CHAPTER III

ON A NONLINEAR SECOND ORDER INTEGRALDIFFERENTIAL EQUATION OF VOLterra TYPE
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ABSTRACT

In this chapter we study the existence, uniqueness and other properties of a nonlinear second order integro-
differential equation of Volterra type

\[ u''(t) = f(t, u(t), u'(t)) + \int_0^t g(t, s, u(s), u'(s))ds \]

\[ u(i) = C_i, \quad i = 0, 1, \]

Banach fixed point theorem and integral inequalities established by Pachpatte are the main tools employed in our analysis.
3.1 **INTRODUCTION**

In the present chapter we consider the equation

\[
\begin{align*}
\ddot{u}(t) f(t,u(t),\dot{u}(t)) + \int_0^t g(t,s,u(s),\dot{u}(s))ds \\
(1) \\
u(0) = C_i, \quad i = 0,1,
\end{align*}
\]  

(3.1.1)

where \(u(0) \in \mathbb{R}, f(t,u(t),\dot{u}(t)) \in C[R_+ \times \mathbb{R}, \mathbb{R}], \) and \(g(t,s,u(s),\dot{u}(s)) \in C[R_+ \times \mathbb{R}, \mathbb{R}], R_+ = [0,\infty) \) and \(\mathbb{R}\) is a real line.

Equation of the type (3.1.1) and its various special forms arise in many areas of applied mathematics and mathematical models of physical processes e.g. see [1,2,4,5] and references given therein. The problems of existence, uniqueness and other properties of the solutions of various special forms of (3.1.1) have been studied by many authors e.g. see [3,6]. However our approach and conditions on nonlinear functions \(f\) and \(g\) are entirely different than those of [2,3,4,5]. Banach fixed point theorem [9] and integral inequalities established by Pachpatte in [6,7,8] are used as a main tools in this chapter to obtain our results.
The main aim of this chapter is to study the existence, uniqueness and other properties of the solutions of (3.1.1) under suitable conditions on nonlinear functions $f$ and $g$. We need the following lemmas in our subsequent discussion.

**Lemma 3.1.1** (See, Pachpatte [7, pp. 1157-1175]): Let $u(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued continuous functions defined on $\mathbb{R}_+$ for which the inequality

$$ u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(z)u(z)dz)ds $$

$$ + \int_0^t f(s)(\int_0^z g(r)(\int_0^r h(u)du)dr)dz)ds, \quad t \in \mathbb{R}_+ $$

holds, where $u_0$ is a nonnegative constant. Then

$$ u(t) \leq u_0 \left[ 1 + \int_0^t f(s)\exp\left(\int_0^s f(v)dv\right)X \right. $$

$$ \left. \left[ 1 + \int_0^t g(r)\exp\left(\int_0^r [g(z)+h(z)]dz\right)dr\right]ds \right] . $$

**Lemma 3.1.2** (See, Pachpatte [8, pp. 798-802]): Let $u(t)$, $p(t)$, $q(t)$ and $x(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$ and $p(t)$ be a positive, monotonic nondecreasing continuous functions defined on $I$, for which the inequality
\[ u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left( \int_0^s g(z)u(z)dz \right)ds \]
\[ + \int_0^t f(s)\left( \int_0^s g(z)\left( \int_0^z h(r)u(r)dr \right)dz \right)ds, \quad t \in I \]

holds, then

\[ u(t) \leq n(t) \left\{ 1 + \int_0^t \exp\left( \int_0^s f(v)dv \right)X \right\} \]
\[ \left[ 1 + \int_0^s g(r)\exp\left( \int_0^r [g(z)+h(z)]dz \right)dr \right]ds \]

, \quad t \in I.

3.2. MAIN RESULTS:

**THEOREM 3.2.1** Suppose \( f \) is continuous real-valued function defined on a suitable domain in \( R_+^3 \) and \( g \) is a continuous real-valued function defined on a suitable domain in \( R_+^2 \) and \( \alpha > 0 \). Then the following are equivalent.

(i) \( u \) is a differentiable function defined on \([0,\alpha]\) such that \( u \) satisfies (3.1.1).
(ii) \( u \) is continuous function defined on \([0, α]\) such that

\[
\begin{align*}
u(t) & \leq C_0 + C_1 t + \int_0^t \left( \int_0^s f(z,u(z), u'(z)) dz \right) ds \\
& \quad + \int_0^t \left( \int_0^z g(z,r,u(r), u'(r)) dr \right) dz ds, \quad \ldots \\
\end{align*}
\]

\((3.2.1)\)

**Proof:** Suppose that \( u(t) \) is a solution of \((3.1.1)\). Then by definition of solution \( u(t) \) is a continuous function of \( t \).

By integrating \((3.1.1)\) from \( 0 \) to \( t \) twice and using \( U^{(i)}(0) = C_i \), \( i = 0,1 \), we get \((3.2.1)\). Thus any solution of \((3.1.1)\) is a continuous solution of \((3.2.1)\).

Conversely if \( u(t) \) is a continuous solution of \((3.2.1)\) then we get \( U^{(i)}(0) = C_i \), \( i = 0,1 \), and when \((3.2.1)\) is differentiated twice we get \((3.1.1)\). This completes the proof of Theorem 3.2.1.

**Theorem 3.2.2** Assume that for all \( t, s, z \in [0, α] \) and \( x_i, y_i, z_i \in \mathbb{R}^+ \) for \( i=1,2 \), there exist nonnegative constants \( L_i, i=1,2 \), such that

\[
(A_1) \quad |f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L_1[|x_1-x_2| + |y_1-y_2|]
\]
\((A_2)\) \[|g(t,s,y_1,z_1) - g(t,s,y_2,z_2)| \leq L_2[|y_1-y_2| + |z_1-z_2|]\]

\((A_3)\) \[\left[ L_1 + (L_1+L_2) \frac{\alpha}{2!} + L_2 \frac{\alpha^2}{3!} \right] \alpha < 1.\]

Then for \(u^{(i)}(o) \in \mathbb{R}, i=0,1\), the initial value problem (3.1.1)
has a unique solution \(u \in C([0,\alpha];\mathbb{R})\) for \(t \geq 0\) such that \(0 \leq t \leq \alpha\).

**Proof:** In the space \(C = C([0,\alpha];\mathbb{R})\) we define the norm

\[\|u\|_C = \max[|u(t)| + |u'(t)|]\]

... (3.2.2)

It can be seen that \(C\) with the norm defined in (3.2.2) is a
Banach space. We define mapping \(T: C \to C\) by

\[(Tu)(t) = C_o + C_1 t + \int_0^t \left( \int_0^s f(z, u(z), u'(z))dz \right)ds'

+ \int_0^t \left( \int_0^s \int g(z,r,u(z),u'(r))drdz \right)ds \quad (3.2.3) \quad 0 \leq t \leq \alpha, \quad t \in I.

Clearly the solution of the equation (3.1.1) is a fixed point
of the operator equation \(Tu = u\). Let \(u, v \in C\). From (3.2.2),
(3.2.3) and assumptions \((A_1) - (A_3)\), we get
\[(T_u(t)) - (T_v(t))| \leq \int_0^t (\int_0^s \left[|u(z) - v(z)| + |u'(z) - v'(z)|\right]dz)ds
\]
\[+ \int_0^t (\int_0^s \left[|u(r) - v(r)| + |u'(r) - v'(r)|\right]dr)dz)ds. \quad (3.2.4)\]

\[T'_u(t) = C_1 + \int_0^t f(s,u(s),u'(s))ds + \int_0^t (\int_0^s g(r,z,u(z),u'(z))dz)ds\]

\[T'_v(t) = C_1 + \int_0^t f(s,v(s),v'(s))ds + \int_0^t (\int_0^s g(s,z,v(z),v'(z))dz)ds\]

\[\therefore T'_u(t) - T'_v(t) = \int_0^t [f(s,u(s),u'(s)) - f(s,v(s),v'(s))]ds\]
\[+ \int_0^t (\int_0^s [g(s,z,v(z),u'(z)) - g(s,z,v(z),v'(z))]dz)ds\]

\[\therefore |T'_u(t) - T'_v(t)| \leq \int_0^t L_1[|u(s) - v(s)| + |u'(s) - v'(s)|]ds\]
\[+ \int_0^t (\int_0^s [u(z) - v(z)] + |u'(z) - v'(z)|]dz)ds \quad \ldots (3.2.5)\]

Adding (3.2.4) and (3.2.5)

\[|T_u(t) - T_v(t)| + |T'_u(t) - T'_v(t)| \leq \int_0^t L_1[|u(s) - v(s)|\]
\[+ |u'(s) - v'(s)|]ds + \int_0^t (\int_0^s [u(z) - v(z)] + |u'(z) - v'(z)|]dz)ds\]
\[ 
+ \int_0^t \left( \int_{L_1} \left[ |u(z) - v(z)| + |u'(z) - v'(z)| \right] \, dz \right) \, ds 
+ \int_0^t \left( \int_{L_2} \left[ |u(r) - v(r)| + |u'(r) - v'(r)| \right] \, dr \right) \, dz \, ds 
\]

\[ 
\therefore \ |T_u(t) - T_v(t)| \leq \int_0^t \left[ |u(s) - v(s)| + |u'(s) - v'(s)| \right] \, ds 
+ \int_0^t \left( \int_{L_1+L_2} \left[ |u(z) - v(z)| + |u'(z) - v'(z)| \right] \, dz \right) \, ds 
+ \int_0^t \left( \int_{L_2} \left[ |u(r) - v(r)| + |u'(r) - v'(r)| \right] \, dr \right) \, dz \, ds 
\]

\[ 
|T_u - T_v| \leq \int_0^t \left[ ||u-v|| C \right] \, ds + \int_0^t \left( \int_{L_1+L_2} \left[ ||u-v|| C \right] \, dz \right) \, ds 
+ \int_0^t \left( \int_{L_2} \left[ ||u-v|| C \right] \, dr \right) \, dz \, ds 
\]
\[
\begin{align*}
\therefore \| Tu - Tv \|_C & \leq L_1 \| u - v \|_C + (L_1 + L_2) \| u - v \|_C \frac{a^2}{2} \\
& \quad + L_2 \| u - v \|_C \frac{a^3}{3} \\
\therefore \| Tu - Tv \|_C & \leq [L_1 a + (L_1 + L_2) \frac{a^2}{2} + L_2 \frac{a^2}{3}] \| u - v \|_C \\
\therefore \| Tu - Tv \|_C & \leq [L_1 + (L_1 + L_2) \frac{a}{2} + L_2 \frac{a^2}{3}] a \| u - v \|_C
\end{align*}
\]

However, by choice of \(a\), we have

\[
[L_1 + (L_1 + L_2) \frac{a}{2} + L_2 \frac{a^2}{3}] a < 1
\]

and so \(T\) is a contraction in \(C\). It follows from Banach fixed point theorem that the operator \(T\) has a unique fixed point in \(C\). This fixed point is the desired solution of (3.1.1). This completes the proof of Theorem 3.2.2.

**Theorem 3.2.3** Assume that the equation (3.1.1) satisfy \((A_1) - (A_3)\) and \(0 \leq t \leq \alpha\) then the solutions of (3.1.1) are bounded if
\[
\int_0^\infty L_1 \exp\left( \int_0^s L_1 \, dr \right) X \left\{ 1 + \int_0^s \frac{(L_1 + L_2)}{L_1} \right. \\
\left. \exp\left( \int_0^z \left[ \frac{(L_1 + L_2)}{L_1} + \frac{L_2}{(L_1 + L_2)} \right] \, dr \, dz \right) \right\} \, ds < \infty
\]

**Proof:** By Theorem 3.2.1 equation (3.1.1) can be written as

\[
u(t) = C_0 + C_1 t + \int_0^t \left( \int_0^s f(z, u(z), u'(z)) \, ds \right)
\]

\[
+ \int_0^t \left( \int_0^s \left( \int_0^z g(z, r, u(r), u'(r)) \, dr \, dz \right) \, ds \right) \quad t \in I.
\]

and

\[
u'(t) = C_1 + \int_0^t f(s, u(s), u'(s)) \, ds + \int_0^t g(s, z, u(z), u'(z)) \, dz \, ds
\]

\[
\Rightarrow \quad \nu(t) + \nu'(t) = C_0 + C_1 (1 + t) + \int_0^t f(s, u(s), u'(s)) \, ds
\]

\[
+ \int_0^t \left( \int_0^s f(z, u(z), u'(z)) \, dz \, ds \right) + \int_0^t \left( \int_0^s g(s, z, u(z), u'(z)) \, dz \, ds \right)
\]

\[
+ \int_0^t \left( \int_0^z g(z, r, u(r), u'(r)) \, dr \, dz \, ds \right)
\]

Therefore
\[
[|u(t)| + |u'(t)|] \leq |C_0| + |C_1(1+t)| + \int_0^t |f(s, u(s), u'(s)) - f(s, o, o)| ds \\
+ \int_0^t |f(s, o, o)| ds + \int_0^s (\int_0^t |f(z, u(z), u'(z)) - f(z, o, o)| dz) ds \\
+ \int_0^t (\int_0^s |f(z, o, o)| dz + \int_0^s (\int_0^z |g(z, r, u(r), u'(r)) - g(z, r, o, o)| dr) dz) ds \\
+ \int_0^t (\int_0^s \int_0^z |g(z, r, o, o)| dr) dz) ds \\
\]

\[\therefore |u(t)| + |u'(t)| \leq A + \int_0^t L_1[|u(s)| + |u'(s)|] ds + \int_0^t (\int_0^s L_1[|u(z)| + |u'(z)|] dz) ds + \int_0^t (\int_0^s L_2[|u(z)| + |u'(z)|] ds + \int_0^t (\int_0^s (L_1 + L_2)[|u(z)| + u'(z)] | dz) ds + \int_0^t (\int_0^s L_2[|u(r)| + |u'(r)|] dr) dz) ds \]
\[ \therefore |u(t)| + |u'(t)| \leq A + \int_0^t L_1 \left[ |u(s)| + |u'(s)| \right] ds \]
\[ + \int_0^t L_1 \left[ \int_0^s \frac{(L_1 + L_2)}{L_1} \left[ |u(z)| + |u'(z)| \right] dz \right] ds \]
\[ + \int_0^t L_1 \left[ \int_0^s \frac{(L_1 + L_2)}{L_1} \left( \int_0^z \frac{L_2}{L_1 + L_2} \left[ |u(r)| + |u'(r)| \right] dr \right) dz \right] ds. \]

Now an application of Lemma 3.1.1 yields

\[ \left[ |u(t)| + |u'(t)| \right] \leq A \left\{ 1 + \int_0^t L_1 \exp \left( \int_0^s L_1 dr \right) X \right\} \]
\[ \left[ 1 + \int_0^s \frac{(L_1 + L_2)}{L_1} \exp \left( \int_0^z \left[ \frac{(L_1 + L_2)}{L_1} + \frac{L_2}{L_1 + L_2} \right] dr \right) dz \right] ds < \infty. \]

Since

\[ \int_0^\infty L_1 \exp \left( \int_0^s L_1 dr \right) X \left[ 1 + \int_0^s \frac{(L_1 + L_2)}{L_1} \exp \left( \int_0^z \left[ \frac{(L_1 + L_2)}{L_1} + \frac{L_2}{L_1 + L_2} \right] dr \right) dz \right] ds < \infty. \]

Therefore the solutions of (3.1.1) are bounded. This completes the proof of Theorem 3.2.3.

**THEOREM 3.2.4.** Assume that the functions \( f \) and \( g \) satisfy

(i) \( |f(t,u(t),u'(t)| \leq \left[ |u(t)| + |u'(t)| \right] e^{\omega t} \)
(ii) \[ |g(t,s,u(s),u'(s))| \leq \left[ |u(s)| + |u'(s)| \right] e^{ws} \]

(iii) \[
\int_0^s e^{\int_0^r e^{wr} dr} x \left[ \int_0^s e^{\int_0^r (e^{wz} + 1) dz} dr \right] ds < M
\]

for all \( t, s \in [0, \infty) \), \( w > 0 \) is a constant. Then all solutions of (3.1.1) approach to \( |A| (1+M) \) as \( t \to \infty \).

**Proof:** By Theorem 3.2.1 equation (3.1.1) can be written as

\[
u'(t) = C_1 + \int_0^t f(s, u(s), u'(s)) ds + \int_0^t \int_0^s g(s, z, u(z), u'(z)) dz ds \]

\[
u(t) = C_0 + C_1 t + \int_0^t \left( \int_0^s f(z, u(z), u'(z)) dz \right) ds + \int_0^t \int_0^s g(z, r, u(r), u'(r)) dr dz ds + \int_0^t \int_0^s \left( f(z, u(z), u'(z)) + g(s, z, u(z), u'(z)) \right) dz ds + \int_0^t \int_0^s g(z, r, u(r), u'(r)) dr dz ds
\]

\[.\quad u(t) + u'(t) = C_0 + C_1 (1+t) + \int_0^t f(s, u(s), u'(s)) ds + \int_0^t \left( \int_0^s f(z, u(z), u'(z)) dz \right) ds + \int_0^t \int_0^s g(z, r, u(r), u'(r)) dr dz ds
\]
\[ |u(t) + u'(t)| \leq |A| + \int_0^t e^{\int_0^s w(s) \left[ |u(s)| + |u'(s)| \right] ds} \]

\[ + \int_0^t e^{\left( \int_0^s 2(\int_0^z \frac{1}{2} \left[ |u(r)| + |u'(r)| \right] dr) dz \right) ds} \]

\[ + \int_0^t e^{\left( 2 \int_0^s (\int_0^z \frac{1}{2} \left[ |u(r)| + |u'(r)| \right] dr) dz \right) ds} \]

\[ \therefore |u(t) + u'(t)| \leq |A| + \int_0^t e^{\int_0^s w(s) \left[ \int_0^r w(r) \exp(\int_0^r w(r) dr) dr ds \right] \left[ 1 + \int_0^s 2 \exp(\int_0^z (2 + \frac{1}{2}) dz) dr ds \right]} \]

\[ \therefore |u(t) + u'(t)| \leq |A| \left\{ 1 + \int_0^t e^{\int_0^s w(s) \left[ \int_0^r w(r) \exp(\int_0^r w(r) dr) dr ds \right]} \right\} \]
Since

$$\int_0^\infty \exp\left( \int_0^s \exp\left( \int_0^r \exp\left( \int_0^\infty (e^{z+1})dz\right)dr\right)ds\right)<M$$

\[.\quad |u(t) + u'(t)| \text{ is asymptotic to } |A|(1+M)\]

as \( t \to \infty \).

This completes the proof of Theorem 3.2.4.

**Theorem 3.2.5** Assume that for all \( t, s, z \in [0, \alpha] \) and \( x_i, y_i, z_i \in \mathbb{R}_+ \) for \( i = 1, 2 \), there exist nonnegative constants \( L_i, i = 1, 2 \) such that

\[(A_1) \quad |f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1[|x_1 - x_2| + |y_1 - y_2|] \]

\[(A_2) \quad |g(t, s, y_1, z_1) - g(t, s, y_2, z_2)| \leq L_2[|y_1 - y_2| + |z_1 - z_2|] \]

\[(A_3) \quad \left( \frac{L_1}{2} + \frac{L_2 \alpha}{3} \right) \alpha < 1 \]

\[(A_4) \quad [|u(t)| + |u'(t)|] < \infty.\]

Then \( ||u|| \to 0 \) as \( t \to \infty \).
**Proof:** In the space \( C = C([0, a], \mathbb{R}) \) we define the norm

\[
\| u \|_C = \max_{t} [|u(t)| + |u'(t)|] \quad \ldots \quad (3.2.1)
\]

It can be shown that \( C \) with the norm defined in (3.2.1) is a Banach space. We define the mapping \( T : C \to C \) by

\[
(Tu)(t) = C_0 + C_1 t + \int_0^t ( \int_0^s f(z, u(z), u'(z)) \, ds \\
+ \int_0^t ( \int_0^s g(z, r, u(r), u'(r)) dr) \, dz) \, ds, \ldots
\]

\[
\ldots \quad 0 \leq t \leq a, \quad t \in I \quad \ldots \quad (3.2.2)
\]

\[
\| u(t) \|_C \leq A + L_1 \int_0^t \| u \|_C \, ds + \int_0^t ( \int_0^s (L_1 + L_2) \| u \|_C dz) \, ds \\
+ \int_0^t ( \int_0^s L_2 \| u \| \, dr) \, dz) ds
\]
\[ \| u(t) \| \leq A + L_1 \| u \| t + (L_1 + L_2) \| u \| \frac{t^2}{2!} + L_2 \| u \| \frac{t^3}{3!} \]

\[ \leq A[L_1 + (L_1 + L_2) \frac{t}{2!} + L_2 \frac{t^2}{3!}] t \| u \| \]

\[ \| u(t) \| \leq A + Bt \| u \| \]

\[ \therefore \| u \| - Bt \| u \| t \leq A \]

\[ \| u \| (1 - Bt) \leq A \]

\[ \therefore \| u \| \leq \frac{A}{1 - Bt} \]

As \( t \to \infty \)

\[ \| u \| \to 0 \]

This completes the proof of Theorem 3.2.5.
3.3 REFERENCES


