ ......(5.36) \]

\[ C = \] term in function of \( b/a \) given in Appendix A.
5.4.2 Trough

From Fig 5.2

\[ C_f = \frac{b}{2} \int_{0}^{x'} f_c \, dx' \] ..........(5.37)

where

\[ f_c = f_{c1} - y_1 \] ..........(5.38)

The equation of the parabolic stress diagram with the most compressed fibre as origin, is

\[ y = \frac{f_{c1}}{x_n^2} x^2 \] ..........(5.39)

and

\[ y_1 = \left( \frac{f_{c1}}{x_n^2} \right) \left( \frac{(x')^2}{2} \right) \] ..........(5.40)

Integrating Eq. (5.37) after substituting for \( f_c \) from Eq. (5.38) and Eq. (5.40)

\[ C_f = \frac{2bt}{3} (2f_{c1} + f_{c2}) \] ..........(5.41)

\[ C_w = f_{c2} \text{ at} \] ..........(5.42)
where $f_{c1}$, $f_{c2}$ are defined by Eq. (5.26) to Eq. (5.27) and

$$x' = \frac{1}{x_n} \left( x_n - \frac{b}{\sqrt{2}} \right) \quad \ldots \ldots (5.43)$$

where $x_n =$ position of neutral axis

Longitudinal load is sum of Eq. (5.41) and Eq. (5.42)

$$P = \frac{4}{3} f_{c1} bt + f_{c2} + at \quad \ldots \ldots (5.44)$$

Moment equilibrium about centroidal axis yy is

$$P_{ex} = 2\int_0^b f_{ct} \left( x_1 - \frac{x'}{\sqrt{2}} \right) \, dx' - C_w x_2 \quad \ldots \ldots (5.45)$$

where $C_w = f_{c2} at \quad \ldots \ldots (5.46)$

Integrating Eq. (5.45) after substituting for $f_c$ from Eq. (5.38) and Eq. (5.40)

$$P_{ex} = C_f (x_1 - \bar{x}) - (C_w x_2) \quad \ldots \ldots (5.47)$$

where

$$\bar{x} = \frac{3b(f_{c1} + f_{c2})}{4\sqrt{2}(2f_{c1} + f_{c2})} \quad \ldots \ldots (5.48)$$
Substituting Eq. (5.44) in Eq. (5.47) and rewriting

\[
\frac{e_x}{a} = \left( \frac{C_f}{f_{cc}bt} \right) \left[ kb_1 - k \left( \frac{x}{a} \right) \right] - \left( \frac{C_w}{f_{cc}at} \right) B_2 \\
\quad + k \left( \frac{C_f}{f_{cc}bt} \right) + \frac{C_w}{f_{cc}at}
\]

......(5.49)

where \( B_1 \) and \( B_2 \) are functions of \((b/a)\) provided in Appendix A.

The limiting value of small eccentricity is determined from Eq. (5.49) when \( x_n = b/\sqrt{2} \)

Writing the longitudinal load (Eq. (5.44)) in a nondimensional form

\[
\frac{P}{P_x} = N_{el} \left( \frac{1}{a} \right)^2 
\]

......(5.50)

where \( P_x, E_t \) and \( I_x \) are given by Eq. (5.10), Eq. (5.22) and Eq. (5.35) and

\[
N_{el} = \frac{1}{2\pi^2 C (e_o - e_1)} \left[ \frac{4k}{3} (2e_o e_1 - e_1^2) \right. \\
\quad + \left. \left( \frac{2k}{3} + 1 \right) \left\{ 2e_o (e_1 X') - (e_1 X')^2 \right\} \right] 
\]

......(5.51)

where \( C = \) function of \((b/a)\) provided in Appendix A.
5.4.3 Angle

From Fig. 5.3 the equation for the forces in the legs of the angle is derived similar to that of the force in the flange of a trough as

\[ C_f = \frac{2at}{\left(2fc_1 + fc_2\right)} \quad \ldots \ldots (5.52) \]

where \( fc_1, fc_2, x' \) are defined by Eq. (5.26) to Eq. (5.27) and Eq. (5.43)

Longitudinal load in the member is

\[ P = \frac{2at}{3} \left(2fc_1 + fc_2\right) \quad \ldots \ldots (5.53) \]

By considering the moment equilibrium about the centroidal axis \( yy \), similar to the moment of forces in the flange of a trough

\[ Pe_x = C_f (x_1 - \bar{x}) \quad \ldots \ldots (5.54) \]

where \( \bar{x} \) is given by Eq. (5.48)

Substituting Eq. (5.53) in Eq. (5.54) and rewriting

\[ \frac{e_x}{a} = \frac{(C_f/f_{cc}at) [B_1 - (x/a)]}{(C_f/f_{cc}at)} \quad \ldots \ldots (5.55) \]

where \( B_1 \) = function of \( a \) provided in Appendix A.
The limiting value of small eccentricity is determined from Eq. (5.55) when \( x_n = a/\sqrt{2} \)

Writing the longitudinal load (Eq. (5.53)) in a nondimensional form

\[
\frac{P}{P_x} = N_{e1} \left( \frac{1}{a} \right)^2
\] .....

where \( P_x \) \( E_t \) and \( I_x \) are given by Eq. (5.10), Eq. (5.22) and Eq. (5.35) and

\[
N_{e1} = \frac{1}{3\pi^2 C(e_0-e_1)} \left[ 2(2e_0e_1e_1^2) + 2e_0e_1x'-(e_1x')^2 \right]
\] .....

where the value of \( C \) is provided in Appendix A

5.4.4 Circular arc

From Fig. 5.4 the longitudinal load \( P \) is

\[
P = 2Rt \int_0^\alpha f_C \, d\theta
\] .....

where \( R \) = radius of the section

\( \alpha \) = semi-central angle

Substituting for \( f_C \) and \( y_1 \) from Eq. (5.38) and Eq. (5.40) where

\[
x' = R(\cos\theta - \cos\alpha)
\] .....

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and integrating

$$P = 2Rt f_{c1} \left[ \alpha - \left( \frac{R}{x_n} \right)^2 T_1 \right]$$

.....(5.60)

where

$$T_1 = \frac{\alpha}{2} + \alpha \cos^2 \alpha - \left( \frac{3}{4} \right) \sin 2\alpha$$

.....(5.61)

Moment equilibrium about centroidal axis yy is

$$P_{ex} = 2Rt \int_{0}^{\alpha} f_c(x_1 - x') \, d\theta$$

.....(5.62)

Integrating Eq. (5.62) after substituting for $f_c$ from Eq. (5.38), Eq. (5.40) and $x'$ from Eq. (5.59)

$$P_{ex} = PRB_1 - 2R^2t f_{c1} \left[ \alpha B_1 - \left( \frac{R}{x_n} \right)^2 T_2 \right]$$

.....(5.63)

where $B_1$ = function of $(b/a)$ given in Appendix A.

$$T_2 = \frac{\sin 3\alpha}{12} + \frac{3 \sin \alpha}{4} - \alpha \cos^3 \alpha + \frac{3}{3} \sin \alpha \cos^2 \alpha - \frac{3}{2} \alpha \cos \alpha$$

.....(5.64)

Substituting Eq. (5.60) in Eq. (5.62) and rewriting

$$\frac{e_x}{R} = B_1 - \frac{\alpha B_1 - \left( \frac{R}{x_n} \right)^2 T_2}{\alpha - \left( \frac{R}{x_n} \right)^2 T_1}$$

.....(5.65)
The limiting value of small eccentricity is determined from Eq. (5.65) where \( x_n = R(1 - \cos \alpha) \).

Writing the longitudinal load (Eq. (5.60)) in a nondimensional form

\[
\frac{P}{P_x} = N_{e1} \left( \frac{1}{R} \right)^2 
\]

(5.66)

where \( P_x \), \( E_t \) and \( I_x \) are given by Eq. (5.10), Eq. (5.22) and Eq. (5.35) and

\[
N_{e1} = \frac{2e_0 e_1 - e_1^2}{\pi^2 C(e_0 - e_1)} \left[ \alpha - \left( \frac{R}{x_n} \right)^2 T_1 \right] 
\]

(5.67)

\( C \) = function of \((\alpha)\) provided in Appendix A

5.4.5 Z-Section

Case i: \( \tan \theta < b/a \) (Fig. 5.5)

Force in flange \( F_1 = C_{f1} = \int_{ak_1}^{ak_2} f_c t \, dx' \) \hspace{1cm} (5.68)

where \( f_c \) is given by Eq. (5.38) and

\[
k_1 = k - \tan \theta 
\]

(5.69)

\[
k_2 = 2k - \tan \theta 
\]

(5.70)
integrating

\[ C_{f1} = f_{cal} \left[ -(a/x)^2 \frac{\cos^2 \theta}{3} (k_2^3 - k_1^3) + (k_2 - k_1) \right] \]

\[ \ldots \ldots (5.71) \]

Moment equilibrium about centroidal axis yy

\[ M_{f1} = \int \left[ \frac{(2bc\cos \theta - a\sin \theta)/2 - x' \cos \theta}{f_c} t \right] dx' \]

\[ \frac{ak_2}{ak_1} \]

\[ \ldots \ldots (5.72) \]

where \( f_c \), \( k_1 \) and \( k_2 \) are given by Eq. (5.38), Eq. (5.69), Eq. (5.70) and integrating

\[ M_{f1} = f_{cal} a^2 t \left[ (a/x)^2 \left\{ \frac{\cos^3 \theta}{4} (k_2^4 - k_1^4) \right. \right. \]

\[ \left. \left. - \frac{k_3 \cos^2 \theta}{3} (k_2^3 - k_1^3) \right\} - \frac{\cos \theta}{2} (k_2^2 - k_1^2) \right] + k_3 (k_2 - k_1) \]

\[ \ldots \ldots (5.73) \]

where

\[ k_3 = \frac{2k\cos \theta - \sin \theta}{2} \]

\[ \ldots \ldots (5.74) \]

Similar to Eq. (5.71) for flange \( F_2 \) and web, expressions for forces \( C_{f2} \) and \( C_w \) and for moments \( M_{f2} \) and \( M_w \) can be derived
where for flange F₂
\[ k_1 = 0 \] \hspace{1cm} (5.75)
\[ k_2 = k \] \hspace{1cm} (5.76)
\[ k_3 = \frac{2k \cos \theta - \sin \theta}{2} \] \hspace{1cm} (5.77)

and for web
\[ k_1 = (k \cot \theta - 1) \] \hspace{1cm} (5.78)
\[ k_2 = k \cot \theta \] \hspace{1cm} (5.79)
\[ k_3 = \frac{2k \cos \theta - \sin \theta}{2} \] \hspace{1cm} (5.80)

case ii: \( \tan \theta > b/a \) (Fig. 5.6)

Derivations similar to case (i) prove that expressions
for forces \( C_{f1}, C_{f2}, C_w \) and moments \( M_{f1}, M_{f2}, M_w \) take the
form of Eq. (5.71) and Eq. (5.73), where for flange F₁
\[ k_1 = 0 \] \hspace{1cm} (5.81)
\[ k_2 = k \] \hspace{1cm} (5.82)
\[ k_3 = \frac{\sin \theta}{2} \] \hspace{1cm} (5.83)

for flange F₂
\[ k_1 = \tan \theta - k \] \hspace{1cm} (5.84)
\[ k_2 = \tan \theta \] \hspace{1cm} (5.85)
\[ k_3 = \frac{\sin \theta}{2} \] \hspace{1cm} (5.86)

for web
\[ k_1 = 0 \] \hspace{1cm} (5.87)
\[ k_2 = 1 \] \hspace{1cm} (5.88)
\[ k_3 = \frac{\sin \theta}{2} \] \hspace{1cm} (5.89)
**case iii:** \( \tan \theta = k \)

Similar derivations as done in case (i) lead to Eq. (5.71) for forces and Eq. (5.73) for moment in component plates where

for flange \( F_1 \)

\[
\begin{align*}
  k_1 &= 0 & \quad \text{.....(5.90)} \\
  k_2 &= k & \quad \text{.....(5.91)} \\
  k_3 &= \frac{\sin \theta}{2} & \quad \text{.....(5.92)}
\end{align*}
\]

for flange \( F_2 \)

\[
\begin{align*}
  k_1 &= 0 & \quad \text{.....(5.93)} \\
  k_2 &= k & \quad \text{.....(5.94)} \\
  k_3 &= \frac{\sin \theta}{2} & \quad \text{.....(5.95)}
\end{align*}
\]

for web

\[
\begin{align*}
  k_1 &= 0 & \quad \text{.....(5.96)} \\
  k_2 &= 1 & \quad \text{.....(5.97)} \\
  k_3 &= \frac{\sin \theta}{2} & \quad \text{.....(5.98)}
\end{align*}
\]

In each of the above three cases the longitudinal load \( P \) is

\[
P = C_{f1} + C_{f2} + C_w \quad \text{.....(5.99)}
\]
Substituting for $C_{f1}$, $C_{f2}$ and $C_w$, the expression simplifies to the form

$$P = f_{clat} [T_3 - (a/x_n)^2 T_4] \quad \ldots \ldots (5.100)$$

where $T_3$, $T_4$ are functions of $k$ and $\theta$ and expression for moment

$$P_{ex} = M_{f1} + M_{f2} + M_w \quad \ldots \ldots (5.101)$$

and simplifies as

$$P_{ex} = f_{clat} a^2 [T_5 + (a/x_n)^2 T_6] \quad \ldots \ldots (5.102)$$

where $T_5$, $T_6$ are functions of $k$ and $\theta$

Substituting Eq. (5.100) in Eq. (5.102) and rewriting

$$\frac{e_x}{a} = \frac{T_5 + (a/x_n)^2 T_6}{T_3 - (a/x_n)^2 T_4} \quad \ldots \ldots (5.103)$$

The limiting value of small eccentricity is determined from Eq. (5.103) where for case (i)

$$x_n = a (2k \cos \theta - \sin \theta) \quad \ldots \ldots (5.104)$$

for case (ii) and case (iii)

$$x_n = a \sin \theta \quad \ldots \ldots (5.105)$$

Writing the longitudinal load (Eq. (5.100)) in a nondimensional form

81
\[
\frac{P}{P_X} = N_{e1} \left( \frac{1}{a} \right)^2
\]  
\[\text{.....(5.106)}\]

where \(P_X, E_t, \) and \(I_X\) are given by Eq. (5.10), Eq. (5.22) and Eq. (5.35) and

\[
N_{e1} = \frac{2e_0e_1 - e_1^2}{2\pi^2C(e_0 - e_1)} [T_3 - (a/x_n)^2T_4]
\]  
\[\text{.....(5.107)}\]

where

\[C = \text{function of } (b/a) \text{ provided in Appendix A.}\]

Curves for \((e_X/a)\) versus \((x_n/a)\) for plated columns and \((e_X/R)\) versus \((x_n/R)\) are given in Fig. 5.7 to Fig. 5.11 to facilitate the calculation process.

5.5 Beam - Column

For columns with small eccentricity, when the applied load is nearly equal to Euler load, very large lateral deformation occurs. The load corresponding to very large deflection can very well be termed as buckling load, and this approach provides a good approximation, with values almost equal to Euler load.

Equating the expression for longitudinal load in the form \(P/P_X\) for the section under consideration and the Euler load in the form \(P_Y/P_X\) an expression for the critical strain can be determined. This strain provides an approximate limiting value to columns with small eccentricity.
5.6 Torsional - Flexural Buckling

Torsional - flexural buckling under eccentric loading for monosymmetric sections is determined by equating the determinant in Eq. (5.20) to zero.

The governing equation is

\[ P_{cr}^2 \gamma - P_{cr} (P_x \alpha_1 + P_\phi) + P_x P_\phi = 0 \]  \hspace{1cm} (5.108)

where

\[ \gamma = -(x_0 - e_x)^2 \frac{1}{r_o^2} + \alpha_1 \]  \hspace{1cm} (5.109)

\[ P_\phi = \alpha_1 P_\phi' \]  \hspace{1cm} (5.110)

\[ \alpha_1 = 1 + \frac{e_x \beta_2}{r_o^2} \]  \hspace{1cm} (5.111)

\[ P_\phi' \] is given by Eq. (5.12) where

\[ r_1^2 = \alpha_1 r_o^2 \]  \hspace{1cm} (5.112)

The lower root of Eq. (5.108) yields torsional - flexural buckling load for columns subjected to small eccentricity in positive x direction and the solution is

\[ P_{cr} = \frac{1}{2} \left[ (P_\phi + \alpha_1 P_x) - \sqrt{(P_\phi + \alpha P_x)^2 - 4 \frac{1}{2} P_x P_\phi} \right] \]  \hspace{1cm} (5.113)
Writing Eq. (5.113) in a nondimensional form

\[
\frac{P_{cr}}{P_x} = \frac{1}{2} \left[ \left( \frac{P_f}{P_X} + \alpha_1 \right) - \sqrt{\left( \frac{P_f}{P_X} + \alpha_1 \right)^2 - 4 \gamma \frac{P_f}{P_X}} \right]
\]

\[\ldots (5.114)\]

where \( P_f/P_X \), \( \gamma \) and \( \alpha \) for channel, trough, angle and Z sections, after substituting for various parameters in terms of \( l, t, b \) and \( a \)

\[
\frac{P_f}{P_X} = R_1 \left( \frac{t l}{a^2} \right)^2 + S_1
\]

\[\ldots (5.115)\]

\[
\gamma = - \left( \sqrt{B} - \frac{e_x}{a} \right)^2 \frac{1}{F} + \left( 1 + A_3 \frac{e_x}{a} \right)
\]

\[\ldots (5.116)\]

\[
\alpha_1 = 1 + \frac{A_2}{F} \left( \frac{e_x}{a} \right)
\]

\[\ldots (5.117)\]

where

\[
t = \text{thickness of the member}
\]

\[
b = \text{width of the flange}
\]

\[
a = \text{width of the web}
\]

\( R_1, S_1, B, F, A_2 \) and \( A_3 \) are terms in function of \( (b/a) \)

provided in Appendix A.
Substituting Eq. (5.115) in Eq. (5.114)

\[
\frac{P_{cr}}{P_x} = M_4
\]  

…..(5.118)

where

\[
M_4 = \frac{1}{2} \left\{ \alpha_1 + R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right\} \left[ \alpha_1 + R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right] - 4 \left\{ R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right\} 
\]

…..(5.119)

Similarly for column with circular arc section, the equation for torsional-flexural buckling load is derived as

\[
\frac{P_{cr}}{P_x} = M_5
\]  

…..(5.120)

where

\[
M_5 = \frac{1}{2} \left\{ \alpha_1 + R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right\} \left[ \alpha_1 + R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right] - 4 \left\{ R_1 \left( \frac{t}{R} \right)^2 \left( \frac{1}{R} \right)^2 + S_1 \right\} 
\]

…..(5.121)

\[
\gamma = - \left( \sqrt{B} - \frac{e_x}{R} \right) \frac{1}{F} + \left( 1 + A_3 \frac{e_x}{R} \right)
\]

…..(5.122)
\[ \alpha_1 = 1 + \frac{A_2}{F} \left( \frac{e_x}{R} \right) \]  

.....(5.123)

where

\[ R = \text{radius of the circular arc section} \]

\[ R_1, S_1, B, F, A_2 \text{ and } A_3 \text{ are terms in function of } \alpha \text{ provided in Appendix A} \]

At the point of bifurcation of equilibrium equating the longitudinal load for the column under consideration and the critical load and after rearrangement the critical strain \( e_{cr} = e_1 \) for torsional - flexural buckling is obtained as

\[ e_{cr} = \frac{f_1 - \sqrt{f_1^2 - \frac{4n_1g_1}{n_1^2}}}{2n_1} \]  

.....(5.124)

where

for channel and trough

\[ f_1 = N + 8ke_0/3 + \left( \frac{2k}{3} + 1 \right) 2e_0X' \]  

.....(5.125)

\[ g_1 = Ne_0 \]  

.....(5.126)

\[ n_1 = \frac{4k}{3} + \left( \frac{2k}{3} + 1 \right) (X')^2 \]  

.....(5.127)

\[ N = M_4 \frac{2\pi^2C}{(1/a)^2} \]  

.....(5.128)
for angle

\[ f_1 = N + 4e_0 + 2e_0 X' \]  \hspace{2cm} \ldots (5.129)

\[ g_1 = N e_0 \]  \hspace{2cm} \ldots (5.130)

\[ n_1 = (X')^2 + 2 \]  \hspace{2cm} \ldots (5.131)

\[ N = M_4 \frac{3\pi^2 C}{(1/a)^2} \]  \hspace{2cm} \ldots (5.132)

for circular arc

\[ f_1 = N + 2e_0 \]  \hspace{2cm} \ldots (5.133)

\[ g_1 = N e_0 \]  \hspace{2cm} \ldots (5.134)

\[ n_1 = \alpha - \left( \frac{R}{X_n} \right)^2 \]  \hspace{2cm} \ldots (5.135)

\[ N = M_4 \frac{\pi^2 C}{(1/R)^2} \]  \hspace{2cm} \ldots (5.136)

for Z - section

\[ f_1 = N + 2e_0 \]  \hspace{2cm} \ldots (5.137)

\[ g_1 = N e_0 \]  \hspace{2cm} \ldots (5.138)

\[ n_1 = T_3 - \left( \frac{a}{X_n} \right)^2 \]  \hspace{2cm} \ldots (5.139)

\[ N = M_4 \frac{2\pi^2 C}{(1/a)^2} \]  \hspace{2cm} \ldots (5.140)
The curve of critical strain $e_{cr}$ versus slenderness ratio (1/a) for channel, trough, angle and Z-section and slenderness ratio (1/R) for circular arc section are shown in Fig. 5.12 to Fig. 5.16.

Moment about minor axis and torsional - flexural buckling load interaction are studied for sections under investigation. Typical interaction characteristics are presented in Fig. 5.17 to Fig. 5.19.

5.7 Concluding Remarks

The theory available for linearly elastic materials is extended to reinforced concrete for monosymmetric and point-symmetric sections taking into account the nonlinear constitutive relation of concrete using tangent modulus approach. The strain variation is assumed to be linear and the tangent modulus corresponding to maximum strain is taken. A criterion to determine the limiting strain when the column is a beam-column is also presented. Expressions for critical strains for sections under study are presented for torsional-flexural buckling. To facilitate the calculation process, curves relating eccentricity of load and position of neutral axis are provided. The variation of critical strain with slenderness ratio for cross sections under investigation for torsional-flexural buckling is presented.
FIGURE 5.1 STRAIN AND STRESS VARIATION - CHANNEL

FIGURE 5.2 STRAIN AND STRESS VARIATION - TROUGH
FIGURE 5.3 STRAIN AND STRESS VARIATION
-ANGLE
Figure 5.4 Strain and Stress Variation - Circular Arc
FIGURE 5.5 STRAIN AND STRESS VARIATION - Z SECTION
($\tan \theta < b/a$)

FIGURE 5.6 STRAIN AND STRESS VARIATION - Z SECTION
($\tan \theta > b/a$)
FIGURE 5.7 \( (e_x/a) \) vs. \( (x_n/a) \)

\[
\frac{e_x}{a} = 0.29 \\
\frac{e_x}{a} \text{ lim} = 0.29 \\
e_x = \text{ ECCENTRICITY} \\
x_n = \text{ POSITION OF NA}
\]
FIGURE 5.8 \( (e_x/a) \text{ Vs. } (x_n/a) \)

TRough Section

\[ b/a = 1 \]

\[ (\frac{e_x}{d})_{\text{lim}} = 0.206 \]

\[ e_x = \text{eccentricity} \]

\[ x_n = \text{position of NA} \]
FIGURE 5.9 $(ex/a)$ Vs. $(x_n/a)$

$$(ex/a)_{lim} = 0.008$$

$e_x = \text{ECCENTRICITY}$

$x_n = \text{POSITION OF NA}$
FIGURE 5.10 \( (e_x/R) \) Vs. \( (x_n/R) \)

CIRCULAR ARC SECTION

\[ \alpha = 37^\circ \]

\[ \left( \frac{e_x}{R} \right)_{\text{lim}} = 0.436 \]

\( e_x = \) ECCENTRICITY

\( x_n = \) POSITION OF NA

FIGURE 5.10 \( (e_x/R) \) Vs. \( (x_n/R) \)
FIGURE 5.12  CRITICAL STRAIN Vs. (l/a) RATIO

CHANNEL SECTION

b/a = 1.0
a/t = 17.6
e_x/a = 0.215
f_ck = 17 N/mm²
p = 0.00677
FIGURE 5.14 CRITICAL STRAIN Vs. (l/a) RATIO

ANGLE SECTION

a/t = 15.70
\( \varepsilon_x/a = 0.078 \)
\( f_{ck} = 25 \text{ N/mm}^2 \)
p = 0.006

FIGURE 5.14 CRITICAL STRAIN Vs. (l/a) RATIO
FIGURE 5.15  CRITICAL STRAIN Vs. (l/R) RATIO
FIGURE 5.16 CRITICAL STRAIN Vs. (l/a) RATIO

$\frac{b}{a} = 1$
$\frac{a}{t} = 15.5$
$\frac{e_x}{a} = 0.30$
$f_{ck} = 17.5 \text{N/mm}^2$
$p = 0.0059$
$tan \Theta < \frac{b}{a}$
FIGURE 5.17 (TORSIONAL-FLEXURAL BUCKLING LOAD)-(MOMENT) INTERACTION DIAGRAM
FIGURE 5.18  (TORSIONAL-FLEXURAL BUCKLING LOAD-MOMENT) INTERACTION DIAGRAM
Figure 5.19 (Torsional-Flexural Buckling Load)-(Moment) Interaction Diagram

- \( a = 200 \text{ mm} \)
- \( t = 12.67 \text{ mm} \)
- \( l = 930 \text{ mm} \)
- \( f_{cc} = 20 \text{ N/mm}^2 \)
- \( p = 0.006 \)
- \( (e_x)_{lim} = 17.5 \text{ mm} \)