CHAPTER XII

FLOW PAST AN IMPULSIVELY STARTED INFINITE VERTICAL PLATE IN A ROTATING FLUID

1. Introduction

The first exact solution of the Navier-Stokes equation was given by Stokes (1851) which explains the motion of a viscous incompressible fluid past an impulsively started infinite horizontal plate in its own plane. This is known as Stokes' first problem in the literature. Lateron, Stewartson (1951) studied this problem for a semi-infinite horizontal plate. Hall (1969) studied this problem for a semi-infinite plate by finite-difference technique.

If the plate is held in a vertical direction and given an impulsive motion in its own plane in a stationary fluid, how the motion is affected by buoyancy forces? This was first studied by Soundalgekar (1977) by the Laplace-transform technique and the effects of heating or cooling of the plate by free-convection currents were discussed. Now if the stationary mass of fluid is made to rotate about this vertical moving plate, how the motion part this moving plate is affected by buoyancy forces, centrifugal forces and Coriolis forces? This has not been studied in the literature. From the physical point of view, this is very complex situation, and then an exact solution is not possible. However, if the system is rotating very slowly, the square and higher order terms in the centrifugal forces can be neglected and then the system is acted only by the thermal buoyancy force and the Coriolis force give by $-\rho_o \beta \bar{g}(T-T_\infty)$ and $-2\rho_o \bar{\Omega} \times \bar{V}$, where $\bar{g}$ is the gravitation vector force, $\bar{\Omega}$, $\bar{V}$ are rotation and velocity vectors respectively. A more general case where all these forces are taken into account is studied by Ker and Lin (1996) while studying the combined convection in a rotating cubic cavity. In practical situation, the slowly rotating system is found useful in the application of rotating disk electrode. Hence the motivation to undertake this study. In Section 2, the mathematical analysis is presented and in Section 3 the conclusions are summarised.
2. Mathematical Analysis:

Consider an infinite vertical plate held in a stationary mass of viscous, incompressible fluid, in a stationary condition and at the same temperature $T_\infty$, coinciding with the plane $z' = 0$. The $x'$-axis is taken along the plate in the vertically upward direction and the $z'$-axis is taken normal to the plate while $y'$-axis is assumed to be in the plane of the plate and normal to both $x'$ and $z'$-axes (Fig. A). Initially, the fluid and the plate rotate in unison with a uniform angular velocity $\Omega'$ about the $z'$-axis. Relative to the rotating fluid, the plate is given an impulsive motion so that it moves with a velocity $U_0$ in its own plane along the $x'$-axis and the plate temperature is raised or lowered to $T_w'$. Then the flow can be shown to be governed by the following equations under usual Boussinesq’s approximation as all the physical variables are functions of $z'$ and $t'$ only.

\[
\frac{\partial u'}{\partial t'} - 2\Omega' V' = \nu \frac{\partial^2 u'}{\partial z'^2} + g\beta (T' - T_\infty') \cos \Omega't' \quad \cdots(1)
\]

\[
\frac{\partial V'}{\partial t'} + 2\Omega' u' = \nu \frac{\partial^2 V'}{\partial z'^2} - g\beta (T' - T_\infty') \sin \Omega't' \quad \cdots(2)
\]

\[
\rho C_p \frac{\partial T'}{\partial t'} = k \frac{\partial^2 T'}{\partial z'^2} \quad \cdots(3)
\]

with following initial and boundary conditions:

\[
\begin{align*}
  u' = V' = 0, & \quad T' = T_\infty', \text{ for all } z', t' \leq 0 \\
  u' = U_0, & \quad V' = 0, \quad T' = T_w' \text{ at } z' = 0 \\
  u' = 0, & \quad V' = 0, \quad T' \rightarrow T_\infty' \text{ as } z' \rightarrow \infty \quad t > 0
\end{align*}
\]

\[
\quad \cdots(4)
\]

All the physical variables are defined in the Nomenclature.

On introducing the following non-dimensional quantities,

\[
\begin{align*}
  u, v = \left(u', v'\right)/U_0, & \quad z = z'U_0 / \nu, \quad t = t'U_0^2 / \nu, \\
  \theta = (T - T')/(T_w' - T_\infty''), & \quad Gr = \frac{\nu g\beta (T_w' - T_\infty'')}{U_0^3}, \\
  Pr = \frac{\mu C_p}{k}, & \quad \Omega' = \Omega' \nu / U_0^2
\end{align*}
\]

\[
\quad \cdots(5)
\]

in equations (1) to (4), we have

\[
\frac{\partial q}{\partial t} + 2i\Omega q = \frac{\partial^2 q}{\partial z^2} + Gr \theta e^{-i\Omega t} \quad \cdots(6)
\]

\[\text{186}\]
\[
\frac{Pr \partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2}
\]

with following initial and boundary conditions:
\[
\begin{align*}
q &= \theta = 0, \quad \text{for all } z, \quad t \leq 0, \\
q &= 1, \theta = 1, \quad \text{at } z = 0 \\
q &= 0, \quad \theta = 0, \quad \text{as } z \to \infty
\end{align*}
\]  \(t > 0\)  \(\ldots(8)\)

Here \(q = u + iV\).

Equations (6) and (7) subject to the initial and boundary conditions (8) are solved by the usual Laplace-transform technique and the solutions are as follows:

\[
\therefore \quad q = \frac{1}{2} \left[ e^{z \sqrt{i \Omega}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} + \sqrt{i \Omega t} \right) + e^{-z \sqrt{i \Omega}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} - \sqrt{i \Omega t} \right) \right]
\]
\[
+ \frac{G}{i \Omega} \left[ e^{ibt} \left[ 2 \left[ e^{z \sqrt{i c}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} + \sqrt{ict} \right) + e^{-z \sqrt{i c}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} - \sqrt{ict} \right) \right] \right]
\]
\[
- \frac{e^{-i \Omega t}}{2} \left[ e^{z \sqrt{i \Omega t}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} + \sqrt{i \Omega t} \right) + e^{-z \sqrt{i \Omega t}} \text{erfc} \left( \frac{z}{2 \sqrt{t}} - \sqrt{i \Omega t} \right) \right]
\]
\[
- \frac{e^{ibt}}{2} \left[ e^{z \sqrt{i \Omega t}} \text{erfc} \left( \frac{z \sqrt{Pr}}{2 \sqrt{t}} + \sqrt{iat} \right) + e^{-z \sqrt{i \Omega t}} \text{erfc} \left( \frac{z \sqrt{Pr}}{2 \sqrt{t}} - \sqrt{iat} \right) \right]
\]
\[
+ e^{-i \Omega t} \text{erfc} \left( \frac{z \sqrt{Pr}}{2 \sqrt{t}} \right) \right] \quad \ldots(9)
\]

where \(a = \frac{\Omega}{Pr - 1}, \quad b = \frac{\Omega(2 - Pr)}{Pr - 1}, \quad c = \frac{\Omega Pr}{Pr - 1}\)

\[
\theta = \text{erfc} \left( \frac{z \sqrt{Pr}}{2 \sqrt{t}} \right) \quad \ldots(10)
\]

We have separated \(q = u + iV\) into its real and imaginary parts and the numerical values of \(u\) and \(V\) are computed. During computation, it is observed that the argument of the error function is a complex quantity and hence to separate it into real and imaginary parts, we have employed the following formula:

\[
\text{erf}(X + iY) = \text{erf}(X) + \frac{e^{-X^2}}{2\pi X} \left[ (\cos 2XY) + \sin 2XY \right]
\]
\[
+ \frac{2}{\pi} e^{-X^2} \sum_{n=1}^{\infty} \frac{e^{-X^2/n^4}}{n^2 + 4X^2} \left[ f_n(X,Y) + ig_n(X,Y) \right] \in (X,Y) \quad \ldots(11)
\]
where \( f_n(X,Y) = 2X - 2X \cosh ny \cos 2XY + n \sinh ny \sin 2XY \)
\[ g_n(X,Y) = 2X \cosh ny \sin 2XY + n \sinh ny \cos 2XY \]
\[ |e(X,Y)| \approx 10^{-16} |\text{erf}(X+iY)| \]

The axial velocity \( u \) is plotted on Figs. 1 - 3. The Grashof number \( Gr \) depends upon \( (T'_w - T_{\infty}) \) which can be positive \( (T_w > T_{\infty}) \) or negative \( (T'_w < T_{\infty}) \) depending upon the temperature of the plate. If the temperature of the plate is greater than that of the surrounding fluid, then the free-convection currents are convected from the hotter plate to the surrounding and then \( Gr > 0 \), which can be interpreted from the physical point of view as cooling of the plate by free convection currents. Then when \( Gr < 0 \), the free convection currents are convected from the surrounding fluid towards the plate and physically, it can be interpreted as heating of the plate by free-convection currents.

On Fig. 1, the axial velocity profiles are shown for air \( (Pr = 0.71), \Omega = 0.4, \ell = 0.2 \) and for different values of \( Gr < 0 \). We observe from this figure that greater cooling of the plate causes a rise in the axial velocity whereas opposite is the effect of the heating of the plate by free convection currents viz., the axial velocity decreases as \((-Gr)\) - values increases. On Fig. 2, the effect of rotating fluid on the axial velocity is shown for air. We observe from this figure that for both \( Gr < 0 \), the axial velocity decreases with increasing the angular velocity. On Fig. 3, the effect of the Prandtl number on the axial velocity profile is shown. We observe from this figure that an increase in the value of the Prandtl number leads to a decrease in the axial velocity when the plate is being cooled \( (Gr > 0) \) by free convection currents whereas under similar conditions, the axial velocity increases when the plate is being heated \( (Gr < 0) \) by free convection currents. On Figs. 4-6, transverse velocity profiles are shown under different conditions. We observe from Fig. 4 that greater cooling of the plate by free convection currents leads to a rise in the transverse velocity whereas greater heating of the plate leads to a fall in the transverse velocity. From Fig. 5, we conclude that an increase in the angular velocity of the rotating fluid leads to a rise in the transverse velocity for both \( Gr < 0 \). From Fig. 6, we observe that when the plate is being cooled by free convection currents, the transverse velocity decreases with the Prandtl number whereas it increases when the plate is being heating by free convection currents.
From the velocity field, we now study the effects of these parameters on the skin-friction. It is given by

\[ \tau' = -\mu \frac{\partial q'}{\partial y'} \bigg|_{z=0} \]  \hspace{1cm} \text{..(12)}

and in non-dimensional form, it becomes

\[ \tau_x = \tau'_x / \rho U_0^2 = -\frac{du}{dz} \bigg|_{z=0} \] \hspace{1cm} \text{..(13)}

and

\[ \tau_y = \tau'_y / \rho U_0^2 = -\frac{dv}{dz} \bigg|_{z=0} \] \hspace{1cm} \text{..(14)}

We have computed numerical values of \( \tau_x \) and \( \tau_y \) and these are listed in Table I. We observe from this table that the axial skin-friction increases with increasing the angular velocity of the fluid \( \Omega \) or owing to an increase in the Prandtl number \( Pr \), but decreases due to grater cooling of the plate. However, when the plate is being heated by free-convection currents, the axial skin-friction \( \tau_x \) increases due to greater heating of the plate by free convection currents or due to increasing \( \Omega \) but decreases with increasing the Prandtl number, for \( Gr < 0 \).

The transverse skin-friction \( \tau_y \) is found to increase due to greater cooling of the plate by free convection currents or due to increasing \( \Omega \) but decreases due to an increase in the Prandtl number, but \( \tau_y \) decreases due to greater heating of the plate by free-convection currents but increases with increasing \( \Omega \) or the Prandtl number \( Pr \).

### Table I

Flow Part an Impulsively Started Vertical Plate in a Rotating Fluid

**Values of Skin-Friction**

<table>
<thead>
<tr>
<th>( Pr )</th>
<th>( t )</th>
<th>( \Omega )</th>
<th>( Gr )</th>
<th>0.8</th>
<th>0.6</th>
<th>0.4</th>
<th>-0.4</th>
<th>-0.6</th>
<th>-0.8</th>
</tr>
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<tr>
<td>0.71</td>
<td>0.2</td>
<td>0.4</td>
<td>1.04886</td>
<td>1.10338</td>
<td>1.15790</td>
<td>1.37599</td>
<td>1.43051</td>
<td>1.48503</td>
<td></td>
</tr>
<tr>
<td>0.71</td>
<td>0.2</td>
<td>0.6</td>
<td>1.05682</td>
<td>1.11103</td>
<td>1.16524</td>
<td>1.38207</td>
<td>1.43627</td>
<td>1.49048</td>
<td></td>
</tr>
<tr>
<td>0.71</td>
<td>0.2</td>
<td>0.8</td>
<td>1.06794</td>
<td>1.12171</td>
<td>1.17548</td>
<td>1.39056</td>
<td>1.44432</td>
<td>1.49809</td>
<td></td>
</tr>
<tr>
<td>7.00</td>
<td>0.2</td>
<td>0.4</td>
<td>1.15664</td>
<td>1.18422</td>
<td>1.21179</td>
<td>1.32210</td>
<td>1.34967</td>
<td>1.37725</td>
<td></td>
</tr>
<tr>
<td>100.0</td>
<td>0.2</td>
<td>0.4</td>
<td>1.23037</td>
<td>1.23951</td>
<td>1.24866</td>
<td>1.28523</td>
<td>1.29438</td>
<td>1.30352</td>
<td></td>
</tr>
</tbody>
</table>

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3. Conclusions:

a) Greater cooling of the plate by free convection currents causes a rise in the axial velocity whereas opposite is the effect of greater heating of the plate by free convection currents.

b) For both $Gr < 0$, the axial velocity decreases with increasing $\Omega$.

c) When the plate is being cooled by free-convection currents, an increase in the Prandtl number leads to a decrease in the axial velocity and opposite is the effect of the Prandtl number when the plate is being heated by free convection currents.

d) The transverse velocity increases due to greater cooling of the plate by free convection currents and opposite is the effect due to greater heating of the plate by free convection currents.

e) Owing to increase in the angular velocity $\Omega$, the transverse velocity always increases for both $Gr > 0$.

f) When $Gr > 0$, an increase in the Prandtl number leads to a decrease in the transverse velocity and opposite is the effect of increasing $Pr$ when $Gr < 0$.

g) An axial skin-friction $\tau_x$ increases when or $Pr$ increases but decreasing owing to greater cooling of the plate by free-convection currents. But $\tau_x$ increases with greater heating of the plate by free convection currents or due to increasing $\Omega$, but decreases with increasing the Prandtl number when $Gr < 0$. 

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h) $\tau_y$ increases due to greater cooling of the plate by free convection currents or due to increasing $\Omega$ but decreases with increasing $Pr$. But $\tau_y$ decreases owing to greater heating of the plate by free convection currents but increases with increasing $\Omega$ or $Pr$.

References:


Fig. A Schematic Diagram
FIG. 1 AXIAL VELOCITY PROFILES

$Pr = 0.71, \Omega = 0.4, t = 0.2$
FIG. 2 AXIAL VELOCITY PROFILES,
Pr = 0.71, t = 0.2
FIG. 3
AXIAL VELOCITY PROFILES,

Pr = 100.0
Pr = 7.0
Pr = 0.71
Gr = -0.4
Gr = 0.4
Ω = 0.4, t = 0.2
FIG. 4. TRANSVERSE VELOCITY PROFILES,
Pr = 0.71, Ω = 0.4, t = 0.2
FIG. 5 TRANSVERS VELOCITY PROFILES,
Pr = 0.71, t = 0.2
FIG. 6 TRANSVERSE VELOCITY PROFILES,
\[ \Omega = 0.4, \ t = 0.2 \]