CHAPTER V

MAXIMUM PRINCIPLES FOR NONLINEAR
FOURTH ORDER ORDINARY DIFFERENTIAL INEQUALITIES
MAXIMUM PRINCIPLES FOR NONLINEAR FOURTH ORDER
DIFFERENTIAL INEQUALITIES

ABSTRACT

In this chapter we prove maximum principles for some nonlinear fourth order ordinary differential equation

\[ u^{(4)}(x) + H(x, u'', u''') = 0, \]

An application of these maximum principles comparison theorems are proved. Uniqueness theorem for two point boundary value problem under consideration is also obtained.

I express my sincere gratitude to Dr. D.B. Dhaigude, Lecturer in Applied Mathematics, for many helpful discussions and suggestions during the preparation of this chapter.
5.1 **INTRODUCTION**

It is a well known fact that if $u$ is a real-valued function of class $C^2$ defined on $I = [a,b]$ and if it satisfies the inequality

$$u''(x) \geq 0 \text{ in } (a,b)$$

then $u$ satisfies maximum principle. More precisely if $u$ attains its maximum at an interior point of $I$ then $u$ is identically constant on $I$. This result is also true if $u$ satisfies the inequality

$$u''(x) + g(x)u'(x) + h(x)u(x) \geq 0, \quad x \in (a,b) \quad \ldots \quad (5.1.1)$$

where $g(x)$ is bounded and $h(x) \geq 0$. The extensive study of maximum principles and its applications for inequality (5.1.1) is to be found in the book of Protter and Weinberger [4].

The above maximum principle is not true for function satisfying higher order inequality follows from the following counterexample.

$$u^{(4)} \leq 0 \text{ in } (-1,1) \quad \ldots \quad (5.1.2)$$
The function $u = -x^2$ satisfies inequality (5.1.2) and yet has a maximum at $x = 0$. On the other hand the maximum principle for inequality (5.1.2) is developed by S.N. Chow, D.R.Dunninger and A. Lasota [1].

**LEMMA 5.1.1 (Maximum Principle - S.N. Chow, D.R. Dunninger and A. Lasota, [1]):** Let $u$ be a real-valued function of class $C^4$ defined on $I$ and satisfies the system of inequalities

\[(4)\]

$$u \leq 0 \quad \forall x \in (a,b)$$

$$u'(a) \leq 0, \quad u'(b) \geq 0$$

and, moreover, attains its maximum at a point $x_0 \in (a,b)$. Then $u$ is identically constant on $I$.

Recently D.R. Dunninger has extended this maximum principle for more general fourth order differential inequalities.

**LEMMA 5.1.2 (Maximum Principle - D.R. Dunninger [3]):** Let $u \in C^4(a,b) \cap C^2[a,b]$ satisfy the differential inequalities

\[(4)\]

$$u + g(x)u'' + h(x)u''' \leq 0, \quad \forall x \in (a,b)$$

$$u'(a) \leq 0, \quad u'(b) \geq 0$$

where the given function $g(x)$ is bounded and $h(x) \leq 0$. 
Then $u$ cannot assume its maximum at an interior point of $(a,b)$ unless it is identically constant.

Moreover, he has obtained an extremum principle near the end points which gives information on the sign of $u''$ at the end points where an extremum is obtained.

**Lemma 5.1.3 (D.R. Dunninger [3]):** Let $u \in C^4(a,b) \cap C^2[a,b]$ be a nonconstant function which has one sided third order derivatives at $a$ and $b$ and which satisfies the system

\[(4) \quad u'' + g(x) u''' + h(x) u'' \leq 0, \quad x \in (a,b)\]

\[u'(a) = 0, \quad u'(b) = 0\]

where the function $g(x)$ is bounded and $h(x) \leq 0$.

If $u$ attains its maximum at $x = a$ then $u''(a) > 0$ whereas if $u$ attains its maximum at $x = b$ then $u''(b) < 0$.

We extend both the lemmas 5.1.2 and 5.1.3 for nonlinear fourth order differential equation

\[F(u) = (4) \quad u'' + H(x,u'',u'''') = 0, \quad x \in (a,b) \quad \ldots \quad (5.1.3)\]

in the subsequent sections.
Maximum principle is an useful tool in the comparison of solutions, a subject of great interest to many researchers. Recently D.B. Dhaigude [2] has obtained a comparison theorem for fourth order ordinary differential equations which states that:

**LEMMA 5.1.4 (Comparison Theorem - D.B. Dhaigude [2]):**

Suppose that

1. $u_1(x)$ and $u_2(x)$ are continuous on $[a,b]$;
2. $(4)$ $u_1^{(4)}(x) \leq u_2^{(4)}(x)$, $x \in (a,b)$;
3. $u_1(a) \leq u_2(a)$, $u_1(b) \leq u_2(b)$; $u_1'(a) \leq u_2'(a)$, $u_1'(b) \geq u_2'(b)$.

Then $u_1(x) \leq u_2(x)$ for $x \in [a,b]$. We extend this result for nonlinear fourth order differential equation (5.1.3) in the section 5.3.

5.2 **MAXIMUM PRINCIPLES FOR NONLINEAR FOURTH ORDER ORDINARY DIFFERENTIAL INEQUALITY**: Consider the nonlinear fourth order ordinary differential equation

$$F(u) = u^{(4)} + H(x, u'', u''') = 0, \quad a < x < b \quad \ldots \quad (5.2.1)$$
The functions $H(x,y,z), \frac{\partial H}{\partial y}(x,y,z), \frac{\partial H}{\partial z}(x,y,z)$ are assumed to be continuous functions of $x,y,z$ throughout their domain of definition.

$$H(x,y,z) \leq H(x,y_2,z) \text{ for } y_1 \geq y_2;$$

or equivalently

$$\frac{\partial H}{\partial y} \leq 0 \quad \cdots \quad (5.2.2)$$

**Theorem 5.2.1**: Suppose that

$$u(x) - w(x) \in C^4(a,b) \cap C^2[a,b]$$

satisfies the differential inequalities

$$F(u) \leq F(w), \quad a \leq x \leq b \quad \cdots \quad (5.2.3)$$

$$u'(a) - w'(a) \leq 0, \quad u'(b) - w'(b) \geq 0 \quad \cdots \quad (5.2.4)$$

where $H, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}$ are continuous functions and $\frac{\partial H}{\partial y} \leq 0$. Then $u(x) - w(x)$ cannot assume a maximum value at an interior point of $(a,b)$ unless $u(x) - w(x)$ is identically constant.

**Proof**: The differential inequality
\[ F(u) \leq F(w) \] gives
\[ (4) \quad u + H(x, u'', u''') - w - H(x, w'', w''') \leq 0 \quad \ldots \quad (5.2.5) \]

Consider the function

\[ v(x) = u(x) - w(x) \]

Applying mean-value theorem to (5.2.5), we have
\[ v^{(4)} + \frac{\partial H}{\partial u} v''' + \frac{\partial H}{\partial y} v'' \leq 0, \quad \ldots \quad a < x < b \]

where the quantities are evaluated at
\[ (x, w'' + \Theta (u'' - w''), w''' + \Theta (u''' - w''')) \] with
\[ 0 < \Theta < 1. \] Thus the function \( v \) satisfies the linear equation with boundary conditions
\[ v'(a) \leq 0, \quad v'(b) \geq 0. \]

The result follows from maximum principle due to D.R. Dunniger, Lemma 5.1.2.
THEOREM 5.2.2: Suppose that

\[ u(x) - w(x) \in C^4(a,b) \cap C^2[a,b] \] is a nonconstant function

which has one sided third derivative at \( a \) and \( b \) and which the system

\[ F(u) \leq F(w), \quad a < x < b \]

\[ u'(a) - w'(a) = 0, \quad u'(b) - w'(b) = 0; \]

where the functions \( H, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z} \) are continuous and \( \frac{\partial H}{\partial y} \leq 0 \) holds.

If \( u(x) - w(x) \) assumes its maximum at \( x = a \) then \( u'''(a) - w'''(a) > 0 \)

whereas if \( u(x) - w(x) \) assumes its maximum at \( x = b \) then

\[ u'''(b) - w'''(b) < 0. \]

PROOF: The differential inequality

\[ F(u) \leq F(w) \]

gives

\[ u^4 + H(x,u'',u''') - w^4 - H(x,w'',w''') \leq 0 \quad \ldots \quad (5.2.6) \]

consider the function
\[ \psi(x) = u(x) = w(x) \]

Applying mean-value theorem to (5.2.6), we have

\[ \psi^{(4)} + \frac{\partial H}{\partial z} \psi'' + \frac{\partial H}{\partial y} \psi'' < 0, \quad a < x < b \]

where the quantities are evaluated at \((x, w'' + \theta(u'' - w''), w'' + \theta(u''' - w''))\) with \(0 < \theta < 1\). Thus the function \(\psi\) satisfies the linear equations with boundary conditions

\[ u'(a) - w'(a) = 0, \quad u'(b) - w'(b) = 0 \]

Then by Lemma 5.1.3, if \(u(x) - w(x)\) assumes its maximum at \(x = a\) then \(u''(a) - w''(a) > 0\) whereas if \(u(x) - w(x)\) assumes its maximum at \(x = b\) then \(u''(b) - w''(b) < 0\).

This completes the proof of the theorem.

**Remark:** The following example shows that Theorem 5.2.1 is false if lower order terms are allowed in the inequality (5.2.4). For simplicity consider a function \(u = \sin x\) which attains its maximum value at \(x = \frac{\pi}{2}\) and yet satisfies
\[ u'' + u = 0, \quad x \in (-\pi, 2\pi) \]
\[ u'(-\pi) = -1; \quad u'(2\pi) = 1 \]

Now we obtain some comparison theorems for nonlinear fourth order ordinary differential equation as an application of the maximum principle obtained in this section and thereby extend the D.B. Dhaigude's Lemma 5.1.4.

5.3 COMPARISON THEOREMS FOR NONLINEAR FOURTH ORDER ORDINARY DIFFERENTIAL INEQUALITY:

Consider the two point boundary value problem (TPBVP) for nonlinear fourth order ordinary differential equation

\[ F(u) = 0, \quad a < x < b \quad \ldots \quad (5.3.1) \]

with boundary conditions

\[ u(a) = r_1, \quad u(b) = r_2, \quad \ldots \quad (5.3.2) \]
\[ u'(a) = r_3, \quad u'(b) = r_4 \]

and mixed boundary conditions
\[ \alpha[u(a)] = \alpha_1 u(a) + \alpha_2 u''(a) = r_1, \ u'(a) = r_2; \]
\[ \beta[u(b)] = \beta_1 u(b) + \beta_2 u''(b) = r_3, \ u'(b) = r_4; \]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are nonnegative constants such that
\( \alpha_1^2 + \alpha_2^2 > 0, \beta_1^2 + \beta_2^2 > 0, \) but not both \( \alpha_1 \) and \( \beta_1 \) are zero.

Now, we prove the following comparison theorems.

**Theorem 5.3.1:** Suppose \( u(x) \) is a solution of TPBVP (5.3.1) - (5.3.2). Suppose that \( H, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z} \) are continuous functions and \( \frac{\partial H}{\partial y} \leq 0. \) If \( u_1(x) \) satisfies

\[ F(u_1) \geq 0, \quad a < x < b \]

\[ u_1(a) \geq r_1, \ u_1(b) \geq r_2; \]
\[ u'_1(a) \geq r_3, \ u'_1(b) \leq r_4; \]

and \( u_2(x) \) satisfies

\[ F(u_2) \leq 0, \quad a < x < b \]
\[ u'_2(a) \leq r_1, \quad u'_1(b) \leq r_2 ; \]
\[ u'_2(a) \leq r_3, \quad u'_2(b) \geq r_4 ; \]

Then

\[ u'_2(x) \leq u(x) \leq u'_1(x), \quad a \leq x \leq b \]

**PROOF:** Define a function

\[ v'_1(x) = u(x) - u'_1(x) \]

From (5.3.1) and (5.3.4)

\[ F(u) \leq F(u'_1), \quad a < x < b \]

and (5.3.2), (5.3.5), we get

\[ u(a) - u'_1(a) \leq 0, \quad u(b) - u'_1(b) \leq 0 ; \]

\[ u'_1(a) - u'_1(a) \leq 0, \quad u'_1(b) - u'_1(b) \geq 0 ; \]

Now apply the maximum principle [Theorem 5.2.1] to the function

\[ u(x) - u'_1(x) \] satisfying the inequalities (5.3.8) and (5.3.9), we get
\[ u(x) - u_{1}(x) \leq 0, \quad a \leq x \leq b \]

Thus

\[ u(x) \leq u_{1}(x), \quad a \leq x \leq b \]

Secondly, consider the function \( u_{2}(x) - u(x) \) we can prove

\[ u_{2}(x) \leq u(x), \quad ... \quad a \leq x \leq b \]

using the inequalities (5.3.1), (5.3.6) and (5.3.2), (5.3.7) on the similar lines.

This completes the proof.

**Theorem 5.3.2** \( \) Suppose that \( u(x) \) is a solution of the TPBVP (5.3.1) - (5.3.3). Assume that \( H, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial s} \) are continuous functions and \( \frac{\partial H}{\partial y} \leq 0 \). If \( u_{1}(x) \) satisfies

\[ F(u_{1}) \geq 0, \quad a < x < b \quad ... \quad (5.3.10) \]

\[ \alpha [ u_{1}(a) ] \geq x_{1}, \quad u_{1}'(a) = x_{2}; \]
\[ \beta [ u_{1}(b) ] \geq x_{3}, \quad u_{1}'(b) = x_{4}; \]

\[ ... \quad (5.3.11) \]
and $u_2(x)$ satisfies

\[ F(u_2) \leq 0, \quad a < x < b; \]
\[ \alpha [u_2(a)] \leq x_1, \quad u_2(a) = x_2; \]
\[ \beta [u_2(b)] \leq x_3, \quad u_2(b) = x_4; \]

Then

\[ u_2(x) \leq u(x) \leq u_1(x), \quad a < x < b. \]

**PROOF:**

Consider the function $u(x) - u_1(x)$ from (5.3.4) and (5.3.10), we get

\[ F(u) \leq F(u_1), \quad a < x < b \]

and (5.3.14), (5.3.11), we get
\[\begin{align*}
\alpha [u(a) - u_1(a)] &\leq 0, \quad u'(a) - u_1'(a) = 0 \\
\beta [u(b) - u_1(b)] &\leq 0, \quad u'(b) - u_1'(b) = 0
\end{align*}\] (5.3.15)

If \(u(x) - u_1(x)\) is ever positive, Theorem 5.2.1 states that its maximum occurs at \(x = a\) or \(x = b\). If it occurs at \(x = a\) then Theorem 5.2.2 states that \(u'''(a) - u_1'''(a) > 0\); contradicting the first inequality in (5.3.15). This may only occur if \(\alpha_1 = 0\) and \(u'''(a) - u_1'''(a) = 0\).

Since \(\beta [u(b) - u_1(b)] \leq 0\).

We conclude that, unless both \(\alpha_1\) and \(\beta_1\) are zero

\[u(x) \leq u_1(x), \quad a \leq x \leq b\]

Lower bounds may be obtained in a similar way.

Thus

\[u_2(x) \leq u(x) \leq u_1(x), \quad a \leq x \leq b\]

This completes the proof.

As an application of maximum principle, we obtain uniqueness for Theorem for TPBVF.
\[ P(u) = f(x), \quad a < x < b \quad \ldots \quad (5.3.16) \]

\[
\begin{align*}
\alpha[u(a)] &= \alpha_1 u(a) + \alpha_2 u''(a) = r_1, \quad u'(a) = r_2; \\
\beta[u(b)] &= \beta_1 u(b) + \beta_2 u''(b) = r_3, \quad u'(b) = r_4;
\end{align*}
\]

\[
(5.3.17)
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are nonnegative constants such that

\[ \alpha_1 + \alpha_2 > 0, \quad \beta_1 + \beta_2 > 0, \quad \text{but not both } \alpha_1 \text{ and } \beta_1 \text{ are zero and} \]

\( f(x) \) is defined and continuous on \([a, b]\).

**Theorem 5.3.3 (Uniqueness Theorem):** Suppose \( u_1(x) \) and \( u_2(x) \)

are solutions of \((5.3.16) - (5.3.17)\) then

\[ u_1(x) = u_2(x) \quad \forall x \in [a, b] \]

**Proof:** Define a function \( v(x) \) such that

\[ v(x) = u_1(x) - u_2(x) \]

then \( v \) satisfies
\[ v + \frac{\partial H}{\partial x} v'' + \frac{\partial H}{\partial y} v'' = 0 \]

and

\[ \alpha_1 v(a) + \alpha_2 v''(a) = 0, \quad v'(a) = 0; \]
\[ \beta_1 v(b) + \beta_2 v''(b) = 0, \quad v'(b) = 0; \]

If \( v \) were ever positive, it would have a positive maximum. By Theorem 5.2.1 the maximum must occur at \( x = a \) or \( x = b \). If \( v(a) \) is positive and is nonconstant then by Theorem 5.2.2

\[ v''(a) > 0 \]

which contradicts the boundary condition at \( x = a \).

Similarly if \( v(b) > 0 \)

then \[ v''(b) < 0 \]

which contradicts the boundary condition at \( x = b \).

Thus either \( v \) is constant or \( v \leq 0 \) in \([a, b]\).

Applying the same argument to \(-v\) we see that \( v \) must be constant. Lastly no constant other than zero satisfies (5.3.17)
unless $\alpha_1 = \beta_1 = 0$ in which case any constant satisfies the system

$$v(x) = 0 \quad \text{in} \quad [a, b]$$

$$\therefore u_1(x) = u_2(x) \quad \text{in} \quad [a, b] .$$

This completes the proof.
5.4 REFERENCES


