Chapter 3

On some modular relations and 2-, 4-dissections of Ramanujan’s continued fraction of order six
3.1 Introduction

The famous Göllnitz-Gordon functions $A(q)$ and $B(q)$ are defined by

$$A(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_{\infty} (q^3; q^8)_{\infty} (q^7; q^8)_{\infty}}, \quad (3.1.1)$$

$$B(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2 + 2n}}{(q^2; q^2)_n} = \frac{1}{(q^3; q^8)_{\infty} (q^5; q^8)_{\infty}}, \quad (3.1.2)$$

where the two equalities on the right-hand sides of (3.1.1) and (3.1.2) are the celebrated Göllnitz-Gordon identities [46, 47].

We note that

$$H_1(q) := \frac{q^{1/2}}{1 + q} + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \cdots = q^{1/2} B(q) \over A(q).$$

Ramanujan established the following two identities for $H_1(q)$, [29, 67]:

$$\frac{1}{H_1(q)} - H_1(q) = \frac{(-q^2; q^4)^2 (q^4; q^8)_\infty (q^4; q^8)_\infty}{q^{1/2} (q^2; q^8)_\infty}, \quad (3.1.3)$$

$$\frac{1}{H_1(q)} + H_1(q) = \frac{(-q^2; q^2)^2 (q^2; q^8)_\infty (q^4; q^8)_\infty}{q^{1/2} (q^8; q^8)_\infty}. \quad (3.1.4)$$

Recently, B. Yuttanan [88] established factorizations for $H_1(q)$, namely,

$$\frac{1}{\sqrt{H_1(q)}} - \sqrt{H_1(q)} = \frac{\chi(q^2) \chi(-q^4)}{q^{1/4} \sqrt{\chi(-q)}} \prod_{n=1}^{\infty} \left( 1 - \left( \frac{n}{2} \right) q^{n/2} \right) \quad (3.1.5)$$

and

$$\frac{1}{\sqrt{H_1(q)}} + \sqrt{H_1(q)} = \frac{\chi(q^2) \chi(-q^4)}{q^{1/4} \sqrt{\chi(-q)}} \prod_{n=1}^{\infty} \left( 1 + \left( \frac{n}{2} \right) q^{n/2} \right), \quad (3.1.6)$$

where $\left( \frac{n}{2} \right)$ is the Kronecker symbol.

In his first two letters to Hardy [68, pp.xxvii, xxviii], Ramanujan communicated several theorems about $R(q)$ and $S(q) = -R(-q)$. Richmond and Szekeres [71]
studied asymptotically the power series coefficients of a large class of infinite products including \( R^*(q) := q^{-1/5}R(q) \). In particular, if \( R^*(q) = \sum_{n=0}^{\infty} a(n)q^n \), they proved that there exists \( N_0 \) such that for any \( n \geq N_0 \),

\[
a(5n), \ a(5n+2) > 0 \ \text{and} \ a(5n+1), \ a(5n+3), \ a(5n+4) < 0.
\]

Andrews [19], further showed that the above inequalities hold for all \( n \) except that \( a(3)=a(8)=a(13)=a(23)=0 \) by considering the formulas for \( \sum_{n=0}^{\infty} a(5n+j)q^n, \ 0 \leq j \leq 4 \), which were recorded in Ramanujan’s Lost Notebook [69]. In [18], Andrews also considered the 2-dissections of the Rogers-Ramanujan continued fraction and its reciprocal.

Ramanujan’s cubic continued fraction has been studied by Srivastava [77] and Hirschhorn [55], the Ramanujan-Göllnitz-Gordon continued fraction has been considered by Hirschhorn [54], Xia and Yao [84], general infinite products have been investigated by K. Alladi and B. Gordon [17], Andrews and Bressoud [21], S. H. Chan and Yesilyurt [38] and Ramanathan [66].

In his Second Notebook [1, p.24], Ramanujan recorded the following beautiful continued fraction identity:

\[
\frac{(a^2 q^3; q^4)_\infty (b^2 q^3; q^4)_\infty}{(a^2 q; q^4)_\infty (b^2 q; q^4)_\infty} = \frac{1}{1 - ab + \frac{(a - bq)(b - aq)}{(1 - ab)(q^2 + 1)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1)} + \cdots}, \quad |ab| < 1. \tag{3.1.7}
\]

For \( |ab| > 1 \), Lisa Jacobsen [58] has shown that,

\[
- \frac{1}{ab} \frac{(q^3/a^2; q^4)_\infty (q^3/b^2; q^4)_\infty}{(q/a^2; q^4)_\infty (q/b^2; q^4)_\infty} = \frac{1}{1 - ab + \frac{(a - bq)(b - aq)}{(1 - ab)(q^2 + 1)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1)} + \cdots}. \tag{3.1.8}
\]
Changing $q$ to $q^{3/2}$ and then putting $a = q^{1/4}$, $b = q^{-7/4}$ in above continued fraction, we obtain

$$
\frac{1}{X(q)} := q^{-1/4} \prod_{n=1}^{\infty} \frac{(1 - q^{6n-2})(1 - q^{6n-4})}{(1 - q^{6n-1})(1 - q^{6n-5})} = \frac{q^{-1/4}(q^{1/2} - q^{-3/2})}{(1 - q^{-3/2})} + \frac{(q^{1/4} - q^{-1/4})(q^{-7/4} - q^{7/4})}{(1 - q^{-3/2})(1 + q^3)} + \cdots .
$$

Motivated by the above works, in this chapter, we derive factorizations for $X(q)$ which are similar to (3.1.3), (3.1.4), (3.1.5) and (3.1.6) and we also establish several modular relations for $X(q)$. In Section 3.4, we obtain $2$- and $4$-dissections of $X^*(q) := q^{-1/4}X(q)$ and $\frac{1}{X^*(q)}$. We also show that the sign of the coefficients in the power series expansion of $X^*(q)$ and its reciprocal are periodic with period 2 and 6, respectively.
3.2 Preliminaries

The following lemmas are useful to prove our main results.

Lemma 3.2.1. [7]. Let $m$, $n$, $s$, and $t$ be odd positive integers, then we have

$$f^2(-q^{3n+m}, -q^{n+3m}) - q^2n f^2(-q^{m-n}, -q^{5n+3m}) = f(-q^{2n}, -q^{2m}) \varphi(q^{n+m}). \quad (3.2.1)$$

Lemma 3.2.2. [1]. We have

$$\varphi(q) \psi(q^2) = \psi^2(q), \quad (3.2.2)$$

$$\chi(-q) = \frac{f(-q)}{f(-q^2)}, \quad (3.2.3)$$

$$\chi(-q) \chi(q) = \chi(-q^2). \quad (3.2.4)$$

Lemma 3.2.3. [15]. We have

$$\psi^4(q^3) - q \psi^2(q) \psi^2(q^9) = \frac{f_1 f_2 f_6^2 f_9 f_{18}}{f_3^3}. \quad (3.2.5)$$

Lemma 3.2.4. [7]. We have

$$S(q^4) T(q^8) + q T(q^4) S(q^8) = \frac{f_2^2 f_5 f_{48}}{f_1 f_4 f_6 f_{32}}, \quad (3.2.6)$$

$$S(q^4) T(q^8) - q T(q^4) S(q^8) = \frac{f_1 f_6^2 f_{48}}{f_2 f_3 f_{12} f_{32}}, \quad (3.2.7)$$

$$T^2(q^2) - 3q S^2(q^2) = \frac{f_1^3 f_4 f_{12}}{f_2^3 f_3 f_6^2}, \quad (3.2.8)$$

$$T^2(q^2) + 3q S^2(q^2) = \frac{f_2^2 f_3 f_{12}^2}{f_1^2 f_4 f_6^2 f_6^2}. \quad (3.2.9)$$

where $S(q) = \frac{f(-q, -q^5)}{\psi(-q)}$ and $T(q) = \frac{f(-q^2, -q^4)}{\psi(-q)}$. 

3.3 Modular relations for $X(q)$

We have

$$X(q) = q^{1/4} \frac{f(-q, -q^5)}{f(-q^2, -q^4)} = q^{1/4} \frac{\psi(q^3)}{\psi(q)}.$$  \hspace{1cm} (3.3.1)

In this section, we derive identities involving $X(q)$, which are similar to identities (3.1.3), (3.1.4), (3.1.5) and (3.1.6).

**Theorem 3.3.1.** We have

$$\frac{1}{X(q)} - X(q) = \frac{\chi(q^{1/2})\psi(q^{3/2})\varphi(q^{3/2})}{q^{1/4}f(-q)\psi(q^3)},$$ \hspace{1cm} (3.3.2)

$$\frac{1}{X(q)} + X(q) = \frac{\chi(q^{1/2})\psi(-q^{3/2})\varphi(-q^{3/2})}{q^{1/4}f(-q)\psi(q^3)},$$ \hspace{1cm} (3.3.3)

$$\frac{1}{\sqrt{X(q)}} + \sqrt{X(q)} = \frac{f(q^{1/4}, -q^{5/4})}{q^{1/8}\sqrt{f(-q)\psi(q^3)}},$$ \hspace{1cm} (3.3.4)

$$\frac{1}{\sqrt{X(q)}} - \sqrt{X(q)} = \frac{f(-q^{1/4}, q^{5/4})}{q^{1/8}\sqrt{f(-q)\psi(q^3)}}.$$ \hspace{1cm} (3.3.5)

**Proof.** Using (3.3.1), we have

$$\frac{1}{X(q)} - X(q) = \frac{f^2(-q^2, -q^4) - q^{1/2}f^2(-q, -q^5)}{q^{1/4}f(-q^2, -q^4)f(-q, -q^5)}.$$ \hspace{1cm} (3.3.6)

Putting $n = 1$ and $m = 5$ in (3.2.1), we deduce that

$$f^2(-q^8, -q^{16}) - q^2f^2(-q^4, -q^{20}) = f(-q^2, -q^{10})\varphi(q^6).$$ \hspace{1cm} (3.3.7)

Changing $q$ to $q^{1/4}$ in (3.3.7), employing the resulting identity in (3.3.6), we obtain

$$\frac{1}{X(q)} - X(q) = \frac{f(-q^{1/2}, -q^{5/2})\varphi(q^{3/2})}{q^{1/4}f(-q^2)f(-q, -q^5)}.$$ \hspace{1cm} (3.3.8)

Using (2.2.8) and (3.2.3) in (3.3.8), we obtain (3.3.2).

The proof of (3.3.3) is similar to (3.3.2) and we omit the details.

Using (3.3.1), we have

$$\frac{1}{\sqrt{X(q)}} + \sqrt{X(q)} = \frac{f(-q^2, -q^4) + q^{1/4}f(-q, -q^5)}{q^{1/8}\sqrt{f(-q^2, -q^4)f(-q, -q^5)}}.$$ \hspace{1cm} (3.3.9)
Putting \( a = q^{1/4} \) and \( b = -q^{5/4} \) in (2.2.16), we obtain
\[
f(q^{1/4}, -q^{5/4}) = f(-q^2, -q^4) + q^{1/4} f(-q, -q^5). \tag{3.3.10}
\]

Employing (3.3.10) and (3.2.3) in (3.3.9), we obtain (3.3.4).

The proof of (3.3.5) is similar to the proof of (3.3.4).

**Corollary 3.3.2.** We have
\[
\frac{1}{X^2(q)} - X^2(q) = \chi^3(-q) \psi(\varphi(q^3/2) \psi(-q^{3/2})) \frac{q^{1/2} \psi^2(q)}{q^{1/2} \psi^2(q)}.
\tag{3.3.11}
\]

**Proof.** It is easy to check that
\[
\frac{\varphi(-q)}{\psi(q)} = \chi^3(-q). \tag{3.3.12}
\]

From (3.3.2) and (3.3.3), we get
\[
\left( \frac{1}{X(q)} - X(q) \right) \left( \frac{1}{X(q)} + X(q) \right) = \chi(q^{1/2}) \psi(-q^{3/2}) \varphi(-q^{3/2}) \chi(-q^{1/2}) \psi(q^{3/2}) \varphi(q^{3/2}) \frac{q^{1/2} \psi^2(q)}{q^{1/2} \psi^2(q)}. \tag{3.3.13}
\]

Employing (3.2.4), (2.2.9), (3.3.12), (3.2.3) and (2.2.15) in (3.3.13), we obtain (3.3.11).

**Theorem 3.3.3.** We have
\[
\frac{X^2(-q) - X^2(-q^3)}{X^2(q) - X^2(q^3)} = i \chi^3(-q^2) \chi(-q^{18}) \chi^6(-q) \chi^2(-q^9), \tag{3.3.14}
\]
where \( i = \sqrt{-1} \).

**Proof.** From (3.3.1), we have
\[
X^2(q) - X^2(q^3) = q^{1/2} \psi^4(q^3) - q^3 \psi^2(q) \psi^2(q) \frac{\psi^2(q^3) \psi^2(q)}{\psi^2(q)}. \tag{3.3.15}
\]
Using (3.2.5) and (1.1.8) in (3.3.15), we get
\[ X^2(q) - X^2(q^3) = q^{1/2} \frac{f_3^3 f_9 f_{18}}{f_2^3 f_6}. \]  
(3.3.16)

Changing \( q \) to \(-q\) in (3.3.16), we obtain
\[ X^2(-q) - X^2(-q^3) = (-q)^{1/2} \frac{f_2^3 f_{18}}{f_1^3 f_4^3 f_6 f_{36}}. \]  
(3.3.17)

Dividing (3.3.17) by (3.3.16) and using Lemma 2.2.1 in resulting identity, we obtain (3.3.14).

\[ \square \]

**Theorem 3.3.4.** We have
\[ \frac{(X(q^4) + X(q^8))^4}{(X(q^4) - X(q^8))^4} = \frac{-X^4(-q)}{X^4(q)}. \]  
(3.3.18)

**Proof.** From (3.2.6) and (3.2.7), we have
\[ \frac{S(q^4)T(q^8) + qT(q^4)S(q^8)}{S(q^4)T(q^8) - qT(q^4)S(q^8)} = \frac{f_2^3 f_3^3 f_{12}}{f_1^3 f_4^3 f_6}. \]  
(3.3.19)

Expressing the left-hand side of (3.3.19) in terms of \( X(q) \) and using Lemma 2.2.1 on right-hand side of the (3.3.19), we obtain
\[ \frac{X(q^4) + X(q^8)}{X(q^4) - X(q^8)} = \frac{\psi(q) \psi(-q^3)}{\psi(q^3) \psi(-q)}. \]  
(3.3.20)

Employing (3.3.1) on the right-hand side of (3.3.20) and taking fourth power on both side, we get (3.3.18).

\[ \square \]

**Theorem 3.3.5.** We have
\[ \frac{X(-q) - 3X(-q)X^2(q^2)}{X(q) + 3X(q)X^2(q^2)} = \frac{X^4(-q)}{\sqrt{i} \chi^2(-q^2)}. \]  
(3.3.21)

**Proof.** From (3.2.8) and (3.2.9), we have
\[ \frac{T^2(q^2) - 3qS^2(q^2)}{T^2(q^2) + 3qS^2(q^2)} = \frac{f_1^3 f_4^3 f_{12}}{f_2^3 f_3^3 f_6}. \]  
(3.3.22)
Expressing the left-hand side of (3.3.22) in terms of $X(q)$ and using Lemma 2.2.1 on right-hand side, we obtain
\[
\frac{1 - 3X^2(q^2)}{1 + 3X^2(q^2)} = \frac{\psi(q^3)\psi(-q)\chi^4(-q)}{\psi(-q^3)\psi(q)\chi^2(-q^2)}.
\] (3.3.23)

Employing (3.3.1) on the right-hand side of (3.3.23), we obtain (3.3.21).

**Theorem 3.3.6.** We have
\[
\frac{(1 - X^2(q^2))^2(1 - 3X^2(q^2))^2}{(1 + X^2(q^2))^2(1 + 3X^2(q^2))^2} = -\frac{X^2(q)X^2(q^3)}{X^2(-q)X^2(-q^3)}.
\] (3.3.24)

**Proof.** For each $k = 1, 2$, we obtain an identity from (3.3.2) by changing $q^{1/2}$ to $\omega^k q^{1/2}$, where $\omega = e^{i\pi}$. Multiplying these two identities, we obtain
\[
\prod_{k=1,2} \left( \frac{1}{\omega k X(q^2)} - \omega^k X(q^2) \right) = \prod_{k=1,2} \frac{\chi(-\omega^{2k}q)\psi(\omega^{6k}q^3)\varphi(\omega^{6k}q^3)}{\omega^k q^{1/2} f(-\omega^{4k}q^2)\psi(\omega^{12k}q^6)}.
\] (3.3.25)

Using (2.2.15) and (3.2.2) in (3.3.25), we deduce that
\[
\frac{1 - 2X^2(q^2) + X^4(q^2)}{X^2(q^2)} = \frac{\psi^6(q^3)}{q\psi^4(q^6)\psi^2(q)}.
\] (3.3.26)

In a similar way, using (3.3.3), we obtain
\[
\frac{1 + 2X^2(q^2) + X^4(q^2)}{X^2(q^2)} = \frac{\psi^2(-q^3)\varphi^2(-q^3)}{q\psi^2(q^6)\psi^2(-q)}.
\] (3.3.27)

Dividing (3.3.26) by (3.3.27), we deduce that
\[
\frac{1 - 2X^2(q^2) + X^4(q^2)}{1 + 2X^2(q^2) + X^4(q^2)} = \frac{\psi^6(q^3)\psi^2(-q)}{\psi^2(q^6)\psi^2(-q^3)\varphi^2(-q^3)}
\[
= \frac{\psi^6(q^3)\psi^2(-q)}{\psi^2(q)\psi^6(-q^3)}.
\] (3.3.28)

Using (3.3.1) in (3.3.28), we obtain
\[
\frac{X^2(-q)(1 - X^2(q^2))^2}{X^2(q)(1 + X^2(q^2))^2} = \frac{\psi^4(q^3)}{i\psi^4(-q^3)}.
\] (3.3.29)
It is easy to check that
\[ \frac{\psi^2(q^3)}{\psi^2(-q^3)} = \frac{\chi^2(-q^6)}{\chi^4(-q^3)}. \] (3.3.30)

Changing \( q \) to \( q^3 \) in (3.3.21) and employing (3.3.30) in resulting identity, we obtain
\[ \frac{X(-q^3)(1 - 3X^2(q^6))}{X(q^3)(1 + 3X^2(q^6))} = \frac{\psi^2(-q^3)}{\sqrt{\psi^2(q^3)}}. \] (3.3.31)

Squaring (3.3.31), then employing resulting identity in (3.3.29), we obtain (3.3.24). \( \Box \)
3.4 2- and 4-dissections of the continued fraction $X^*(q)$

In [61], Bernard L. S. Lin studied 2-, 3-, 4-, 6- and 12-dissections of a continued fraction of order twelve. In this section, we give 2- and 4-dissections of the continued fraction $X^*(q)$.

We shall first establish the 2-dissections of $X^*(q)$ and its reciprocal.

**Theorem 3.4.1.** If

$$X^*(q) = \sum_{n=0}^{\infty} u_n q^n = \frac{f(-q, -q^5)}{f(-q^2, -q^4)} \quad \text{and} \quad \frac{1}{X^*(q)} = \sum_{n=0}^{\infty} v_n q^n = \frac{f(-q^2, -q^4)}{f(-q, -q^5)},$$

then

$$\sum_{n=0}^{\infty} u_{2n} q^n = \frac{f(q^4, q^8)}{-f(q)}, \quad (3.4.1)$$

$$\sum_{n=0}^{\infty} u_{2n+1} q^n = -\frac{\chi(q^2)\psi(-q^6)}{f(q)}, \quad (3.4.2)$$

$$\sum_{n=0}^{\infty} v_{2n} q^n = \frac{f(-q^2)f(q^4, q^8)}{\psi(q^3)\varphi(-q^3)}, \quad (3.4.3)$$

$$\sum_{n=0}^{\infty} v_{2n+1} q^n = \frac{f(-q^2)\chi(q^2)\psi(-q^6)}{\psi(q^3)\varphi(-q^3)}. \quad (3.4.4)$$

**Proof.** We have

$$\sum_{n=0}^{\infty} u_{2n} q^{2n} = \frac{1}{2} [X^*(q) + X^*(-q)]$$

$$= \frac{1}{2} \left[ \frac{S(q)T(-q) + S(-q)T(q)}{T(q)T(-q)} \right]. \quad (3.4.5)$$

Putting $a = q$ and $b = q^5$ in (2.2.2), we obtain

$$f(q, q^5) + f(-q, -q^5) = 2f(q^8, q^{16}). \quad (3.4.6)$$
Employing (3.4.6) in (3.4.5), we obtain

\[ \sum_{n=0}^{\infty} u_{2n}q^{2n} = \frac{f(q^8, q^{16})}{f(-q^2)}. \]  \hfill (3.4.7)

Changing \( q \) to \( q^{1/2} \) in (3.4.7), we obtain (3.4.1).

\[ \sum_{n=0}^{\infty} u_{2n+1}q^{2n+1} = \frac{1}{2} \left[ X^*(q) - X^*(-q) \right] \]
\[ = \frac{1}{2} \left[ \frac{S(q)T(-q) - S(-q)T(q)}{T(q)T(-q)} \right] \]
\[ = \frac{f(-q,-q^5) - f(q,q^5)}{2f(-q^2,-q^4)}. \]  \hfill (3.4.8)

Putting \( a = -q \) and \( b = -q^5 \) in (2.2.3), we obtain

\[ f(-q,-q^5) - f(q,q^5) = -2qf(q^4,q^{20}). \]  \hfill (3.4.9)

Employing (3.4.9) in (3.4.8), we obtain

\[ \sum_{n=0}^{\infty} u_{2n+1}q^{2n} = -\frac{f(q^4,q^{20})}{f(-q^2)}. \]  \hfill (3.4.10)

Employing (2.2.8) in (3.4.10) and changing \( q \) to \( q^{1/2} \) in resulting identity, we obtain (3.4.2).

Proof of (3.4.3) and (3.4.4) are similar and we omit the details. \hfill \Box

**Theorem 3.4.2.** We have

\[ \sum_{n=0}^{\infty} u_{2n}q^n = -\sum_{n=0}^{\infty} v_{2n}q^n. \]

\[ \sum_{n=0}^{\infty} u_{2n+1}q^n = -\sum_{n=0}^{\infty} v_{2n+1}q^n. \]

**Proof.** Proof follows from Theorem 3.4.1. \hfill \Box
Now, we shall establish the 4-dissections of \( X^*(q) \) and its reciprocal.

**Theorem 3.4.3.** We have

\[
\sum_{n=0}^{\infty} u_{4n} q^n = \frac{f(q^2, q^4) f(q^3, q^5)}{f^2(-q)}, \quad (3.4.11)
\]

\[
\sum_{n=0}^{\infty} u_{4n+1} q^n = -\frac{\chi(q) \psi(-q^3) f(q^3, q^5)}{f^2(-q)}, \quad (3.4.12)
\]

\[
\sum_{n=0}^{\infty} u_{4n+2} q^n = \frac{f(q^2, q^4) f(q, q^7)}{f^2(-q)}, \quad (3.4.13)
\]

\[
\sum_{n=0}^{\infty} u_{4n+3} q^n = -\frac{\chi(q) \psi(-q^3) f(q, q^7)}{f^2(-q)}, \quad (3.4.14)
\]

where \( u'_n \)'s are as defined in Theorem 3.4.1.

**Proof.** We have

\[
\sum_{n=0}^{\infty} u_{2n} q^n = \frac{f(q^4, q^8)}{f(-q)} = \frac{f(q^4, q^8) \varphi(-q^3)}{f(-q, -q^2) f(q, q^2) \chi(-q)}. \quad (3.4.15)
\]

Now, employing (2.2.15) in (3.4.15), we obtain

\[
\sum_{n=0}^{\infty} u_{2n} q^n = \frac{f(q^4, q^8) \psi(q)}{f^2(-q^2)}. \quad (3.4.16)
\]

Putting \( a = q \) and \( b = q^3 \) in (2.2.16), we get

\[
\psi(q) = f(q, q^3) = f(q^6, q^{10}) + q f(q^2, q^{14}). \quad (3.4.17)
\]

Employing (3.4.17) in (3.4.16), we obtain

\[
\sum_{n=0}^{\infty} u_{2n} q^n = \frac{f(q^4, q^8)}{f^2(-q^2)} \left\{ f(q^6, q^{10}) + q f(q^2, q^{14}) \right\}.
\]

It follows immediately that
\[ \sum_{n=0}^{\infty} u_{4n}q^{2n} = \frac{f(q^4, q^8)f(q^6, q^{10})}{f^2(-q^2)}, \]  
(3.4.18)
\[ \sum_{n=0}^{\infty} u_{4n+2}q^{2n} = \frac{f(q^4, q^8)f(q^2, q^{14})}{f^2(-q^2)}. \]  
(3.4.19)

Changing \( q \) to \( q^{1/2} \) in (3.4.18) and (3.4.19), we deduce (3.4.11) and (3.4.13), respectively.

Proofs of (3.4.12) and (3.4.14) are similar and we omit the details.

Theorem 3.4.4. We have
\[ \sum_{n=0}^{\infty} v_{4n}q^n = \frac{f(-q)f(q^2, q^4)f(-q^6)f(q^9, q^{15})}{\psi(-q^3)\varphi(-q^6)f^2(-q^3)}, \]  
(3.4.20)
\[ \sum_{n=0}^{\infty} v_{4n+1}q^n = \frac{f(-q)\chi(q)f(-q^6)f(q^9, q^{15})}{\varphi(-q^6)f^2(-q^3)}, \]  
(3.4.21)
\[ \sum_{n=0}^{\infty} v_{4n+2}q^n = q \frac{f(-q)f(q^2, q^4)f(-q^6)f(q^3, q^{21})}{\psi(-q^3)\varphi(-q^6)f^2(-q^3)}, \]  
(3.4.22)
\[ \sum_{n=0}^{\infty} v_{4n+3}q^n = q \frac{f(-q)\chi(q)f(-q^6)f(q^3, q^{21})}{\varphi(-q^6)f^2(-q^3)}, \]  
(3.4.23)

here \( v_n \)'s are as defined in Theorem 3.4.1.

Proof. On using (2.2.1), (3.2.3) and (2.2.15), we have
\[ \sum_{n=0}^{\infty} v_{2n}q^n = \frac{f(-q^2)f(q^4, q^8)}{\psi(q^3)\varphi(-q^3)} \]
\[ = \frac{f(-q^2)f(q^4, q^8)}{\psi(q^3)\varphi(-q^3)\psi(q^3)} \]
\[ = f(-q^2)f(q^4, q^8)\psi(-q^3) \]
\[ = \psi(-q^6)\varphi(-q^12)\psi(-q^3) \]
\[ = \frac{f(-q^2)f(q^4, q^8)f(-q^{12})}{\psi(-q^6)\varphi(-q^{12})f(-q^3)} \]
\[ = \frac{f(-q^2)f(q^4, q^8)f(-q^{12})}{\psi(-q^6)\varphi(-q^{12})\chi(-q^3)f(-q^6)} \]
\[ = \frac{f(-q^2)f(q^4, q^8)f(-q^{12})\psi(q^3)}{\psi(-q^6)\varphi(-q^{12})f^2(-q^6)}. \]  
(3.4.24)
Putting $a = q^3$ and $b = q^9$ in (2.2.16), we get
\[
\psi(q^3) = f(q^3, q^9) = f(q^{18}, q^{30}) + q^3 f(q^6, q^{42}).
\] (3.4.25)

Employing (3.4.25) in (3.4.24), we deduce that
\[
\sum_{n=0}^{\infty} v_{2n} q^n = \frac{f(-q^2) f(q^4, q^8) f(-q^{12}) f(q^{18}, q^{30})}{\psi(-q^6) \varphi(-q^{12}) f^2(-q^6)} \{ f(q^{18}, q^{30}) + q^3 f(q^6, q^{42}) \}.
\] (3.4.26)

It follows immediately that
\[
\sum_{n=0}^{\infty} v_{4n} q^{2n} = \frac{f(-q^2) f(q^4, q^8) f(-q^{12}) f(q^{18}, q^{30})}{\psi(-q^6) \varphi(-q^{12}) f^2(-q^6)},
\] (3.4.27)

Changing $q$ to $q^{1/2}$ in (3.4.26) and (3.4.27), we obtain (3.4.20) and (3.4.22), respectively.

Proof of (3.4.21) and (3.4.23) are similar and we omit the details.

**Theorem 3.4.5.** The coefficients of $u_n$ satisfy the inequalities

\[
 u_{2n} > 0 \quad \text{and} \quad u_{2n+1} < 0.
\]

**Proof.** We have
\[
\sum_{n=0}^{\infty} u_n (-1)^n q^n = \frac{(-q; q^{12})_\infty (-q^5; q^{12})_\infty}{(q^2; q^{12})_\infty (q^4; q^{12})_\infty}.
\]

From the above equality, we obtain
\[
 u_{2n} > 0.
\]

Similarly, we can determine the sign of the remaining subsequence for $u_n$.

**Theorem 3.4.6.** We have $v_3$, $v_{6n+2}$, $v_{6n+5} = 0$. The remaining coefficients of $v_n$ satisfy the inequalities $v_{6n}$, $v_{6n+1} > 0$ and $v_{6n+3}$, $v_{6n+4} < 0$. 
Proof. From (3.4.3), we have

\[
\sum_{n=0}^{\infty} v_{2n} q^n = \frac{(q^2; q^2)_\infty (-q^4; q^{12})_\infty (-q^8; q^{12})_\infty}{(q^6; q^{12})_\infty (q^2; q^6)_\infty (q^{12}; q^{12})_\infty} = \frac{(q^2, q^4, q^6, q^8, q^{10}, q^{12})_\infty}{(q^3, q^6, q^9, q^{12})_\infty}.
\]

From the above equality, it follows that \( v_{6n} > 0, \ v_{6n+2} = 0 \) and \( v_{6n+4} < 0 \).

From (3.4.4), we have

\[
\sum_{n=0}^{\infty} v_{2n+1} q^n = \frac{(q^2; q^2)_\infty (-q^2; q^{12})_\infty (-q^{10}; q^{12})_\infty}{(q^6; q^{12})_\infty (q^2; q^6)_\infty (q^{12}; q^{12})_\infty} = \frac{(q^2, -q^2, q^4, q^8, q^{10}, q^{12})_\infty}{(q^3, q^6, q^9, q^{12})_\infty}.
\]

From the above identity, it is easy to see that \( v_{6n+1} > 0, \ v_{6n+5} = 0 \) and \( v_{6n+3} < 0 \ (n \neq 0) \).

\( \square \)