CHAPTER IV

MONOTONE METHOD FOR FINITE DIFFERENCE
EQUATIONS OF REACTION DIFFUSION

4.1 INTRODUCTION:

The well-known monotone method or the method of upper-lower solutions for continuous parabolic problem is extended to finite difference system of nonlinear time degenerate parabolic initial boundary value problem (IBVP). The central idea of this method is to develop iterative scheme based on the notion of upper and lower solutions. Using upper and lower solutions as distinct initial iterations, two monotone sequences are constructed. It is shown that these two sequences converge monotonically from above and below to maximal and minimal solutions respectively which lead to the Existence-Comparison and Uniqueness results for the solution of the nonlinear finite difference system. Positivity Lemma is the main ingredient used in the proof of these results.

Recently, Pao [29], Dhaigude and Kiwne [7] and Kiwne [18] developed this method for finite difference equations of nonlinear parabolic and elliptic boundary value problems. Here, we develop the monotone scheme for finite difference system of nonlinear, time degenerate parabolic problems.
We plan the chapter as follows:

In section 4.2, finite difference system of nonlinear time degenerate parabolic Dirichlet initial boundary value problem is formulated from the corresponding continuous problem under consideration. Section 4.3 is devoted for the monotone scheme for the discrete problem. Using upper and lower solutions as distinct initial iterations, two monotone sequences are constructed, which converge monotonically from above and below to maximal and minimal solutions respectively. Also the Existence-Comparison and Uniqueness results for discrete problem are discussed in the section 4.4. In the last section concluding remark is given.

4.2. FINITE DIFFERENCE EQUATIONS:

Consider the nonlinear time degenerarte Dirichlet initial boundary value problem (IBVP)

\[ d(x,t)u_t - D(x,t)\nabla^2 u = f(x,t,u); \quad \text{in } D_T \]

Boundary Condition \[ u(x,t) = h(x,t); \quad \text{on } S_T \] (4.2.1)

Initial Condition \[ u(x,0) = \psi(x); \quad \text{in } \Omega \]

Now we write the discrete version of the above continuous IBVP (4.2.1)by conveting it into finite difference equations as in [1, 8, 10, 11, 20]
Let \( i = (i_1, i_2, \ldots, i_p) \) be a multiple index with \( i_v = 0, 1, 2, \ldots, M_v + 1 \) and let \( x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_p}) \) be an arbitrary mesh point in \( \Omega_p \) where \( M_v \) is the total number of interior mesh points in the \( x_v \) co-ordinate direction. Denote by \( \Omega_p, \overline{\Omega}_p, \partial \Omega_p, \Lambda_p, \) and \( S_p \) the sets of mesh points in \( \Omega, \overline{\Omega}, \partial \Omega, \Omega \times (0, T] \) and \( \partial \Omega \times (0, T] \) respectively and \( \overline{\Lambda}_p \) denote the set of all mesh points in \( \overline{\Omega} \times [0, T] \) where \( \overline{\Omega} \) is the closure of \( \Omega \). Let \((i, n)\) be used to represent the mesh point \((x_i, t_n)\).

Set

\[
\begin{align*}
    u_{i,n} &\equiv u(x_i, t_n) \\
    g_{i,n}(u_{i,n}) &\equiv g(x_i, t_n, u(x_i, t_n)) \\
    f_{i,n}(u_{i,n}) &\equiv f(x_i, t_n, u(x_i, t_n)) \\
    d_{i,n} &\equiv d(x_i, t_n) \\
    D_{i,n} &\equiv D(x_i, t_n) \\
    h_{i,n} &\equiv h(x_i, t_n) \\
    u_{i,0} &\equiv u(x_i, 0) \\
    \psi_i &\equiv \psi(x_i)
\end{align*}
\]

Let \( k_n = t_n - t_{n-1} \) be the \( n \)th time increment for \( n = 1, 2, \ldots, N \) and \( h_v \) be the spatial increment in the \( x_v \) co-ordinate direction. Let \( e_v = (0, \ldots, 1, \ldots, 0) \) be the unit vector in \( \mathbb{R}^p \) where the constant 1 appears in the \( v \)th component and zero elsewhere. The standard second order difference approximation \( \Delta^{(v)} u_{i,n} \) is
\[ \Delta^n u(x_i, t_n) = h_n^{-2} \left[ u(x_i + x, e_v, t_n) - 2u(x_i, t_n) + u(x_i - x, e_v, t_n) \right] \]

[cf.][1],[2]. Then the continuous IBVP (4.2.1) becomes

\[ d_{i,n} k_n^{-1}(u_{i,n} - u_{i,n-1}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} u_{i,n} = f_{i,n}(u_{i,n}), \quad (i, n) \in \Lambda_p \]

\[ u_{i,n} = h_{i,n}, \quad (i, n) \in S_p \quad (4.2.2) \]

\[ u_{i,0} = \psi_i, \quad i \in \Omega_p \]

To prove our main results, we develop monotone scheme for the finite difference equation (4.2.2). The following positivity lemma is a discrete version of the positivity lemma 2.2.1 for the continuous problem which play an important role in the construction of monotone sequences.

**LEMMA 4.2.1 [Positivity Lemma]**: Suppose that \( w_{i,n} \) satisfies the following inequalities

\[ d_{i,n} k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} w_{i,n} + c_{i,n} w_{i,n} \geq 0, \quad (i, n) \in \Lambda_p \]

\[ Bw_{i,n} = \alpha(x_i, t_n) |x_i - \hat{x}_i|^{-1} \left[ w(x_i, t_n) - w(\hat{x}_i, t_n) \right] + \beta(x_i, t_n) w(x_i, t_n) \geq 0, \quad (i, n) \in S_p \quad (4.2.3) \]

\[ w_{i,0} \geq 0, \quad i \in \Omega_p \]

where \( c_{i,n} \geq 0, d_{i,n} \geq 0, \hat{x}_i \) is a suitable point in \( \Omega_p \) and \( |x_i - \hat{x}_i| \) is the distance between \( x_i \) and \( \hat{x}_i \).

Then

\[ w_{i,n} \geq 0 \quad \text{in} \quad \bar{\Lambda}_p \]
**Proof:** We prove the Lemma by contradiction method.

Assume that the conclusion of the lemma is false.

i.e. \( w_{i,n} < 0 \) in \( \Lambda_p \)

Then there exists a mesh point \((i_0, n_0) \in \Lambda_p\) such that \( w_{i_0,n_0} < 0 \) in \( \Lambda_p \). So \( w_{i_0,n_0} \) is a negative minimum.

By initial and boundary inequalities in (4.2.3) \((i_0, n_0)\) is an interior point of \( \Lambda_p \).

Then, we have

\[
(w_{i_0,n_0} - w_{i_0,n_0-1}) \leq 0
\]

\[
\Delta^{(v)} w_{i_0,n_0} \geq 0.
\]

Therefore, first inequality (4.2.3) implies

\[
c_{i_0,n_0} w_{i_0,n_0} \geq 0.
\]

This is impossible if \( c_{i_0,n_0} \geq 0 \). Hence the contradiction. When \( c_{i_0,n_0} = 0 \), the first inequality in (4.2.3) can hold only when \( w_{i_0,n_0} - w_{i_0,n_0-1} = 0 \) and \( \Delta^{(v)} w_{i_0,n_0} = 0 \), for all \( v = 1, 2, \ldots, p \).

In this case all the neighbouring mesh points of \((i_0, n_0)\) are minimum
points of \( w_{i,n} \). Clearly \( w_{i,n} \) is negative in \( \Lambda_p \).

This contradicts to \( w_{i,0} \geq 0 \).

Hence the result.

4.3. MONOTONE ITERATIVE SCHEME:

Now, we develop monotone scheme for discrete problem (4.2.2). We define upper and lower solutions of the time degenerate discrete problem (4.2.2).

**DEFINITION 4.3.1:** A function \( \tilde{u}_{i,n} \) in \( \Lambda_p \) is called upper solution of discrete problem (4.2.2) if

\[
d_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\tilde{u}_{i,n} \geq f_{i,n}(\tilde{u}_{i,n}); \quad (i,n) \in \Lambda_p
\]

\[
\tilde{u}_{i,n} \geq h_{i,n}; \quad (i,n) \in S_p
\]

\[
\tilde{u}_{i,0} \geq \psi_i; \quad i \in \Omega_p
\]

**DEFINITION 4.3.2:** A function \( \hat{u}_{i,n} \) in \( \Lambda_p \) is called lower solution of (4.2.2) if

\[
d_{i,n}k_n^{-1}(\hat{u}_{i,n} - \hat{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\hat{u}_{i,n} \leq f_{i,n}(\hat{u}_{i,n}); \quad (i,n) \in \Lambda_p
\]

\[
\hat{u}_{i,n} \leq h_{i,n}; \quad (i,n) \in S_p
\]

\[
\hat{u}_{i,0} \leq \psi_i; \quad i \in \Omega_p
\]

**DEFINITION 4.3.3:** A functions \( \tilde{u}_{i,n} \) and \( \hat{u}_{i,n} \) are called ordered upper and lower solutions if \( \tilde{u}_{i,n} \geq \hat{u}_{i,n} \) in \( \Lambda_p \).
DEFINITION 4.3.4 : For any ordered upper and lower solutions $\tilde{u}_{i,n}, \hat{u}_{i,n}$ the sector is denoted by $S_{i,n}^*$ and is defined as

$$S_{i,n}^* = \{u_{i,n} \in \Lambda_p; \hat{u}_{i,n} \leq u_{i,n} \leq \tilde{u}_{i,n}\}. \quad (4.3.3)$$

Adding $c_{i,n}u_{i,n}$ on both the sides of first equation in(4.2.2), we get

$$d_{i,n}k_n^{-1}\left(u_{i,n} - u_{i,n-1}\right) - \sum_{v=1}^{p} D_{i,n}^{(v)}u_{i,n} + c_{i,n}u_{i,n} = c_{i,n}u_{i,n} + f_{i,n}\left(u_{i,n}\right)$$

Denote $L_3[u_{i,n}] = d_{i,n}k_n^{-1}\left(u_{i,n} - u_{i,n-1}\right) - \sum_{v=1}^{p} D_{i,n}^{(v)}u_{i,n} + c_{i,n}u_{i,n}$

Consider the following iteration process

$$L_3[u_{i,n}^{(k)}] = c_{i,n}u_{i,n}^{(k-1)} + f_{i,n}\left(u_{i,n}^{(k-1)}\right); \quad in \ \Lambda_p$$

$$u_{i,n}^{(k)} = h_{i,n}; \quad in \ \Lambda_p$$

$$u_{i,0}^{(k)} = \psi_i; \quad in \ \Omega_p$$

Choose suitable initial iteration $u_{i,n}^{(0)}$ for $k = 1$ we have

$$L_3[u_{i,n}^{(1)}] = c_{i,n}u_{i,n}^{(0)} + f_{i,n}\left(u_{i,n}^{(0)}\right); \quad in \ \Lambda_p$$

$$u_{i,n}^{(1)} = h_{i,n}; \quad in \ \Lambda_p$$

$$u_{i,0}^{(1)} = \psi_i; \quad in \ \Omega_p$$

Since $u_{i,n}^{(0)}$ is known, the R.H.S. is known. The existence theory for linear time degenerate parabolic initial boundary value problem [15 ] implies that $u_{i,n}^{(0)}$ exists.
Similarly, for $k = 2$ we have

\[
L_3[u_{i,n}^{(2)}] = c_{i,n} u_{i,n}^{(1)} + f_{i,n} (u_{i,n}^{(1)}); \quad \text{in } \Lambda_p \\
u_{i,n}^{(2)} = h_{i,n}; \quad \text{in } S_p \\
u_{i,n}^{(2)} = \psi_i; \quad \text{in } \Omega_p
\]  

(4.3.6)

Since $u_{i,n}^{(1)}$ is known, the R.H.S. is known. The existence theory for linear time degenerate parabolic initial boundary value problem [15] implies that $u_{i,n}^{(2)}$ exists.

Thus for $k = 3, 4, 5, \ldots$ we get $u_{i,n}^{(3)}, u_{i,n}^{(4)}, u_{i,n}^{(5)}, \ldots$.

Thus we construct a sequence $\{u_{i,n}^{(1)}, u_{i,n}^{(2)}, u_{i,n}^{(3)}, \ldots\}$. We denote it by $\{u_{i,n}^{(k)}\}$. The sequence $\{u_{i,n}^{(k)}\}$ so obtained is well defined. We choose initial iteration $u_{i,n}^{(0)} = \tilde{u}_{i,n}$ and denote the sequence by $\{\tilde{u}_{i,n}^{(k)}\}$. We also choose initial iteration $u_{i,n}^{(0)} = \hat{u}_{i,n}$ and denote the sequence by $\{\hat{u}_{i,n}^{(k)}\}$. Thus choosing an upper solution or lower solution as the initial iterations, we get upper and lower sequences $\{\tilde{u}_{i,n}^{(k)}\}$ and $\{\hat{u}_{i,n}^{(k)}\}$ respectively.
LEMMA 4.3.1 [Monotone Property]: Suppose that

(i)  $\tilde{u}_{i,n}, \hat{u}_{i,n}$ are ordered upper and lower solutions of discrete problem (4.2.2),

(ii) $f_{i,n}$ satisfies the one sided Lipschitz condition in $u_{i,n}$

$$f_{i,n}(u_{i,n}^{(1)}) - f_{i,n}(u_{i,n}^{(2)}) \geq -c_{i,n}(u_{i,n}^{(1)} - u_{i,n}^{(2)}) \quad \text{for} \quad u_{i,n}^{(1)}, u_{i,n}^{(2)} \in S_{i,n}$$

Then the sequences $\{\hat{u}_{i,n}^{(k)}\}, \{\tilde{u}_{i,n}^{(k)}\}$ possess the monotone property,

$$\hat{u}_{i,n} \leq u_{i,n}^{(1)} \leq \cdots \leq u_{i,n}^{(k)} \leq u_{i,n}^{(k+1)} \leq \hat{u}_{i,n} \leq \tilde{u}_{i,n} \leq u_{i,n}^{(k)} \leq \cdots \leq u_{i,n} \leq \hat{u}_{i,n} \quad \text{in} \ \Lambda_{\rho}$$

(4.3.7)

PROOF: Define, $w_{i,n} = \tilde{u}_{i,n}^{(0)} - u_{i,n}^{(1)}$, where $\tilde{u}_{i,n}^{(0)} = \hat{u}_{i,n}$

$$\therefore \ w_{i,n} = \hat{u}_{i,n} - \tilde{u}_{i,n}^{(1)}$$

Since $\tilde{u}_{i,n}$ is an upper solution, we have by definition 4.3.1

$$d_{i,n}k_n^{-1}(\tilde{u}_{i,n} - u_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}\tilde{u}_{i,n} \geq f_{i,n}(\tilde{u}_{i,n}) \quad \text{in} \ \Lambda_{\rho}$$

$$\hat{u}_{i,n} \geq h_{i,n} \quad \text{on} \ S_{\rho}$$

$$\hat{u}_{i,0} \geq \psi_i \quad \text{in} \ \Omega_{\rho}$$

(4.3.8)

$$\therefore \ d_{i,n}k_n^{-1}(w_{i,n} - w_{i,n-1}) = d_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \hat{u}_{i,n}) - d_{i,n}k_n^{-1}(\tilde{u}_{i,n}^{(1)} - \hat{u}_{i,n}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}w_{i,n} = -\sum_{\nu=1}^{p} D_{i,n}^{(\nu)}\tilde{u}_{i,n} + \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}\tilde{u}_{i,n}^{(1)}$$

$$c_{i,n}w_{i,n} = c_{i,n}\hat{u}_{i,n} - c_{i,n}\tilde{u}_{i,n}^{(1)}.$$
By adding these we get,
\[
L_3[w_{i,n}]
\]
\[
= \left[ d_{i,n} k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} \tilde{u}_{i,n} + c_{i,n} \tilde{u}_{i,n} \right]
\]
\[
- \left[ d_{i,n} k_n^{-1}(u_{i,n}^{(l)} - u_{i,n-1}^{(l)}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} u_{i,n}^{(l)} + c_{i,n} u_{i,n}^{(l)} \right]
\]
\[
= \left[ d_{i,n} k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} \tilde{u}_{i,n} + c_{i,n} \tilde{u}_{i,n} \right]
\]
\[
- \left[ c_{i,n} \tilde{u}_{i,n}^{(0)} + f_{i,n}(\tilde{u}_{i,n}^{(0)}) \right], \ (By \ 4.3.4)
\]
\[
\geq 0
\]
\[
\therefore L_3[w_{i,n}]
\]
\[
= d_{i,n} k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} w_{i,n} + c_{i,n} w_{i,n} \geq 0 \ in \ \Lambda_p
\]

Also,
\[
w_{i,n} = \tilde{u}_{i,n} - \tilde{u}_{i,n}^{(1)}
\]
\[
= \tilde{u}_{i,n} - h_{i,n} \geq 0; \quad on \ S_p
\]

and
\[
w_{i,0} = \tilde{u}_{i,0} - \psi_i \geq 0; \quad in \ \Omega_p
\]

Now applying the Lemma 4.2.1 we get,
\[
w_{i,n} \geq 0 \quad in \ \bar{\Lambda}_p.
\]

This implies
\[
\tilde{u}_{i,n}^{(1)} \leq \tilde{u}_{i,n}^{(0)}. \quad (4.3.9)
\]
We also know that $\hat{u}_{i,n}$ is a lower solution.

Define

$$w_{i,n} = u_{i,n}^{(1)} - u_{i,n}^{(0)}$$

and using $u_{i,n}^{(0)} = \hat{u}_{i,n}$ we have

$$w_{i,n} = u_{i,n}^{(1)} - \hat{u}_{i,n}.$$}

Using the above argument we get

$$u_{i,n}^{(1)} \geq u_{i,n}^{(0)} \text{ in } \overline{\Lambda}_p$$

(4.3.10)

Next we define

$$w_{i,n}^{(1)} = \overline{u}_{i,n}^{(1)} - \overline{u}_{i,n}^{(1)}.$$

\[ d_{i,n} k_n^{-1} (w_{i,n}^{(1)} - w_{i,n+1}^{(0)}) = d_{i,n} k_n^{-1} (\overline{u}_{i,n}^{(1)} - \overline{u}_{i,n+1}^{(1)}) - d_{i,n} k_n^{-1} (u_{i,n}^{(1)} - u_{i,n+1}^{(1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} w_{i,n}^{(1)} = -\sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \overline{u}_{i,n}^{(1)} + \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} u_{i,n}^{(1)}
\]

\[ \zeta_{i,n} w_{i,n} = \zeta_{i,n} \overline{u}_{i,n}^{(1)} - \zeta_{i,n} u_{i,n}^{(1)}.
\]

By adding these we get,

\[ L_3[w_{i,n}^{(1)}] = d_{i,n} k_n^{-1} (\overline{u}_{i,n}^{(1)} - \overline{u}_{i,n+1}^{(1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \overline{u}_{i,n}^{(1)} + \zeta_{i,n} \overline{u}_{i,n}^{(1)}
\]

\[ -d_{i,n} k_n^{-1} (u_{i,n}^{(1)} - u_{i,n+1}^{(1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} u_{i,n}^{(1)} + \zeta_{i,n} u_{i,n}^{(1)}
\]

\[ = \zeta_{i,n} \overline{u}_{i,n}^{(0)} + f_{i,n} \left( \overline{u}_{i,n}^{(0)} \right) - \zeta_{i,n} u_{i,n}^{(0)} + f_{i,n} \left( u_{i,n}^{(0)} \right) \text{ (By 4.3.4)}
\]

\[ = \zeta_{i,n} (\hat{u}_{i,n} - \hat{u}_{i,n}) + f_{i,n}(\hat{u}_{i,n}) - f_{i,n}(\hat{u}_{i,n}) \geq 0
\]

\[ (\overline{u}_{i,n}^{(0)} = \hat{u}_{i,n}, u_{i,n}^{(0)} = \hat{u}_{i,n} \text{ and using Lipschitz condition})
\]

\[ \therefore L_3[w_{i,n}^{(1)}] = d_{i,n} k_n^{-1} (w_{i,n}^{(1)} - w_{i,n+1}^{(0)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} w_{i,n}^{(1)} + \zeta_{i,n} w_{i,n}^{(1)} \geq 0;
\]

in $\Lambda_p$
\[ w_{i,n}^{(1)} = h_{i,n} - h_{i,n} = 0; \text{ on } S_p \]
\[ w_{i,0}^{(1)} = \psi_j - \psi_j = 0; \text{ in } \Omega_p \]

Applying the Lemma 4.2.1 we get,
\[ w_{i,n}^{(1)} \geq 0 \text{ in } \bar{\Lambda}_p \]

This shows that \[ \bar{u}_{i,n}^{(l)} \leq \bar{u}_{i,n}^{(l)} \].

Thus we conclude that
\[ u_{i,n}^{(0)} \leq u_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(0)}. \] (4.3.11)

Assume by induction
\[ u_{i,n}^{(k-1)} \leq u_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k-1)} \text{ in } \bar{\Lambda}_p \]. (4.3.12)

Define a function \[ w_{i,n}^{(k)} = \bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)} \]

\[ \therefore d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) = d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k-1)}) - d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k+1)}) \]
\[ - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} w_{i,n}^{(k)} = - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} \bar{u}_{i,n}^{(k)} + \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} \bar{u}_{i,n}^{(k+1)} \]
\[ c_{i,n} w_{i,n}^{(k)} = c_{i,n} \bar{u}_{i,n}^{(k)} - c_{i,n} \bar{u}_{i,n}^{(k+1)}. \]

By adding these we get
\[ L_3[w_{i,n}^{(k)}] = \left[ d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} \bar{u}_{i,n}^{(k)} + c_{i,n} \bar{u}_{i,n}^{(k)} \right] \]
\[ - \left[ d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k+1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} \bar{u}_{i,n}^{(k+1)} + c_{i,n} \bar{u}_{i,n}^{(k+1)} \right] \]
\[ = \left[ c_{i,n} \bar{u}_{i,n}^{(k+1)} + f_{i,n}(\bar{u}_{i,n}^{(k-1)}) \right] - \left[ c_{i,n} \bar{u}_{i,n}^{(k)} + f_{i,n}(\bar{u}_{i,n}^{(k)}) \right] \geq 0; \text{ in } \Lambda_p \]

\[ L_3[w_{i,n}^{(k)}] = d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}w_{i,n}^{(k)} + c_{i,n}w_{i,n}^{(k)} \geq 0; \]

\[ \text{in } \Lambda_p \]

Also \[ w_{i,n}^{(k)} = h_{i,n} - h_{i,n} = 0; \text{ on } S_p \]

\[ w_{i,0}^{(k)} = \psi_i - \psi_i = 0; \text{ in } \Omega_p \]

Applying the Lemma 4.2.1 we get

\[ w_{i,n}^{(k)} \geq 0 \]

i.e.

\[ \bar{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k)} \text{ in } \Lambda_p \] (4.3.13)

Define a function \[ w_{i,n}^{(k)} = u_{i,n}^{(k+1)} - u_{i,n}^{(k)} \]

\[ \therefore d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) = d_{i,n}k_n^{-1}(u_{i,n}^{(k+1)} - u_{i,n}^{(k)}) - d_{i,n}k_n^{-1}(u_{i,n}^{(k)} - u_{i,n}^{(k-1)}) \]

\[ - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}w_{i,n}^{(k)} = - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}u_{i,n}^{(k+1)} + \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}u_{i,n}^{(k)} \]

\[ c_{i,n}w_{i,n}^{(k)} = c_{i,n}u_{i,n}^{(k+1)} - c_{i,n}u_{i,n}^{(k)} \]

By adding these we get,

\[ L_3[w_{i,n}^{(k)}] = \left[ d_{i,n}k_n^{-1}(u_{i,n}^{(k+1)} - u_{i,n}^{(k)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}u_{i,n}^{(k+1)} + c_{i,n}u_{i,n}^{(k+1)} \right] \]

\[ - \left[ d_{i,n}k_n^{-1}(u_{i,n}^{(k)} - u_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}u_{i,n}^{(k)} + c_{i,n}u_{i,n}^{(k)} \right] \]

\[ = \left[ c_{i,n}u_{i,n}^{(k)} + d_{i,n}(u_{i,n}^{(k)}) \right] - \left[ c_{i,n}u_{i,n}^{(k+1)} + d_{i,n}(u_{i,n}^{(k+1)}) \right] \geq 0 \]

\[ L_3[w_{i,n}^{(k)}] = d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)}w_{i,n}^{(k)} + c_{i,n}w_{i,n}^{(k)} \geq 0; \text{ in } \Lambda_p \]
Also,
\[ w_{i,n}^{(k)} = h_{i,n} - h_{i,n} = 0; \text{ on } S_p \]
\[ w_{i,0}^{(k)} = \psi_i - \psi_i = 0; \text{ in } \Omega_p. \]

By applying the Lemma 4.2.1 we have,
\[ w_{i,n}^{(k)} \geq 0 \text{ in } \bar{\Lambda}_p \]
i.e.
\[ u_{i,n}^{(k+1)} \geq u_{i,n}^{(k)}. \quad (4.3.14) \]

Define a function \[ w_{i,n}^{(k+1)} = \bar{u}_{i,n}^{(k+1)} - u_{i,n}^{(k+1)} \]

\[ \therefore d_{i,n} k_n^{-1} (w_{i,n}^{(k+1)} - w_{i,n}^{(k+1)}) = d_{i,n} k_n^{-1} (\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k+1)}) \]
\[ - d_{i,n} k_n^{-1} (u_{i,n}^{(k+1)} - u_{i,n}^{(k+1)}) \]
\[ - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} w_{i,n}^{(k+1)} = \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} \bar{u}_{i,n}^{(k+1)} + \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} u_{i,n}^{(k+1)} \]
\[ \mathcal{C}_{i,n} w_{i,n}^{(k+1)} = \mathcal{C}_{i,n} \bar{u}_{i,n}^{(k+1)} - \mathcal{C}_{i,n} u_{i,n}^{(k+1)} \]

By adding these we get

\[ L_3[w_{i,n}^{(k+1)}] = \left[ d_{i,n} k_n^{-1} (\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k+1)}) - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} \bar{u}_{i,n}^{(k+1)} + \mathcal{C}_{i,n} \bar{u}_{i,n}^{(k+1)} \right] \]
\[ - \left[ d_{i,n} k_n^{-1} (u_{i,n}^{(k+1)} - u_{i,n}^{(k+1)}) - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} u_{i,n}^{(k+1)} + \mathcal{C}_{i,n} u_{i,n}^{(k+1)} \right] \]
\[ = \left[ \mathcal{C}_{i,n} \bar{u}_{i,n}^{(k)} + f_{i,n}(\bar{u}_{i,n}^{(k)}) \right] - \left[ \mathcal{C}_{i,n} u_{i,n}^{(k)} + f_{i,n}(u_{i,n}^{(k)}) \right] \geq 0 \]

\[ \therefore L_3[w_{i,n}^{(k+1)}] = d_{i,n} k_n^{-1} (w_{i,n}^{(k+1)} - w_{i,n}^{(k+1)}) - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} w_{i,n}^{(k+1)} \]
\[ + \mathcal{C}_{i,n} w_{i,n}^{(k+1)} \geq 0; \text{ in } \Lambda_p \]
Also,
\[ w_{i,n}^{(k+1)} = h_{i,n} - h_{i,n} = 0; \text{ on } S_p \]
\[ w_{i,0}^{(k+1)} = \psi_i - \psi_i = 0; \text{ in } \Omega_p \]

By applying the Lemma 4.2.1 we have,
\[ w_{i,n}^{(k+1)} \geq 0. \text{ in } \overline{\Lambda}_p \]
\[ u_{i,n}^{(k+1)} \leq \overline{u}_{i,n}^{(k+1)} \] (4.3.15)

Thus we have from principle of induction
\[ \hat{u}_{i,n} \leq u_{i,n}^{(1)} \leq \cdots \leq u_{i,n}^{(k)} \leq \overline{u}_{i,n}^{(k+1)} \leq u_{i,n}^{(k)} \leq \overline{u}_{i,n}^{(k)} \leq \cdots \]
\[ \leq \cdots \leq \overline{u}_{i,n}^{(1)} \leq \tilde{u}_{i,n} \text{ in } \overline{\Lambda}_p, \text{ for } k=1,2,3,\ldots \]

Hence the result.
4.4 APPLICATIONS:

**THEOREM 4.4.1 [Existence-Comparison Theorem]:**

Suppose that

(i) \( \hat{u}_{i,n}, \tilde{u}_{i,n} \) are upper and lower solutions of the time degenerate discrete IBVP (4.2.2)

\[
d_{i,n}k_n^{-1}(u_{i,n}, -u_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)}u_{i,n} = f_{i,n}(u_{i,n}); \quad (i,n) \in \Lambda_p
\]

\[
u_{i,n} = h_{i,n}; \quad (i,n) \in S_p
\]

\[
u_{i,0} = \psi_i; \quad i \in \Omega_p
\]

with \( \tilde{u}_{i,n} \geq \hat{u}_{i,n} \),

(ii) \( f_{i,n}(u_{i,n}) \) satisfies the one sided Lipschitz condition in \( u_{i,n} \)

\[
f_{i,n}(u_{i,n}^{(1)}) - f_{i,n}(u_{i,n}^{(2)}) \geq -c_{i,n}(u_{i,n}^{(1)} - u_{i,n}^{(2)}) \quad \text{for} \quad u_{i,n}^{(1)}, u_{i,n}^{(2)} \in S_{i,n}^* \quad (4.4.1)
\]

Then the maximal sequence \( \{\tilde{u}_{i,n}^{(k)}\} \) converges monotonically from above to a solution \( \tilde{u} = \tilde{u}_{i,n} \) and the minimal sequence \( \{\hat{u}_{i,n}^{(k)}\} \) converges monotonically from below to a solution \( \hat{u} = \hat{u}_{i,n} \) of the discrete IBVP (4.2.2). Moreover, \( \hat{u}_{i,n} \) and \( \tilde{u}_{i,n} \) satisfy the relation

\[
\hat{u}_{i,n} \leq \tilde{u}_{i,n}^{(1)} \leq \ldots \leq u_{i,n} \leq \hat{u}_{i,n} \leq \ldots \leq \tilde{u}_{i,n}^{(1)} \leq \tilde{u}_{i,n}, \quad (i,n) \in \Lambda_p \quad (4.4.2)
\]
**Proof:** We show the monotone convergence of the maximal and minimal sequences \( \{\bar{u}_{i,n}^{(k)}\} \) and \( \{\underline{u}_{i,n}^{(k)}\} \) respectively.

Suppose

\[
w_{i,n} = \bar{u}_{i,n}^{(0)} - \bar{u}_{i,n}^{(1)} \quad \text{where} \quad \bar{u}_{i,n}^{(0)} = \tilde{u}_{i,n}.
\]

Since \( \tilde{u}_{i,n} \) is an upper solution, we have by definition 4.3.1

\[
d_{i,n}k_{n}^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\tilde{u}_{i,n} \geq f_{i,n}(\tilde{u}_{i,n}); \quad (i,n) \in \Lambda_{p}
\]

\[
\tilde{u}_{i,n} \geq h_{i,n}; \quad \text{on} \ S_{p}
\]

\[
\tilde{u}_{i,0} \geq \psi_{i}; \quad \text{in} \ \Omega_{p}
\]

\[
\therefore \ d_{i,n}k_{n}^{-1}(w_{i,n} - w_{i,n-1}) = d_{i,n}k_{n}^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1})
\]

\[
\sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}w_{i,n} = \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\tilde{u}_{i,n} + \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(1)}
\]

\[
\mathcal{L}_{i,n} w_{i,n} = \mathcal{L}_{i,n} \tilde{u}_{i,n} - \mathcal{L}_{i,n} \underline{u}_{i,n}\]

By adding these and using the definition 4.3.1 and iterative system (4.3.5) we have,

\[
L_{3}[w_{i,n}] = [d_{i,n}k_{n}^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\tilde{u}_{i,n} + \mathcal{L}_{i,n} \tilde{u}_{i,n}]
\]

\[
- [d_{i,n}k_{n}^{-1}(\bar{u}_{i,n}^{(1)} - \bar{u}_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(1)} + \mathcal{L}_{i,n} \bar{u}_{i,n}^{(1)}]
\]

\[
= d_{i,n}k_{n}^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\tilde{u}_{i,n} + \mathcal{L}_{i,n} \tilde{u}_{i,n}
\]

\[
- [\mathcal{L}_{i,n} \bar{u}_{i,n}^{(0)} + f_{i,n}(\bar{u}_{i,n})]
\]

\[
\therefore L_{3}[w_{i,n}] = d_{i,n}k_{n}^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\nabla^{(\nu)}w_{i,n} + \mathcal{L}_{i,n} w_{i,n} \geq 0; \text{in} \ \Lambda_{p}
\]
Also,
\[ w_{i,n} = (\tilde{u}_{i,n} - \bar{u}_{i,n}^{(1)}) \]
\[ = \tilde{u}_{i,n} - h_{i,n} \geq 0; \text{ on } S_p \]
and
\[ w_{i,0} = \bar{u}_{i,0}^{(0)} - \bar{u}_{i,n}^{(1)} \]
\[ = \tilde{u}_{i,n} - \psi_i \geq 0; \text{ in } \Omega_p \]

By using the Lemma 4.2.1 we get,
\[ w_{i,n} \geq 0 \text{ in } \bar{\Lambda}_p \]
\[ \bar{u}_{i,n}^{(0)} \geq \bar{u}_{i,n}^{(1)} \quad (4.4.3) \]

Similarly, if we consider
\[ w_{i,n} = u_{i,n}^{(1)} - u_{i,n}^{(0)} \]

Then we can prove
\[ u_{i,n}^{(1)} \geq u_{i,n}^{(0)} \text{ in } \bar{\Lambda}_p \]

Next suppose that
\[ w_{i,n}^{(1)} = \bar{u}_{i,n}^{(1)} - u_{i,n}^{(1)} \quad (4.4.4) \]

\[ : d_{i,n} k_n^{-1} (w_{i,n}^{(1)} - w_{i,n-1}^{(1)}) = d_{i,n} k_n^{-1} (\bar{u}_{i,n}^{(1)} - \bar{u}_{i,n-1}^{(1)}) - d_{i,n} k_n^{-1} (u_{i,n}^{(1)} - u_{i,n-1}^{(1)}) \]
\[ - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} w_{i,n} = - \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} \bar{u}_{i,n}^{(1)} + \sum_{\nu=1}^{P} D_{i,n} \Delta^{(\nu)} u_{i,n}^{(1)} \]
\[ \bar{c}_{i,n} w_{i,p} = \bar{c}_{i,n} \bar{u}_{i,n}^{(1)} - \bar{c}_{i,n} u_{i,n}^{(1)} \]

By adding these and by using Lipschitz condition and iterative system we have,
\[ L_3[w_{i,n}^{(1)}] = \left[ d_{i,n}k_n^{-1} (\bar{u}_{i,n}^{(1)} - u_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} \bar{u}_{i,n}^{(1)} + c_{i,n} \bar{u}_{i,n}^{(1)} \right] \\
- \left[ d_{i,n}k_n^{-1} (u_{i,n}^{(1)} - u_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} u_{i,n}^{(1)} + c_{i,n} \bar{u}_{i,n}^{(1)} \right] \\
= (f_{i,n}(\hat{u}_{i,n}) + c_{i,n} \bar{u}_{i,n}) - (f_{i,n}(\hat{u}_{i,n}) + c_{i,n} \bar{u}_{i,n}) \\
= f_{i,n}(\bar{u}_{i,n}) + f_{i,n}(\hat{u}_{i,n}) + c_{i,n}(\bar{u}_{i,n} - \hat{u}_{i,n}) \geq 0. \\
\text{(By using Lipschitz condition).} \\
\therefore L_3[w_{i,n}^{(1)}] = d_{i,n}k_n^{-1}(w_{i,n}^{(1)} - w_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(\nu)} w_{i,n}^{(1)} \\
+ c_{i,n} w_{i,n}^{(1)} \geq 0. \quad \text{in } \Lambda_p \]

Also
\[ w_{i,n}^{(1)} = \bar{u}_{i,n}^{(1)} - u_{i,n}^{(1)} = h_{i,n} - h_{i,n} = 0; \quad \text{on } S_p \]

and \[ \therefore w_{i,0}^{(1)} = \bar{u}_{i,0}^{(1)} - u_{i,0}^{(1)} = \psi_i - \psi_i = 0 \quad \text{in } \Omega_p \]

Applying the Lemma 4.2.1, we get,
\[ w_{i,n}^{(1)} \geq 0 \quad \text{in } \bar{\Lambda}_p \]

\[ \therefore \bar{u}_{i,n}^{(1)} \geq u_{i,n}^{(1)} \quad \text{in } \bar{\Lambda}_p \quad (4.4.5) \]

Thus from inequalities (4.4.3), (4.4.4) and (4.4.5), we have,
\[ u_{i,n}^{(0)} \leq u_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(0)} \quad \text{in } \bar{\Lambda}_p \]

Assume that
\[ u_{i,n}^{(k-1)} \leq u_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k-1)} \quad \text{in } \bar{\Lambda}_p. \]
Suppose \( w_{i,n}^{(k)} = \bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)} \)

\[
\therefore d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) = d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k-1)}) - d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k+1)}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}w_{i,n}^{(k)} = -\sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(k)} + \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(k+1)}
\]

\[
\therefore \bar{c}_{i,n}w_{i,n}^{(k)} = \bar{c}_{i,n}\bar{u}_{i,n}^{(k)} - \bar{c}_{i,n}\bar{u}_{i,n}^{(k+1)}
\]

By adding we get

\[
L_3[w_{i,n}^{(k)}] = \left[ d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(k)} + \bar{c}_{i,n}\bar{u}_{i,n}^{(k)} \right] - \left[ d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}\bar{u}_{i,n}^{(k+1)} + \bar{c}_{i,n}\bar{u}_{i,n}^{(k+1)} \right] = \left[ \bar{c}_{i,n}\bar{u}_{i,n}^{(k-1)} + f_{i,n}\bar{u}_{i,n}^{(k-1)} \right] - \left[ \bar{c}_{i,n}\bar{u}_{i,n}^{(k)} + f_{i,n}\bar{u}_{i,n}^{(k)} \right] \geq 0
\]

\[
\therefore L_3[w_{i,n}^{(k)}] = d_{i,n}k_n^{-1}(w_{i,n}^{(k)} - w_{i,n}^{(k-1)}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}w_{i,n}^{(k)} + \bar{c}_{i,n}w_{i,n}^{(k)} \geq 0; \quad \text{in } \Lambda_p.
\]

Also,

\[
w_{i,n}^{(k)} = (\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)}) = h_{i,n} - h_{i,n} = 0; \quad \text{on } S_p
\]

\[
w_{i,0}^{(k)} = (\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)}) = \psi_{i,0} - \psi_{i} = 0; \quad \text{on } \Omega_p
\]
By applying the Lemma 4.2.1 we can obtain,

\[ w^{(k)}_{i,n} \geq 0 \]

\[ \bar{u}^{(k)}_{i,n} \geq \bar{u}^{(k+1)}_{i,n} \quad (4.4.6) \]

Suppose \( \omega^{(k)}_{i,n} = \omega^{(k+1)}_{i,n} - \omega^{(k)}_{i,n} \)

\[ d_{i,n}k_{n}^{-1}(\omega^{(k)}_{i,n} - \omega^{(k+1)}_{i,n}) = d_{i,n}k_{n}^{-1}(\omega^{(k)}_{i,n} - \omega^{(k)}_{i,n-1}) \]

\[ -\sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} w^{(k)}_{i,n} = -\sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} u^{(k)}_{i,n} + \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} u^{(k+1)}_{i,n} \]

\[ c_{i,n} w^{(k)}_{i,n} = c_{i,n} u^{(k)}_{i,n} - c_{i,n} u^{(k+1)}_{i,n} \]

By adding these we get,

\[ L_{3}[\omega^{(k)}_{i,n}] = \left[ d_{i,n}k_{n}^{-1}(\omega^{(k)}_{i,n} - \omega^{(k)}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} u^{(k)}_{i,n} + c_{i,n} u^{(k)}_{i,n} \right] \]

\[ - \left[ d_{i,n}k_{n}^{-1}(\omega^{(k+1)}_{i,n} - \omega^{(k+1)}_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} u^{(k+1)}_{i,n} + c_{i,n} u^{(k+1)}_{i,n} \right] \]

\[ = \left[ c_{i,n} u^{(k-1)}_{i,n} + f_{i,n}(\omega^{(k-1)}_{i,n}) \right] - \left[ c_{i,n} u^{(k)}_{i,n} + f_{i,n}(u^{(k)}_{i,n}) \right] \geq 0 \]

\[ \therefore L_{3}[\omega^{(k)}_{i,n}] = d_{i,n}k_{n}^{-1}(\omega^{(k)}_{i,n} - \omega^{(k+1)}_{i,n}) - \sum_{\nu=1}^{p} D_{i,n} \Delta^{(\nu)} w^{(k)}_{i,n} + c_{i,n} w^{(k)}_{i,n} \geq 0; \]

in \( \Lambda_{p} \)
Also
\[
\begin{align*}
\omega_{i,n}^{(k+1)} - \omega_{i,n}^{(k)} &= h_{i,n} - h_{i,n} = 0; \text{ on } S_p \\
\omega_{i,0} - \omega_{i,0} &= \psi_i - \psi_i = 0; \text{ in } \Omega_p
\end{align*}
\]

By applying the Lemma 4.2.1 we have,
\[
\omega_{i,n}^{(k)} \geq 0 \text{ in } \Lambda_p
\]

i.e. \( \omega_{i,n}^{(k+1)} \geq \omega_{i,n}^{(k)} \) \( \text{(4.4.7)} \)

Define a function \( \omega_{i,n}^{(k+1)} = \omega_{i,n}^{(k+1)} - \omega_{i,n}^{(k+1)} \)

\[
\begin{align*}
\therefore d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) &= d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) \\
&= -d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) \\
&- \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)} = -\sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)} + \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)} \\
&= c_{i,n} \omega_{i,n}^{(k+1)} = c_{i,n} \omega_{i,n}^{(k+1)} - c_{i,n} \omega_{i,n}^{(k+1)}
\end{align*}
\]

By adding these we get,
\[
L_3[\omega_{i,n}^{(k+1)}] = \begin{bmatrix}
d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)} c_{i,n} \omega_{i,n}^{(k+1)} \\
- d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)} + c_{i,n} \omega_{i,n}^{(k+1)}
\end{bmatrix}
\]
\[
= [c_{i,n} \omega_{i,n}^{(k+1)} + f_{i,n} (\omega_{i,n}^{(k+1)})] - [c_{i,n} \omega_{i,n}^{(k+1)} + f_{i,n} (\omega_{i,n}^{(k+1)})] \geq 0
\]

\[
\therefore L_3[\omega_{i,n}^{(k+1)}] = d_{i,n} k_n^{-1}(\omega_{i,n}^{(k+1)} - \omega_{i,n-1}^{(k+1)}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} \omega_{i,n}^{(k+1)}
\]
\[
+ c_{i,n} \omega_{i,n}^{(k+1)} \geq 0; \text{ in } \Lambda_p
\]
Also,
\[
\begin{align*}
    w_{i,n}^{(k+1)} &= \overline{u}_{i,n}^{(k+1)} - u_{i,n}^{(k+1)} = h_{i,n} - h_{i,n} = 0; \text{ on } S_p \\
    w_{i,0}^{(k+1)} &= \overline{u}_{i,0}^{(k+1)} - u_{i,0}^{(k+1)} = \psi_{i} - \psi_{i} = 0; \text{ in } \Omega_p
\end{align*}
\]

By applying the Lemma 4.2.1 we have,

\[
\begin{align*}
w_{i,n}^{(k+1)} &\geq 0 \text{ in } \overline{\Lambda}_p \\
i.e. \quad u_{i,n}^{(k+1)} &\leq \overline{u}_{i,n}^{(k+1)} \tag{4.4.8}
\end{align*}
\]

Therefore (4.4.6), (4.4.7) and (4.4.8) gives,

\[
\begin{align*}
    u_{i,n}^{(k)} &\leq u_{i,n}^{(k+1)} \leq \overline{u}_{i,n}^{(k+1)} \leq \overline{u}_{i,n}^{(k)} \text{ in } \overline{\Lambda}_p
\end{align*}
\]

Thus monotone property (4.4.2) follows from principle of mathematical induction.

Now we conclude that the sequence \( \{\overline{u}_{i,n}^{(k)}\} \) is monotone nonincreasing and is bounded from below, hence it is convergent. Also the sequence \( \{u_{i,n}^{(k)}\} \) is monotone nondecreasing and is bounded from above, hence it is convergent.

So,
\[
\lim_{k \to \infty} \overline{u}_{i,n}^{(k)} = \overline{u}_{i,n}
\]

and
\[
\lim_{k \to \infty} u_{i,n}^{(k)} = u_{i,n}
\]
exist and are called maximal and minimal solutions respectively, of the discrete time degenerate parabolic initial boundary value problem (4.2.2) and they satisfy the monotone property

\[ \tilde{u}_{i,n} \leq u_{i,n}^{(1)} \leq u_{i,n}^{(2)} \leq \ldots \leq u_{i,n} \leq \tilde{u}_{i,n} \leq \ldots \leq \tilde{u}_{i,n}^{(2)} \leq u_{i,n}^{(1)} \leq \tilde{u}_{i,n}, (i,n) \in \Lambda_p \]

Hence the result.

**UNIQUENESS THEOREM 4.4.2**: Suppose that

(i) \( \tilde{u}_{i,n}, \hat{u}_{i,n} \) are upper and lower solutions of the discrete time degenerate IBVP (4.2.2),

(ii) \( f_{i,n}(u_{i,n}) \) satisfies the Lipschitz condition in \( u_{i,n} \)

\[-c_{i,n}(u_{i,n}^{(1)} - u_{i,n}^{(2)}) \leq f_{i,n}(u_{i,n}^{(1)}) - f_{i,n}(u_{i,n}^{(2)}) \leq \bar{c}_{i,n}(u_{i,n}^{(1)} - u_{i,n}^{(2)})\]

Then the discrete time degenerate IBVP (4.2.2) has a unique solution.

**PROOF**: We know that \( \tilde{u}_{i,n}, \hat{u}_{i,n} \) are maximal and minimal solutions respectively of the discrete time degenerate IBVP (4.2.2).

To prove uniqueness we suppose that,

\[ w_{i,n} = \tilde{u}_{i,n} - u_{i,n} \]


\begin{align*}
& d_{i,n}^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} w_{i,n} = f_{i,n}(\bar{u}_{i,n}) - f_{i,n}(u_{i,n}) \\
& \quad \geq -\varepsilon_{i,n}(\bar{u}_{i,n} - u_{i,n}) \\
& \quad \geq -\varepsilon_{i,n} w_{i,n} \\
\therefore & d_{i,n}^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{v=1}^{p} D_{i,n} \Delta^{(v)} w_{i,n} \geq -\varepsilon_{i,n} w_{i,n} \quad \text{in } \Lambda_p \tag{4.4.9}
\end{align*}

\begin{align*}
& w_{i,n} = \bar{u}_{i,n} - u_{i,n} = h_{i,n} - h_{i,n} = 0; \quad \text{on } S_p \\
& w_{i,0} = \bar{u}_{i,0} - u_{i,0} = \psi_i - \psi_i = 0; \quad \text{in } S_p
\end{align*}

\therefore By using the Lemma 4.2.1 we get,

\begin{align*}
& w_{i,n} \geq 0 \quad \text{in } \bar{\Lambda}_p \\
\therefore & \bar{u}_{i,n} \geq u_{i,n} \tag{4.4.10}
\end{align*}

Also we can show that

\begin{align*}
& \bar{u}_{i,n} \equiv \leq u_{i,n} \quad \text{in } \Lambda_p \\
\text{Inequalities (4.4.10) and (4.4.11) implies that} \\
& \bar{u}_{i,n} \equiv u_{i,n} \quad \text{in } \Lambda_p
\end{align*}

\therefore IBVP (4.2.2) has unique solution.

Hence the result.
4.5 CONCLUDING REMARK: Thus it has been possible to develop monotone method for nonlinear time degenerate parabolic discrete IBVP (4.2.2). It has been also possible to prove qualitative properties such as Existence-Comparison and Uniqueness of solution of discrete IBVP (4.2.2), as an application of monotone method.

Such results were available only for nonlinear parabolic discrete parabolic problems.