CHAPTER I

INTRODUCTION

1.1 INTRODUCTION

Reaction diffusion equations are important equations in the study of Physics, Ecology, Heat and Mass Transfer and many other branches of Science and Engineering. It leads to extensive study of non-linear parabolic differential equations. We study these equations both analytically and numerically by developing monotone method or method of upper and lower solutions. Monotone method is one of the best important method in the theory of differential equations. Now a days, it is one of the successful research areas, which has occupied a central place in the study of both non-linear elliptic and parabolic partial differential equations. This method is a constructive method to prove existence, uniqueness, stability, etc. Its theory is beautiful, its technique is powerful and its impact upon science is also profound.

During past few years various researchers have studied monotone method and its applications. It is well illustrated in the elegant books by Ladde, Lakshminatham and Vatasala [19] and Pao [28] and references therein.
1.2 PARABOLIC OPERATORS:

Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ bounded by $\partial \Omega$. Now, we define the parabolic domain $D_T$ bounded by $S_T$, the parabolic boundary as follows:

$$D_T = \Omega \times (0, T], S_T = \partial \Omega \times (0, T]$$

Consider the general second order linear operator

$$\mathcal{L} \equiv \frac{\partial}{\partial t} - L_1$$

where,

$$L_1 = \sum_{i,j=1}^{n} a_{i,j}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial}{\partial x_i}$$

**DEFINITION 1.2.1. [Pao (28)]:**

The operator $\mathcal{L} \equiv \frac{\partial}{\partial t} - L_1$ is said to be uniformly parabolic in a domain $D_T$, if $L_1$ is uniformly elliptic in a domain $D_T$.

For fixed $t$, $L_1$ is called uniformly elliptic operator if the matrix $(a_{ij})$ is positive definite in $\overline{D_T}$, that is, there exists positive constants $a_0$ and $d_1$ such that for every vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $\mathbb{R}$,

$$d_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t) \xi_i \xi_j \leq d_1 |\xi|^2, (x,t) \in \overline{D_T}$$  \hspace{1cm} (1.2.1)

where,

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2.$$
**EXAMPLE 1.2.1:** The prototype example of uniform parabolic operator in \((n+1)\) dimension is
\[
\frac{\partial}{\partial t} - D(x,t) \nabla^2 \equiv \frac{\partial}{\partial t} - D(x,t) \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)
\] (1.2.2)

Note that
\[
\frac{\partial u}{\partial t} - D(x,t) \nabla^2 u = 0
\]
is a linear second order uniformly parabolic equation in \(\mathbb{R}^{n+1}\).

**DEFINITION 1.2.2 [28]:**

The operator \(\mathcal{L} = \gamma(x,t) \frac{\partial}{\partial t} - L_1\) is said to be time degenerate parabolic in a domain \(D_T\), if \(L_1\) is uniformly elliptic and \(\gamma(x,t) \geq 0\) for some \((x,t) \in D_T\).

**EXAMPLE 1.2.2:** The example of time degenerate parabolic operator (non uniform) in \((n+1)\) dimension is
\[
d(x,t) \frac{\partial}{\partial t} - D(x,t) \nabla^2 \equiv d(x,t) \frac{\partial}{\partial t} - D(x,t) \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)
\] (1.2.3)

Note that \(d(x,t) \frac{\partial u}{\partial t} - D(x,t) \nabla^2 u = 0\) is a linear second order time degenerate (nonuniform) parabolic equation in \(\mathbb{R}^{n+1}\).
The coefficient $d(x, t)$ is nonnegative in $D$. However we will not assume that $d(x, t)$ is bounded away from zero. Since $d(x, t) = 0$ for some $(x, t)$ hence the equation is time degenerate.

The second order, both linear and nonlinear time degenerate parabolic equations are studied by various researchers, such as Ford and Waid [9], Waid [35], Ippolito [15], Dhaigude [5], Dhaigude and Gosavi [6], and Gosavi [12].

A variety of nonlinear degenerate parabolic equations arise in many physical problems, such as thermal boundary layer problem caused by the buoyancy induced flow through porous media saturated with pure or saline water at low temperature, Ramilson and Gebhart [30], Dhaigude and Kasture [4], and problems arising in the study of soft tissue, Holmes [14].

1.3 WAID’S MAXIMUM PRINCIPLES:

THEOREM 1.3.1[Waid’s (35)][Boundary Maximum Principle]:

Suppose that

(i) $u(x, t) \in C^{1,2}(D_T)$ and satisfies the time degenerate parabolic differential inequality

$$d(x,t)u_t - D(x,t)\nabla^2 u + c(x,t)u \leq 0 \text{ in } D_T$$

(ii) $D(x, t)$ is bounded function and

$$d(x,t) \geq 0, c(x,t) \geq 0, M \geq 0, \text{ in } D_T$$
(iii) the nonnegative maximum of \( u(x, t) \) in \( D_r \) is \( M \) and it is attained at some point \( P \) on the boundary \( S_r \),

(iv) a sphere through \( P \) can be constructed whose interior lies entirely in \( D_r \) and in which \( u < M \),

(v) the radial direction from the center of the sphere to \( P \) is not parallel to the \( t \)-axis.

(vi) \( d(x, t) > 0 \) at some point of each component at each horizontal segment.

Then \( \frac{\partial u}{\partial v} > 0 \) at \( P \),

where \( \frac{\partial}{\partial v} \) denotes any directional derivative in an outward direction.

**THEOREM 1.3.2. [Waid's (35)] [Boundary Minimum Principle]:**

Suppose that

(i) \( u(x, t) \in C^{1,2}(D_r) \) and satisfies the time degenerate parabolic differential inequality

\[
d(x,t)u_t - D(x,t)\nabla^2 u + c(x,t)u \geq 0 \text{ in } D_r
\]

(ii) \( D(x, t) \) is bounded function and

\[
d(x,t) \geq 0, c(x,t) \geq 0, m \leq 0, \text{ in } D_r
\]
(iii) the nonpositive minimum of u (x, t) in \( D_r \) is \( m \) and it is attained at some point \( P \) on the boundary \( S_r \),

(iv) a sphere through \( P \) can be constructed whose interior lies entirely in \( D_r \) and in which \( u < m \),

(v) the radial direction from the center of the sphere to \( P \) is not parallel to the \( t \)-axis.

(vi) \( d (x, t) > 0 \) at some point of each component at each horizontal segment.

Then \( \frac{\partial u}{\partial v} < 0 \) at \( P \),

where \( \frac{\partial}{\partial v} \) denotes any directional derivative in an outward direction.

1.4 **FINITE DIFFERENCE METHOD**:

As we know many physical phenomena are governed by reaction diffusion equations. Many times it is difficult to obtain solutions of such equations in the closed form. Therefore we prefer numerical methods and obtain approximate solutions of such equations. Secondly numerical methods are applicable to wider class of non-linear problems also. There are some popular numerical methods, such as Finite Difference Method [1, 8, 10, 11, 13, 20, 21, 23, 29, 31, 34 ], Finite Element Method [1, 20, 23 ], Finite Volume Method [23 ], Boundary Integral Element Method [26 ], etc. Among these Finite Difference Method is powerful and most popular in the literature.
Therefore they are used most gainfully than others. So, we study Finite Difference Method to obtain approximate solutions of our present problem.

The central idea of Finite Difference Method is to replace the derivatives present in the partial differential equations and boundary conditions by finite difference approximations and then solve the resulting linear system of equations by a standard method. We study them as follows:

**[I] DIFFERENCE FORMULAS:**

The first partial derivatives $u_x$ and $u_t$ of the function $u = u(x, t)$ are replaced by respective difference quotients

$$u_x(x, t) \approx \frac{u(x + h, t) - u(x, t)}{h} \quad (1.4.1)$$

$$u_t(x, t) \approx \frac{u(x, t + k) - u(x, t)}{k} \quad (1.4.2)$$

where $h$ and $k$ are increments in $x$ and $t$ respectively. By taking $h$ and $k$ to be very small the difference quotients in (1.4.1) and (1.4.2) can be made more accurate. The difference between the derivative and its approximating difference quotient is called truncation error.

To analyze truncation errors in (1.4.1) and (1.4.2), we require Taylor's Theorem in two variables.
[II] TAYLOR’S THEOREM [8]:

Suppose that

(i) \( u(x, t) \) and its partial derivatives up to \((n+1)^{th}\) order are continuous on \( \Omega_1 \) where

\[
\Omega_1 = [a,b] \times [c,d],
\]

(ii) \((x_0, t_0)\) is a point in \( \Omega_1 \). Then for every \((x, t)\) in \( \Omega_1 \) there exists \( \xi \) between \( x \) and \( x_0; \tau \) between \( t \) and \( t_0 \) such that

\[
u(x, t) = u(x_0, t_0) + \left[ \frac{\partial u}{\partial x}(x_0, t_0)(x - x_0) + \frac{\partial u}{\partial t}(x_0, t_0)(t - t_0) \right]
\]

\[
+ \left[ \frac{\partial^2 u}{\partial x^2}(x_0, t_0)\frac{(x - x_0)^2}{2} + \frac{\partial^2 u}{\partial x \partial t}(x_0, t_0)(x - x_0)(t - t_0) + \frac{\partial^2 u}{\partial t^2}(x_0, t_0)\frac{(t - t_0)^2}{2} \right]
\]

\[
+ \ldots + \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{(n-j)!j!}
\]

\[
\frac{\partial^n u}{\partial x^{n-j} \partial t^j}(x_0, t_0)(x - x_0)^{(n-j)}(t - t_0)^j + R_n
\]

where

\[
R_n = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \frac{(n+1)!}{(n+1-j)!j!} \frac{\partial^{n+1} u}{\partial x^{n+1-j} \partial t^j}(x - x_0)^{(n+1-j)}(t - t_0)^j
\]

is the remainder term.

This theorem can be extended to any finite number of variables.
[III] DERIVATION OF DIFFERENCE FORMULAS:

Assume that \( u(x, t) \) is twice continuously differentiable function. Then from Taylor’s theorem, we have for \( h > 0 \)

\[
\begin{align*}
\phi(x + h, t) &= \phi(x, t) + u_x(x, t)h + u_{xx}(\xi, t)\frac{h^2}{2}, \quad x < \xi < x + h
\end{align*}
\]

Rearranging this equation, we get,

\[
\frac{u_x(x, t)}{h} = \frac{u(x + h, t) - u(x, t)}{h} - u_{xx}(\xi, t)\frac{h}{2} \tag{1.4.3}
\]

It is known as forward difference formula for \( u_x \). If we take \( h < 0 \) in the derivation of (1.4.3), we obtain backward difference formula for \( u_x(x, t) \) as

\[
\frac{u_x(x, t)}{h} = \frac{u(x, t) - u(x - h, t)}{h} + u_{xx}(\xi, t)\frac{h}{2}, \quad x - h < \xi < x \tag{1.4.4}
\]

In the forward and backward difference formulas (1.4.3) and (1.4.4) for \( u_x \) the truncation errors are respectively.

\[
-u_{xx}(\xi, t)\frac{h}{2} \quad \text{and} \quad u_{xx}(\xi, t)\frac{h}{2} \quad \text{which are of O (h).}
\]

Now we use the Taylor’s theorem to derive centered (central) difference formula for \( u_x(x, t) \). By Taylor’s theorem we have,

\[
\begin{align*}
\phi(x + h, t) &= \phi(x, t) + u_x(x, t)h + u_{xx}(x, t)\frac{h^2}{2} - u_{xxx}(\xi_2, t)\frac{h^3}{6}, \quad x - h < \xi_2 < x.
\end{align*}
\]
Subtracting (1.4.6) and (1.4.5) and solving for $u_x$ is given

$$u_x(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} - \frac{1}{12} \left[ u_{xxx}(\xi_1,t) + u_{xxx}(\xi_2,t) \right] h^2$$  \hspace{1cm} (1.4.7)

If $u_{xx}$ is a continuous the truncation error in the difference formula (1.4.7) can be simplified by the intermediate value theorem which gives

$$\frac{1}{2} \left[ u_{xxx}(\xi_1,t) + u_{xxx}(\xi_2,t) \right] = u_{xxx}(\xi,t), \quad \xi_1 < \xi < \xi_2$$  \hspace{1cm} (1.4.8)

so we obtain the centered difference formula for $u_x(x,t)$ as

$$u_x(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} - \frac{1}{6} \left[ u_{xxx}(\xi,t) h^2 \right]$$  \hspace{1cm} (1.4.9)

which involves truncation error $-\frac{1}{6} \left[ u_{xxx}(\xi,t) h^2 \right]$ i.e $O(h^2)$.

To obtain difference formula for $u_{xx}(x,t)$ by Taylor's theorem, we have

$$u(x+h,t) = u(x,t) + u_x(x,t)h + u_{xx}(x,t) \frac{h^2}{2} + u_{xxx}(x,t) \frac{h^3}{6} + u_{xxxx}(\xi_1,t) \frac{h^4}{24}$$

where $x < \xi_1 < x + h$  \hspace{1cm} (1.4.10)
and

\[ u(x-h,t) = u(x,t) - u_x(x,t)h + u_{xx}(x,t)\frac{h^2}{2} - u_{xxx}(x,t)\frac{h^3}{6} + u_{xxxx}(\xi_2,t)\frac{h^4}{24} \]

where \( x-h < \xi_2 < x \)

\[ (1.4.11) \]

Adding (1.4.10) and (1.4.11) gives centered difference formula for \( u_{xx}(x,t) \) as

\[ u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + T.E \]

where

\[ T.E = -\frac{h^2}{24} \left[ u_{xxxx}(\xi_1,t) + u_{xxxx}(\xi_2,t) \right] \]

\[ = -\frac{h^2}{12} u_{xxxx}(\xi,t), \quad x-h < \xi < x+h \]

\[ (1.4.12) \]

Analogous forward, backward and centered difference formulas can be obtained for \( u_t \) respectively as

\[ u_t(x,t) = \frac{u(x,t+k) - u(x,t)}{k} - \frac{k}{2} u_{tt}(x,t); \quad t < \tau < t+k \]

\[ (1.4.13) \]

\[ u_t(x,t) = \frac{u(x,t) - u(x,t-k)}{k} + \frac{k}{2} u_{tt}(x,t); \quad t-k < \tau < t \]

\[ (1.4.14) \]

\[ u_t(x,t) = \frac{u(x,t+k) - u(x,t-k)}{2k} - \frac{1}{6} u_{ttt}(x,t)k^2; \quad t-k < \tau < t+k \]

\[ (1.4.15) \]
[IV] GRID NOTATION:

When the domain is sub-divided in the finite number of points, the finite difference formulae can be written shortly using the grid notation. A grid or mesh in the (x, t)-plane is the set of points 
\[(x_i, t_n) = (x_0 + ih, t_0 + nk)\] where \((x_0, t_0)\) is a reference point and \(i, n\) are integers. The points \((x_i, t_n)\) are called grid or mesh points. The positive values of h and k are called mesh spacing or grid spacing in x and t directions respectively. The value of \(u(x, t)\) at the mesh point \((x_i, t_n)\) is denoted by \(u_{i,n}\), where \(u_{i,n} = u(x_i, t_n)\). There are various types of grids. Some of them are as follows:

If the grid spacing h and k are allowed to vary with i and n respectively then such a grid is called non-uniform grid. If h and k are constants then the grid is called uniform grid. If h = k = constant, then the grid is said to be square.

Now we write the various finite difference formulae using the grid notation [1, 8, 34].

[V] FINITE DIFFERENCE EXPRESSIONS:

In continuation of the above discussion we denote the above forward, backward and centered differences for the \(u_x, u_t\) and centered difference for \(u_{xx}\) using grid notation as follows:

Forward Difference for \(u_x\):

\[
u_x(x_i, t_n) = \frac{u_{i+1,n} - u_{i,n}}{h} + O(h)\]
Backward Difference for $u_x$:

$$u_x(x_i, t_n) = \frac{u_{i,n} - u_{i-1,n}}{h} + O(h)$$

Centered Difference for $u_x$:

$$u_x(x_i, t_n) = \frac{u_{i+1,n} - u_{i-1,n}}{2h} + O(h^2)$$

Centered Difference for $u_{xx}$:

$$u_{xx}(x_i, t_n) = \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{h^2} + O(h^2)$$

$$= \Delta u_{i,n} + O(h^2)$$

Forward Difference for $u_t$:

$$u_t(x_i, t_n) = \frac{u_{i,n+1} - u_{i,n}}{k} + O(k)$$

Backward Difference for $u_t$:

$$u_t(x_i, t_n) = \frac{u_{i,n} - u_{i,n-1}}{k} + O(k)$$

Centered Difference for $u_t$:
\[ u_t(x_i, t_n) = \frac{u_{i,n+1} - u_{i,n-1}}{2k} + O(k^2) \]

In the above expressions \( O(h), O(k), O(h^2), O(k^2) \) are the truncation errors involved due to approximation of derivatives by the corresponding difference quotients. It is observed that these errors are less in centered difference approximation.

1.5 ERRORS:

There are two types of errors involved in the solution of Finite Difference Equations (FDE) Problem.

[I] TRUNCATION ERRORS:

These errors are involved due to approximation of derivatives by its approximating difference quotients. These errors are also called discretization errors and can be minimized by taking space increment and time increment sufficiently small.

[II] ROUNDDING ERRORS:

To obtain a good approximation to a derivative by replacing it corresponding difference quotient, we must take very small values of \( h \) and \( k \). But it is not possible to carry out calculation to an infinite number of decimal places. In carrying out the calculation to finite decimal places involving rounding error in every step. So in total steps these errors add up, called global round off measurement error present in the determination of the \( u_{i,n} \).

If \( U_{i,n} \) is a exact solution of the Partial Differential Equations
(PDE) Problem and \( u_{i,n} \) the solution of corresponding FDE Problem, and the solution computed is not \( u_{i,n} \) but \( N_{i,n} \). Then the solution \( N_{i,n} \) is called numerical solution of the FDE Problem and \( u_{i,n} - N_{i,n} \) the global rounding error. The total error at the \( (i,n)^{th} \) mesh point is

\[
U_{i,n} - N_{i,n} = U_{i,n} - u_{i,n} + u_{i,n} - N_{i,n}
= \text{discretization error} + \text{global rounding error}.
\]

If the truncation error is very small and rounding error is bounded, we can study the consistency, stability and convergence of the FDE Problem \([10]\).

### 1.6 CONSISTENCY, CONVERGENCE AND STABILITY:

When the grid spacing \( h \) and \( k \) are taken sufficiently small then the solution of the FDE is very close to the solution of corresponding PDE. The amount by which the solution of FDE fails to satisfy PDE at the given mesh point is the local truncation error. The FDE is said to be consistent with PDE if this local truncation error becomes zero when \( h, k \to 0 \).

We always require that the solution of FDE should converge to a solution of the corresponding PDE. We say that a FDE is convergent if the limiting value as \( h, k \to 0 \) of the difference between the solution of PDE and the solution of corresponding FDE is zero for all mesh points in the domain. The FDE may be consistent without being convergent.
Sometimes FDE propagates rounding off errors. If while solving the FDE rounding errors grow without bound then unbounded solution of FDE cannot be good approximation to a bounded solution of PDE. We say a FDE is a stable if the rounding error in the solution of FDE is remain bounded. In other words if in the small perturbation in the initial data of the FDE Problem, gives small perturbation in its solution, then that FDE Problem is said to be stable. The most of explicit finite difference approximations are conditionally stable where as most of implicit finite difference approximations are unconditionally stable. Stability is the most important point for any FDE Problem. Moreover it is found that an unstable scheme is not convergent. The concept of stability is concerned with the well-posed problem. The formal statement of the relationship between stability and convergence is known as the Lax Equivalence Theorem [8] stated below.

**LAX EQUIVALENCE THEOREM:**

Given a well-posed linear initial value problem or initial boundary value problem and a finite difference system consistent with it, stability is both necessary and sufficient condition for convergence.

The methods such as Von-Neumann Method and Matrix Method are widely used for determining the stability of finite difference approximations and to develop stability bound.

The detail of stability and convergence of Finite Difference Method for highly non linear problems are discussed in [13, 31].
1.7 TYPES OF FINITE DIFFERENCE METHODS:
Mainly finite difference methods are of two types:

[I] EXPLICIT METHODS:

These methods are computationally simple. But a series drawback is that their stability depends on some restriction on the spacing h and k, so they are conditionally stable. These methods give a formula, which expresses one unknown pivotal value, so they are called explicit methods. Now we give a classic explicit method to obtain solution of the simple model linear parabolic equation

\[ u_t - u_{xx} = 0 \quad (1.7.1) \]

CLASSIC EXPLICIT METHOD:

In this method we use forward difference for \( u_t \) and centered difference for \( u_{xx} \) in the equation (1.7.1) then it turns into

\[ \frac{u_{i,n+1} - u_{i,n}}{k} = \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{h^2} \quad (1.7.2) \]

Rearranging the terms in equation (1.7.2) and writing \( r = \frac{k}{h^2} \) we get the finite difference equation

\[ u_{i,n+1} = ru_{i+1,n} + (1 - 2r)u_{i,n} + ru_{i-1,n} \quad (1.7.3) \]

This equation gives unknown value of u at \((i, n + 1)\)th mesh point in terms of known value of u along \(n\)th time row. Truncation error in this approximation is \(O(2,1)\). This method is conditionally stable as
it is valid only for \( r \leq \frac{1}{2} \). There are many explicit methods which uses different approximations for \( u_i \) and \( u_{xx} \).

**[II] IMPLICIT METHODS:**

Implicit methods are crucial for preserving the qualitative property of the solution of the corresponding continuous system. These methods give a formula of finding two or more unknown pivotal values in terms of known pivotal values. This form a system of algebraic equations in unknowns which are interlinked to each other. So they give better approximate values of unknowns.

We discuss the following some implicit methods:

**(i) BACKWARD IMPLICIT METHOD:**

Here we replace \( u_i \) by forward difference formula and \( u_{xx} \) by centered difference formula at \((n+1)\) level, then the equation (1.7.1) becomes

\[
\frac{u_{i,n+1} - u_{i,n}}{k} = \frac{u_{i+1,n+1} - 2u_{i,n+1} + u_{i-1,n+1}}{h^2}
\]  

(1.7.4)

which turns into

\[
-r u_{i-1,n+1} + (1 + 2r)u_{i,n+1} - ru_{i+1,n+1} = u_{i,n}
\]

(1.7.5)

where \( r = \frac{k}{h^2} \).
Here three unknown values of $u$ appear on the line $n$. This approximation has truncation error of $O(2, 1)$ and it is valid for any value of $r$, it is unconditionally stable \[20\].

(ii) CRANK-NICOLSON IMPLICIT METHOD:

In 1947, Crank and Nicolson proposed and used the method that reduces the steps in calculation and is valid for all finite values of $r = \frac{k}{h^2}$. They replaced in (1.7.1) by the average of centered differences at $(i, n)$ and $(i, n+1)^{th}$ and $u_t$ by forward difference. So by this method equation (1.2.1) can be written into its finite difference approximation as

$$\frac{u_{i,n+1} - u_{i,n}}{k} = \frac{1}{2} \left( \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{h^2} + \frac{u_{i+1,n+1} - 2u_{i,n} + u_{i-1,n+1}}{h^2} \right) \quad (1.7.6)$$

giving

$$-ru_{i-1,n+1} + (2 + 2r)u_{i,n+1} - ru_{i+1,n+1} = ru_{i-1,n} + (2 - 2r)u_{i,n} + ru_{i+1,n} \quad (1.7.7)$$

where $r = \frac{k}{h^2}$

This approximation has truncation error of $O(2, 1)$ and is unconditionally stable \[20\].
Consider the equation

\[ u_t = u_{x_1 x_1} + u_{x_2 x_2}, \quad x = (x_1, x_2) \in \mathbb{R}^2 \] (1.7.8)

To convert this equation into its finite difference approximation let \( i = (i_1, i_2) \) be a multiple index and \( x_i = (x_{i_1}, x_{i_2}) \) be a point in \( \Omega_1 \subset \mathbb{R}^2 \). Let \( k_n = t_n - t_{n-1} \), be the time increment and \( h_v \) be the spatial increment in \( x_v \) coordinate direction and \( e_v \) be the unit vector in \( \mathbb{R}^2 \) where 1 occurs at \( v^{th} \) place and zero elsewhere. The classic explicit approximations for (1.7.8) is

\[
\frac{u_{i,n+1} - u_{i,n}}{k} = \sum_{v=1}^{2} \Delta^{(v)} u_{i,n} \quad (1.7.9)
\]

where

\[
\Delta^{(v)} u_{i,n} = \Delta^{(v)} u(x_i, t_n) = \frac{u(x_i + h_v e_v, t_n) - 2u(x_i, t_n) + u(x_i - h_v e_v, t_n)}{h_v^2}
\]

The backward implicit approximation for (1.7.8) is

\[
\frac{u_{i,n+1} - u_{i,n}}{k_n} = \sum_{v=1}^{2} \Delta^{(v)} u_{i,n+1} \quad (1.7.10)
\]

and the Crank-Nicolson implicit approximation for (1.7.8) is

\[
\frac{u_{i,n+1} - u_{i,n}}{k_n} = \frac{1}{2} \left[ \sum_{v=1}^{2} \Delta^{(v)} u_{i,n+1} + \sum_{v=1}^{2} \Delta^{(v)} u_{i,n} \right] \quad (1.7.11)
\]

This explicit and implicit methods has extended to partial differential equations in higher dimensions \([1, 10, 11]\).
1.8 PURPOSE OF THE THESIS:

The purpose of the thesis is two-fold. It is devoted to, on one hand, to develop Monotone method for both Dirichlet initial boundary value problem and corresponding discrete problem for (nonlinear time degenerate parabolic equations) nonlinear reaction diffusion equations and on other hand, to study the qualitative properties of the solution of such problems. The basic idea of Monotone method is to develop iterative scheme based on notion of upper and lower solutions. With the help of upper and lower solutions as initial iterations two monotone sequences are constructed. It is shown that these two sequences converge monotonically from above and below to maximal and minimal solutions respectively. The novelty of this method is that in the process of iteration, it gives Existence-Comparison and Uniqueness results of the problem at hand.

1.9 PLAN OF THE THESIS:

We organize the thesis as follows:
There are five chapters in the thesis.
Chapter I is introductory. In this chapter some basic results are discussed. They are used as a tool in the developments of the results obtained in the thesis.
Chapter II is devoted for reaction diffusion equations. Monotone iterative scheme is developed and Existence-Comparison and Uniqueness of solution of Dirichlet initial boundary value problem for such equations are proved.
In Chapter III, the results obtained in Chapter II, are extended to Dirichlet initial boundary value problem with nonlinear Dirichlet boundary condition, by applying monotone method.
Chapter IV deals with system of finite difference reaction diffusion equations with initial and linear Dirichlet boundary condition. Monotone method is developed for such a discrete problem. This method is successfully applied and Existence-Comparison and Uniqueness results for discrete problem are obtained.

The thesis concludes with chapter V, in which nonlinear discrete problem with nonlinear Dirichlet boundary condition is studied by applying monotone method and Existence-Comparison and Uniqueness results for discrete problem are obtained.

A comprehensive list of up-to-date references is added in alphabetical order at the end of the thesis.