CHAPTER X

FLOW PAST AN ACCELERATED INFINITE VERTICAL PLATE IN A ROTATING FLUID WITH MASS TRANSFER

1. Introduction

Stokes (1851) was the first to present an exact solution of the Navier-Stokes equation for the flow past an impulsively started infinite horizontal plate in a viscous incompressible fluid. This is discussed in every text-book on viscous flow theory and is known as Stokes's first problem. Stewartson (1951) later on studied this problem for a semi-infinite horizontal plate by analytical method and Hall (1969) presented a finite-difference solution to the same problem presented by Stewartson.

However, if the impulsive motion is given to an infinite vertical plate, how the motion is affected by free-convection current if the plate temperature $T'_{\infty}$ is different from the temperature $T'_{\infty}$ of the fluid far away from the plate? This was studied first by Soundalgekar (1977) by solving the coupled linear equations by the Laplace-transform technique. The effects of heating or cooling of the plate by free-convection currents were discussed. Another interesting and important topic from the practical point of view is to study the effects of a stationary mass of fluid rotating about this vertical moving plate. What is the effect of buoyancy, centrifugal and Coriolis forces on the motion past this vertical moving plate? It is a very complex physical situation and hence an exact solution is not possible. So we have to introduce following assumptions: a) the system is very slowly rotating and hence we can neglect the square and higher order terms in the centrifugal forces. This enables us to study the effects of thermal buoyancy force and the Coriolis force given respectively by $-\rho_0 g \bar{\beta} (T - T_\infty)$ and $-2 \rho_0 \bar{\Omega} \times \bar{V}$, where $\bar{g}$ is the gravitation vector force $\bar{\Omega}$, $\bar{V}$ are rotation and velocity vectors respectively, Ker and Lin (1996) have studied a more general case viz. they considered the effects of all these forces on the combined convection in a rotating cubic cavity. In practical situation, the slowly rotating system is found useful in the application of rotating disk electrode.
Soundalgekear et. al. (to be published) studied the effects of the buoyancy and Cariolis forces on the flow past an impulsively started infinite vertical plate in a slowly rotating fluid. Instead of impulsively started vertical plate, if the plate is accelerated in the vertical direction in a slowly rotating fluid, how the flow is taking its shape? This has not been studied in the literature. Hence the motivation. In Section 2, the mathematical analysis is presented and in Section 3, the conclusions are set out.

2. Mathematical Analysis

We assume that an infinite vertical isothermal plate is surrounded by a stationary mass of viscous incompressible fluid in a stationary condition and both are maintained at the same temperature $T_\infty$. The plate consider with the plane $z' = 0$. The $x'$-axis is taken along the plate in a vertically upward direction with $z'$-axis taken normal to the plate while the $y'$-axis is assumed to be in the plane of the plate and normal to both $x'$-and $z'$-axes. (Fig. A). Initially both the plate and the fluid rotate in uniform with a uniform angular velocity $\Omega'$ about the $z'$-axis. Relative to the rotating fluid, the plate is uniformly accelerated in the vertical direction in its own plane along the $x'$-axis and the plate temperature is raised to $T'_{\infty}$. Then the flow can be shown to be governed by the following system of equations under usual Boussinesq's approximation. Also the plate being infinite in length, the physical variables are functions of $z'$ and $t'$ only.

\[
\frac{\partial u'}{\partial t'} - 2\Omega' v' = \nu \frac{\partial^2 u'}{\partial z'^2} + g\beta (T' - T'_\infty) \cos \Omega' t' + g\beta \ast (c' - c'_\infty) \cos \Omega' t'. \quad (1)
\]

\[
\frac{\partial v'}{\partial t'} - 2\Omega' u' = \nu \frac{\partial^2 v'}{\partial z'^2} + g\beta (T' - T'_\infty) \sin \Omega' t' - g\beta \ast (c' - c'_\infty) \sin \Omega' t'. \quad (2)
\]

\[
\rho \ c_p \frac{\partial T'}{\partial t'} = K \frac{\partial^2 T'}{\partial z'^2}. \quad (3)
\]

\[
\frac{\partial c'}{\partial t'} = D \frac{\partial^2 c'}{\partial z'^2}. \quad (4)
\]

with following initial and boundary conditions:
\[ u' = v' = 0, \quad T' = T'_\infty \quad \text{for all} \quad z', t' \leq 0 \]
\[ u' = c_1 t^*, \quad v' = 0, \quad T = T'_0, \quad C' = C'_0 \quad \text{at} \quad z' = 0 \quad \text{if} \quad t' > 0 \]
\[ u' = 0, \quad v' = 0, \quad T' \to T'_\infty, \quad C' = C'_\infty \quad \text{as} \quad z' \to \infty \]

\[ \text{(5)} \]

All the physical variables are defined in the Nomenclature.

We now introduce the following non-dimensional quantities

\[ u = u'/(v C_1)^{1/3}, \quad v = v'/(v C_1)^{1/3}, \quad t = t'(C_1^2/v)^{1/3} \]
\[ z = z'(C_1/v) \]
\[ \theta = \frac{T' - T'_\infty}{T'_0 - T'_\infty}, \quad C = \frac{C' - C'_0}{C'_\infty - C'_0} \]
\[ \text{Pr} = \frac{\mu C_P}{K}, \quad \text{Sc} = \frac{v}{D}, \quad \Omega = \Omega'(v/C_1^2)^{1/3} \]
\[ Gr = g\beta \Delta T/C_\infty, \quad Gc = g\beta \Delta C/C_1, \quad t^* = t(C_1^2/v)^{1/3} \]

\[ \text{(6)} \]

in equations (1) - (5), which then reduce to

\[ \frac{\partial u}{\partial t} - 2\Omega v = \frac{\partial^2 u}{\partial z^2} + Gr \theta \cos(\Omega t) + Gc C \cos(\Omega t) \]
\[ \text{(7)} \]
\[ \frac{\partial v}{\partial t} + 2\Omega u = \frac{\partial^2 v}{\partial z^2} + Gr \theta \sin(\Omega t) + Gc C \sin(\Omega t) \]
\[ \text{(8)} \]
\[ \text{Pr} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} \]
\[ \text{(9)} \]
\[ \text{Sc} \frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial z^2} \]
\[ \text{(10)} \]

with following initial and boundary conditions:

\[ u = v = 0, \quad \theta = 0, \quad C = 0 \quad \text{for all} \quad z, t \leq 0 \]
\[ u = t, \quad v = 0, \quad \theta = 1, \quad C = 1 \quad \text{at} \quad z = 0 \quad \text{if} \quad t > 0 \]
\[ u = 0, \quad v = 0, \quad \theta = 0, \quad C = 0 \quad \text{as} \quad z \to \infty \]

\[ \text{(11)} \]

We now substitute \( q = u + iv \) and combine equations (8) and (9), which then reduce to

\[ \frac{\partial q}{\partial t} + 2i\Omega q = \frac{\partial^2 q}{\partial z^2} + (Gr \theta + Gc C)e^{-i\Omega t} \]

\[ \text{(12)} \]

Then the initial and boundary conditions reduce to
\[q = 0, \quad \theta = C = 0 \text{ for all } z, t \leq 0\]
\[q = t^*, \quad \theta = 1, \quad C = 1 \text{ at } z = 0\]
\[q = 0, \quad \theta = C = 0 \quad \text{as } z \to \infty\] \(t > 0\)
\[\ldots (13)\]

We apply the Laplace-transform technique to derive the solutions of equations (9), (10) and (12) satisfying the conditions (13) and these are as follows:

\[q = u + iv = \frac{t^*}{2} \left[ \exp(-2\eta \sqrt{i\Omega t}) \text{erfc}(\eta - \sqrt{i\Omega t}) + \exp(2\eta \sqrt{i\Omega t}) \text{erfc}(\eta + \sqrt{i\Omega t}) \right]\]
\[+ i \frac{Gr}{\Omega} \left[ \frac{e^{-i\Omega t}}{2} \left[ \exp(-2\eta \sqrt{i\Omega t}) \text{erfc}(\eta - \sqrt{i\Omega t}) + \exp(2\eta \sqrt{i\Omega t}) \text{erfc}(\eta + \sqrt{i\Omega t}) \right] \right]\]
\[+ i \frac{Gr}{\Omega} \left[ \frac{1}{2} \exp(-i(Pr-2)\Omega t) \left[ \exp(-2\eta \sqrt{\frac{iPr\Omega t}{Pr-1}}) \text{erfc}(\eta - \sqrt{\frac{iPr\Omega t}{Pr-1}}) \right] \right]
\[+ \exp(2\eta \sqrt{\frac{iPr\Omega t}{Pr-1}}) \text{erf}(\eta + \sqrt{\frac{iPr\Omega t}{Pr-1}}))\right]\]
\[+ i \frac{Ge}{\Omega} \left[ \frac{e^{-i\Omega t}}{2} \left[ \exp(-2\eta \sqrt{i\Omega t}) \text{erfc}(\eta - \sqrt{i\Omega t}) + \exp(2\eta \sqrt{i\Omega t}) \text{erfc}(\eta + \sqrt{i\Omega t}) \right] \right]\]
\[\ldots (14)\]
\[
\theta = \text{erfc}\left(\eta \sqrt{\text{Pr}}\right) \quad \ldots(15)
\]
\[
C = \text{erfc}\left(\eta \sqrt{5\text{Sc}}\right) \quad \ldots(16)
\]

where \(\text{erfc}(x)\) is the complementary error function defined by \(\text{erfc}(x) = 1 - \text{erf}(x)\).

In order to understand the effects of the buoyancy forces and mass transfer on the flow pattern, we have carried out the numerical computations for the velocity components \((u, v)\) by separating \(q\) into its real and imaginary parts \((u, v)\). Here the argument of the error function involved in (14) is complex and hence we use the well-known formula [Abramowitz and Stegun [1965] to separate it into real and imaginary parts:

\[
\text{erf}(a + ib) = \text{erf}(a) + \frac{e^{-a^2}}{2\pi a} \left[ 1 - \cos(2ab) + i \sin(2ab) \right] + \frac{2e^{-a^2}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n/4}}{n^2 + 4a^2} \\
+ \left\{ f_n(a,b) + ig_n(a,b) \right\} \in (a,b) \quad \ldots(17)
\]

where

\[
f_n(a,b) = 2a - 2a \cosh(nb) + n \sinh(nb) \sin(2ab)
\]
\[
g_n(a,b) = 2a \cosh(nb) \sin(2ab) + n \sinh(nb) \cos(2ab)
\]

\[|e(a,b)| \approx 10^{-16} |\text{erf}(a + ib)| \quad \ldots(18)
\]

We assume here the fluid surrounding the plate to be air (\(\text{Pr} = 0.71\)) which is commonly occurring in nature whereas the foreign mass present in air is oxygen (\(\text{Sc} = 0.30\)), water- vapour (\(\text{Sc} = 0.60\)) and nitrogen (\(\text{Sc} = 0.78\)). On Fig. 1, both the axial and the transverse velocity profiles are shown for different values of \(\text{Gr}, \text{Gc}, t^*, \Omega\) and presence of water-vapour is assumed in the air. We observe from this figure that an increase in time leads to an increase in both the axial and the transverse velocity.

On Fig. 2, the effects of acceleration of the plate on \(u, v\) are shown. We observe from this figure that due to more and more acceleration of the plate, the axial velocity of air increases whereas the transverse velocity decreases.

On Fig. 3, the effect of Grashof number on the velocity components is shown. We observe from this figure that an increase in the Grashof number \(\text{Gr}\) leads to an increase in the axial velocity and a decrease in the transverse velocity component. On Fig. 4, the effect of modified Grashof number on both the velocity components \(u, v\) is shown. It is observed that the effect of \(\text{Gc}\) is same as that of \(\text{Gr}\).
On Fig. 5, the effect of Schmidt number on the velocity profiles is shown. We observe from that an increase in the Schmidt number $Sc$ leads to a decrease in the axial velocity and an increase in the transverse velocity. On Fig. 6, the effect of the rotation parameter $\Omega$ on the velocity profiles is shown. We observe from this figure that an increase in the rotation parameter $\Omega$ leads to a decrease in the axial and the transverse velocity.

The temperature and concentration profiles are not shown as these are decreasing functions as the Prandtl and Schmidt number increase.

From the velocity field, we can now calculate the skin-friction at the plate which is given by

$$\frac{dq}{d\eta} \bigg|_{\eta=0} = -2i \left[ \sqrt{2i \Omega t} \text{erf}(\sqrt{2i \Omega t}) + \frac{1}{\sqrt{\pi}} \exp(-2i \Omega t) \right]$$

$$- \frac{2i Gr}{\Omega} \exp(-i \Omega t) \left[ \sqrt{i \Omega t} \text{erf}(\sqrt{i \Omega t}) + \frac{1}{\sqrt{\pi}} \exp(-i \Omega t) \right]$$

$$+ \frac{2i Gr}{\Omega} \exp\left(-i \frac{(Pr-2) \Omega t}{Pr-1}\right) \left[ \sqrt{i Pr \Omega t \over \sqrt{Pr-1}} \text{erf}\left(\sqrt{i Pr \Omega t \over \sqrt{Pr-1}}\right) + \frac{1}{\sqrt{\pi}} \exp\left(-i Pr \Omega t \over \sqrt{Pr-1}\right) \right]$$

$$- \frac{2i Gc}{\Omega} \exp(-i \Omega t) \left[ \sqrt{i \Omega t} \text{erf}(\sqrt{i \Omega t}) + \frac{1}{\sqrt{\pi}} \exp(-i \Omega t) \right]$$

$$+ \frac{2i Gc}{\Omega} \exp\left(-i \frac{(Sc-2) \Omega t}{Sc-1}\right) \left[ \sqrt{i Sc \Omega t \over \sqrt{Sc-1}} \text{erf}\left(\sqrt{i Sc \Omega t \over \sqrt{Sc-1}}\right) + \frac{1}{\sqrt{\pi}} \exp\left(-i Sc \Omega t \over \sqrt{Sc-1}\right) \right]$$

$$+ \frac{2i}{\sqrt{\pi} \Omega} \left( Gr \sqrt{Pr} + Gc \sqrt{Sc} \right) \exp(i \Omega t)$$

$$- \frac{2i Gr}{\Omega} \exp\left(-i \frac{(Pr-2) \Omega t}{Pr-1}\right) \left[ \sqrt{i Pr \Omega t \over \sqrt{Pr-1}} \text{erf}\left(\sqrt{i Pr \Omega t \over \sqrt{Pr-1}}\right) + \sqrt{Pr \over \pi} \exp\left(-i Pr \Omega t \over \sqrt{Pr-1}\right) \right]$$

$$- \frac{2i Gc}{\Omega} \exp\left(-i \frac{(Sc-2) \Omega t}{Sc-1}\right) \left[ \sqrt{i Sc \Omega t \over \sqrt{Sc-1}} \text{erf}\left(\sqrt{i Sc \Omega t \over \sqrt{Sc-1}}\right) + \sqrt{Sc \over \pi} \exp\left(-i Sc \Omega t \over \sqrt{Sc-1}\right) \right]$$

(19)

We have computed numerical values of the axial and transverse skin-friction by separating real and imaginary parts of (19) and these are listed in Table I.
Table 1

Values of $\tau_x, \tau_y$

| $t$ | $t^*$ | $\Omega$ | $Gr$ | $Gc$ | $Sc$ | $\frac{du}{d\eta}\big|_{\eta=0}$ | $\frac{dv}{d\eta}\big|_{\eta=0}$ |
|-----|------|--------|-----|-----|-----|----------------|----------------|
| 0.2 | 0.1  | 0.5    | 2.0 | 0.5 | 0.3 | 0.5175         | -0.0979        |
| 0.2 | 0.1  | 0.5    | 2.0 | 0.5 | 0.3 | 0.4990         | -0.0953        |
| 0.2 | 0.1  | 0.5    | 2.0 | 0.5 | 0.78| 0.4918         | -0.0943        |
| 0.2 | 0.1  | 0.5    | 2.0 | 1.0 | 0.6 | 0.6253         | -0.1104        |
| 0.2 | 0.1  | 0.5    | 2.0 | 3.0 | 0.6 | 1.1303         | -0.1706        |
| 0.2 | 0.1  | 0.5    | 4.0 | 0.5 | 0.6 | 0.9854         | -0.1530        |
| 0.2 | 0.1  | 0.5    | 6.0 | 0.5 | 0.6 | 1.4718         | -0.2107        |
| 0.2 | 0.1  | 2.0    | 2.0 | 0.5 | 0.6 | 0.4229         | -0.3684        |
| 0.2 | 0.1  | 5.0    | 2.0 | 0.5 | 0.6 | 0.0517         | -0.7572        |
| 0.2 | 0.1  | 10.0   | 2.0 | 0.5 | 0.6 | 0.6871         | -0.6980        |
| 0.5 | 0.1  | 0.5    | 2.0 | 0.5 | 0.6 | 1.3563         | -0.5047        |
| 0.2 | 0.3  | 0.5    | 2.0 | 0.5 | 0.6 | 0.2719         | -0.1404        |

We observe from this table that an increase in the Schmidt number leads to a decrease in the axial and transverse components of the skin-friction $\tau_x$ and $-\tau_y$. Also an increase $Gr$, $Gc$ also leads to an increase in $\tau_x$, $-\tau_y$, but due to increasing the rotation parameter, $\Omega$, there is a fall in the value of $\tau_x$ and a rise in the value of $-\tau_y$. Due to an increase in time $t$, there is an increase in $\tau_x$ and $-\tau_y$ but due to more acceleration of the plate, both $\tau_x$ and $-\tau_y$ increase.

3. Conclusions:

a) Axial velocity increases due to an increase in time, the Grashof number, the modified Grashof number and more acceleration and decreases with increasing the Schmidt number and the rotation parameter.

b) Transverse velocity decreases with increasing time, the Grashof number, the modified Grashof number and more acceleration of the plate but increases with increasing the Schmidt number and the rotation parameter.

c) $\tau_x$ increases with increasing the Grashof number, modified Grashof number, time $t$ and more acceleration, but decreases with increasing the Schmidt number, rotation parameter and time $t$.  

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d) \{-\tau_3\} increases with increasing \(Gr, \ Gc, \ time \ t\) or more acceleration but decreases due to an increase in \(Sc\) or \(\Omega\).

**Nomenclature**

- \(C'\): species concentration in buoyancy layer
- \(C_1\): a constant
- \(C'_w\): species concentration at the plate
- \(C'_\infty\): species concentration away from the plate
- \(c_p\): specific heat of the fluid at constant pressure
- \(D\): chemical molecular diffusivity
- \(Gr\): Grashof number
- \(g\): gravitational acceleration
- \(Gr_c\): modified Grashof number
- \(K\): thermal conductivity of the fluid
- \(Pr\): Prandtl number
- \(q\): \(u + iv\)
- \(Sc\): Schmidt number
- \(t\): time
- \(T'\): temperature of the fluid
- \(T'_\infty\): temperature of the plate
- \(T'_{\infty}\): temperature of the fluid far away from the plate
- \(u', v'\): velocity components
- \(X', Y', Z'\): coordinate axes
- \(v\): Kinematic viscosity of the fluid
- \(\mu\): viscosity of the fluid
- \(\rho\): density of the fluid
- \(\beta\): volumetric coefficient of thermal expansion
- \(\beta^*\): volumetric coefficient of expansion with concentration
- \(\Omega'\): angular velocity
- \(\Delta T\): \(T'_\infty - T'_\infty\)
- \(\theta\): non-dimensional temperature of the fluid
- \(C\): non-dimensional species concentration
References

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FIG. A SCHEMATIC DIAGRAM
Fig. 1 Axial and Transverse Velocity Profiles

Pr = 0.71
Sc = 0.60
Gr = 2.0
Gc = 0.50
t = 0.10
ω = 0.50
Fig. 2 Axial and Transverse Velocity Profiles

Pr = 0.71
Sc = 0.6
Gx = 2.0
Gc = 0.5
t = 0.2
\eta = 0.5
Fig. 3 Axial and Transverse Velocity Profiles

$Pr = 0.71$
$Sc = 0.60$
$Gc = 0.50$
$\Omega = 0.50$
$t = 0.20$
$t^* = 0.10$
Fig. 4 Axial and Transverse Velocity Profiles

Pr = 0.71
Sc = 0.6
Gx = 2.0
\( \omega = 0.5 \)
t = 0.2
t* = 0.1
Fig. 5 Axial and Transverse Velocity Profiles

- $u$ vs. $n$
- $v$ vs. $n$
- $0.30 = Sc$
- $0.60$
- $0.78$
- $Pr = 0.71$
- $Gx = 2.0$
- $Gc = 0.5$
- $t = 0.2$
- $t^* = 0.1$
- $\Omega = 0.1$
Fig. 6 Axial and Transverse Velocity Profiles

\[ Pr = 0.71 \]
\[ Sc = 0.60 \]
\[ Gr = 2.0 \]
\[ Gc = 0.50 \]
\[ t = 0.20 \]
\[ t^* = 0.10 \]