CHAPTER 3

ON 'USEFUL' RELATIVE INFORMATION MEASURES OF ORDER α AND TYPE β

3.1 Introduction

In this chapter, some new generalized measures of useful relative information have been defined and their particular cases have been studied. From these measures new useful information measures have also been derived and their relations with different measures of entropy have been obtained.

Let $P = \{(p_{1,} p_{2,} ...., p_n), 0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1\}$ be a discrete probability distribution of a set of events $E = (E_{1,} E_{2,} ..., E_n)$ on the basis of an experiment whose predicted probability distribution $Q = \{(q_{1,} q_{2,} ...., q_n), 0 \le q_i \le 1, \sum_{i=1}^{n} q_i = 1\}$ and $U = (u_{1,} u_2, ..., u_n)$, where $u_i > 0$ is the utility distribution. In Information Theory, the following measures are well known:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ , \tag{3.1.1}$$

$$H(P;U) = -\sum_{i=1}^{n} u_i p_i \log p_i \ , \tag{3.1.2}$$

$$H(P/Q) = -\sum_{i=1}^{n} p_i \log(p_i / q_i) \tag{3.1.3}$$

and

$$H(P;Q) = -\sum_{i=1}^{n} p_i \log q_i \ . \tag{3.1.4}$$

The information-theoretic measures given by Eq. (3.1.3) and Eq. (3.1.4) respectively, known as Kullback's [67] relative information and Kerridge's [66] inaccuracy are of great significance in statistical estimation and physics. The Eq. (3.1.1) measures average information but does not take into account the qualitative information of the events. Belis and Guiasu [16] introduced a quantitative-qualitative measure of information $H(P;U) = -\sum_{i=1}^{n} u_i p_i \log p_i$, where $u_i > 0$ is the utility attached to the *i*th event which occurs with probability p_i and they found deep application in theory of questionnaires. The measure defined by Eq. (3.1.2) was called 'useful' information by Longo [79]. The Shannon's [98] measure defined by Eq. (3.1.1) possesses very nice mathematical properties which are very useful in many fields. Specifically, in life sciences, we have diversity of animal and plant types which can be measured using Shannon's model. The greater the value of this measure, greater will be the diversity.

Where Shannon's model gives useful applications, although it has a drawback. It always leads to exponential family of distributions. (The reader can consult the paper [44] pages 594 and 600.) Since there are families of distributions other than exponential and there are laws of population growth other than exponential, we cannot confine ourselves to exponential families only and consequently, Shannon's measure may not be much applicable. Moreover, there are many factors like physiographic, topography, biotic interference, anthropogenic, climatic, edaphic etc. which affect the diversity in plants. Let α, β etc. represent such factors upon which the information regarding cybernetic system $E_i / u_i / p_i / q_i$ depends. In this communication, we develop a generalized 'useful' relative information measure of order α and type β.

3.2 A Generalized Measure of 'Useful' Relative Information

A measure of directed divergence $D(P:Q;U)$ of $(P;U)$ to $(Q;U)$ has to satisfy the following conditions

(i) $D(P:Q;U) \geq 0$.

(ii) $D(P:Q;U) = 0$ iff $p_i = q_i$ for each *i*.

(iii) $D(P:Q;U)$ is a convex or pseudo-convex function of $p_1, p_2, \ldots, p_n$ as well as $q_1, q_2, \ldots, q_n$.

Bhaker and Hooda [17] and Hooda and Ram [42] defined and characterized 'useful' directed divergence measures given as,

$$D(P:Q;U)=\frac{\sum_{i=1}^{n}u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n}u_i p_i} \tag{3.2.1}$$

and

$$D_\alpha(P:Q;U)=\frac{1}{1-\alpha}\left[\phi(1)-\phi\left(\frac{\sum_{i=1}^{n}u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n}u_i p_i}\right)\right]. \tag{3.2.2}$$

Since for all probability distributions P and Q having attached with utility distribu- tion U, $\frac{1}{\alpha-1}\left[\frac{\sum_{i=1}^{n}u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n}u_i p_i}\right]\geq\frac{1}{\alpha-1}$ according to $\alpha>1$ or $\alpha<1$, provided $\sum_{i=1}^{n}p_i u_i\geq\sum_{i=1}^{n}q_i u_i$.

But $\frac{1}{\alpha-1}>0$ for all $\alpha>1$. So we assume $\left[\frac{\sum_{i=1}^{n}u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n}u_i p_i}\right]>1$. $\phi(.)$ being continuous monotonic increasing convex function defined on [1, ∞).

These measures satisfies all three conditions (i) - (iii) under the condition

$\sum_{i=1}^{n}p_i^\beta u_i\geq\sum_{i=1}^{n}q_i u_i p_i^{\beta-1}$.

For all probability distribution P and Q having attached with Utility distribution U, we propose the function:

$$D_{(\alpha,\beta)}(P:Q;U)=\frac{1}{1-\alpha}\left[\phi(1)-\phi\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n}u_i p_i^\beta}\right)\right]. \tag{3.2.3}$$

Lemma 3.2.1 For all probability distribution P and Q having attached with Utility distribution U

$$\left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right] \geq 1 \quad \text{for} \quad \alpha \geq 1 \quad \text{or} \quad \left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right] \leq 1 \text{ for } \alpha \leq 1,$$

provided $\sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1}$.

Proof :- By Holder's inequality

For $\alpha > 1$

$$\sum_{i=1}^{n} u_i p_i^{\beta-1} q_i \geq \left[\sum_{i=1}^{n}\left(u_i^{\frac{\alpha}{\alpha-1}} p_i^{\frac{\beta\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n}\left(u_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha+\beta-1}{1-\alpha}} q_i\right)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}$$

it implies

$$\sum_{i=1}^{n} u_i p_i^{\beta-1} q_i \geq \left[\sum_{i=1}^{n} u_i p_i^{\beta}\right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i\right]^{\frac{1}{1-\alpha}}$$

Since $\sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1}$,

therefore
$$\left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right] \geq 1.$$

For $\alpha = 1$ $\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}} = 1$ for all probability distributions.

Also if $\alpha \neq 1$ and $p_i = q_i$ for each i, i.e. $P = Q$, we have

$$\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}} = 1$$

and in other cases $\dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \neq 1$.

Lemma 3.2.2 $\dfrac{1}{\alpha-1}\dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ is a convex function of Q.

Proof:- we prove this in the following steps.

Step1: Let $S = \dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ if we differentiate S partially w. r. t q_i

taking all p_i and u_i fixed ,then $\sum_{i=1}^{n} u_i p_i^{\beta}$ is fixed and thus

$\sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1}$ is constant.

Hence we can write

$$S = C\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha} \quad , \quad \text{where} \quad \frac{1}{C} = \sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1} > 0$$

it implies
$$\frac{\partial S}{\partial q_i} = (1-\alpha)CRu_i p_i^{\alpha+\beta-1} q_i^{-\alpha} \qquad (3.2.4)$$

and

$$\frac{\partial^2 S}{\partial p_i^2} = \alpha(\alpha-1)Cu_i p_i^{\alpha+\beta-1} q_i^{-\alpha-1} \qquad (3.2.5)$$

For $\alpha > 1$, Eq. (3.2.5) is positive and hence $\dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ is a convex function of Q .

For $0<\alpha<1$, Eq. (3.2.5) is negative and hence $\dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ is a concave function of Q.

Step 2. Since $\dfrac{1}{\alpha-1}>0$ for $\alpha>1$, and $\dfrac{1}{\alpha-1}<0$ for $\alpha<1$,

therefore $\dfrac{1}{\alpha-1}\dfrac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ is a convex function of Q for all $\alpha>0$,

provided $\sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1}, \quad \beta \geq 1$.

Since $\phi(x)$ is a monotonic increasing convex function of x,

then $\alpha>1$ gives

$$\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \geq 1$$

this implies
$$\phi\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) \geq \phi(1)$$

thus we have $D_{(\alpha,\beta)}(P:Q;U) \geq 0$

For $0<\alpha<1$ we have

$$\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \leq 1$$

it implies
$$\phi\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) \geq \phi(1)$$

thus $D_{(\alpha,\beta)}(P:Q;U) \geq 0$.

As $\alpha \to 1$, $D_{(\alpha,\beta)}(P:Q;U) \to \phi'(1)\dfrac{\sum_{i=1}^{n} u_i p_i^{\beta} \log\left(\dfrac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i^{\beta}}$, is a consequence of the fact that the measure defined by Eq. (3.2.3) is a continuous function of α, which becomes a generalized constant multiple 'useful' relative information measure of type β.

Also when $p_i = q_i$ for each i, $D_{(\alpha,\beta)}(P:Q;U) = 0$. It is fact that the Eq. (3.2.3) gives minimum value, if $p_i = q_i$ for $i = 1,2,...,n$.

Since an increasing convex function of a convex function is a convex function and $\phi(x)$ is a monotonic increasing convex function and $\alpha > 1$ therefor, $D_{(\alpha,\beta)}(P:Q;U)$ is a convex function of Q .

Similarly, if $\phi(x)$ is a concave function and $0 < \alpha < 1$, then by the facts that an increasing concave function of a concave function is a concave and the negative of a concave function is a convex function, $D_{(\alpha,\beta)}(P:Q;U)$ is a convex function of Q.

Theorem 3.2.1 $D_{(\alpha,\beta)}(P:Q;U)$ is a measure of useful relative information if either $\alpha > 1$ and $\phi(x)$ is any monotonic increasing twice differentiable convex function of x or if $0 < \alpha < 1$ and $\phi(x)$ is any monotonic decreasing twice differentiable concave function of x.

3.3 Special Cases

(i) If $\phi(x) = x^j$, $(j > 0)$, we have

$$D_{\alpha,\beta,j}(P:Q;U) = \frac{1}{1-\alpha}\left[1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{j}\right], \qquad (3.3.1)$$

where $\beta \geq 1$, $\alpha > 1$ and $j \geq 1$ or $0 < \alpha < 1$ and $0 < j \leq 1$.

Which is generalized j-useful relative information measure of order α and type β. It also satisfies the conditions (i)-(iii) under the condition $\sum_{i=1}^{n} p_i^{\beta} u_i \geq \sum_{i=1}^{n} q_i u_i p_i^{\beta-1}$.

(ii) If $\phi(x) = x$, then Eq. (3.2.3) reduces to

$$D_{(\alpha,\beta)}(P:Q;U) = \frac{1}{1-\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)\right], \qquad (3.3.2)$$

which is a new generalized useful relative information measure of order α and type β.

(iii) If $u_i = 1$ for each i, then Eq. (3.3.2) gives

$$D_{\alpha,\beta,j}(P:Q) = \frac{1}{1-\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^{\beta}}\right)^{j}\right]. \qquad (3.3.3)$$

(iv) $$\lim_{\alpha\to 1} D_{\alpha,\beta,j}(P:Q;U) = j\,\frac{\sum_{i=1}^{n} u_i p_i^{\beta}\log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} u_i p_i^{\beta}}, \qquad (3.3.4)$$

which is generalized j-multiple 'useful' relative information measure of type β.

(v) If $u_i = 1$ for each i, then Eq. (3.3.4) gives $j\,\frac{\sum_{i=1}^{n} p_i^{\beta}\log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^{n} p_i^{\beta}}$,

which is j-multiple of Kullback and Leibler [68] measure of directed divergence.

(vi) If $u_i = 1$ for each i, then Eq. (3.3.2) gives

$$D_{(\alpha,\beta)}(P:Q;U) = \frac{1}{1-\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^{\beta}}\right)\right],$$

which is the generalized measure relative information characterized and studied by Rathie[93].

(vii) If β=1, then Eq. (3.3.2) gives a measure similar to Hooda's [40] 'useful' relative information measure of order α.

(viii) If β=1 and $u_i = 1$ for each i, then Eq. (3.3.2) reduces to

$$D_\alpha(P:Q;U) = \frac{1}{1-\alpha}\left[1 - \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}\right],$$

which is a relative information measure characterized by Rathie and Kannappan [94], Havrda-Charvat [39], Tsallis's [129] and Nath and Mittal [86].

(ix) If $\phi(x) = \log x$, then Eq. (3.3.2) reduces to

$$D_{(\alpha,\beta)}(P:Q;U) = \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right), \tag{3.3.5}$$

which is generalized 'useful' relative information measure of order α and type β studied by Hooda and Sharma [43].

(x) $u_i = 1$ for each i, then Eq. (3.3.5) gives

$$D_{(\alpha,\beta)}(P:Q) = \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^{\beta}}\right), \tag{3.3.6}$$

which is generalized relative information measure characterized and studied by Sharma [99].

(xi) If β=1, Eq. (3.3.5) reduces to

$$D_\alpha(P:Q;U) = \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\right), \tag{3.3.7}$$

which is generalized 'useful' relative information measure of order α studied by Bhaker and Hooda [17].

(xii) $u_i = 1$ for each i, then Eq. (3.3.7) gives

$$D_\alpha(P:Q) = \frac{1}{1-\alpha}\log\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha},$$

which is Reny's measure [96] of directed divergence.

3.4 Measure of 'Useful' Information

Let C= $\left(\frac{1}{n},\frac{1}{n},........,\frac{1}{n}\right)$ be uniform distribution. Then

a) $D_{(\alpha,\beta)}(P:C;U)\geq 0$.

b) $D_{(\alpha,\beta)}(P:C;U)=0$, if $P=C$.

c) $D_{(\alpha,\beta)}(P:C;U)$ is a convex function of C.

We see that

$$D_{(\alpha,\beta)}(P:C;U)=\frac{1}{1-\alpha}\left[\phi(1)-\phi(n^{\alpha-1})\right]-\frac{1}{1-\alpha}\left[\phi\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}n^{\alpha-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)-\phi(n^{\alpha-1})\right].$$

Next we define

$$H_{\alpha,\beta,\phi}(P:C;U)=\frac{1}{1-\alpha}\left[\phi\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}n^{\alpha-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)-\phi(n^{\alpha-1})\right]. \quad (3.4.1)$$

If $P=C$ ie $p_i=\frac{1}{n}$ for each i, then Eq. (3.4.1) gives

$$H_{\alpha,\beta,\phi}(C;U)=\frac{1}{1-\alpha}\left[\phi(1)-\phi(n^{\alpha-1})\right] \quad (3.4.2)$$

and

$$\underset{\alpha\to 1}{Lim}\, H_{\alpha,\beta,\phi}(C;U)=\phi'(1)\log n. \quad (3.4.3)$$

Eq. (3.4.2) and Eq. (3.4.3) are the same result as if utilities are ignored i.e., in case of uniform distributions, utilities play no role in the generalized 'useful' information measure.

Therefore,

$$D_{(\alpha,\beta)}(P:C;U)=H_{\alpha,\beta,\phi}(C;U)-H_{\alpha,\beta,\phi}(P:C;U).$$

So to minimize $D_{(\alpha,\beta)}(P:C;U)$ is equivalent to maximize $H_{\alpha,\beta,\phi}(P:C;U)$.

(d) $H_{\alpha,\beta,\phi}(C;U)\geq H_{\alpha,\beta,\phi}(P:C;U)$.

(e) $H_{\alpha,\beta,\phi}(C;U) = H_{\alpha,\beta,\phi}(P:C;U)$, if $P = C$.

(f) $H_{\alpha,\beta,\phi}(P:C;U)$ is a concave function of C.

Theorem 3.4.1 The function defined, by Eq. (3.4.1) is a useful information measure of P, C and U corresponding to 'useful' directed divergence measure $D_{(\alpha,\beta)}(P:C;U)$.

If $\phi(x) = x^j, (j \geq 1)$, we have

$$H_{\alpha,\beta,j}(P;U) = \frac{1}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^j - (n^{\alpha-1})^j\right]$$

$$= \frac{\left(n^{\alpha-1}\right)^j}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^j - 1\right]. \qquad (3.4.4)$$

The measure defined by Eq. (3.4.4) is non-additive i.e., for the independent distributions P and Q having U and V respectively as utility distributions, then

$$H_{\alpha,\beta,j}(P*Q:U*V) = \frac{1-\alpha}{n^{(\alpha-1)j}} H_{\alpha,\beta,j}(P;U) H_{\alpha,\beta,j}(Q;V) + H_{\alpha,\beta,j}(P;U) + H_{\alpha,\beta,j}(Q;V), \qquad (3.4.5)$$

where $U*V = \{u_i v_j : u_i \in U \quad and \quad v_j \in V\}$ is the product utility distribution associated with the product probability distribution

$$P*Q = \{p_i q_j : p_i \in P \quad and \quad q_j \in Q\}.$$

Then the Eq. (3.4.4) reduces to

$$H_{\alpha,\beta,j}(P*Q:U*V) = \frac{n^{(\alpha-1)j}}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} u_i v_j (p_i q_j)^{\alpha+\beta-1}}{\sum_{i=1}^{n}\sum_{j=1}^{m} u_i v_j (p_i q_j)^{\beta}}\right)^j - 1\right].$$

Since P and Q considered here are independent therefore, we have

$$H_{\alpha,\beta,j}(P*Q:U*V)=\frac{n^{(\alpha-1)j}}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\frac{\sum_{j=1}^{m}v_j q_j^{\alpha+\beta-1}}{\sum_{j=1}^{m}v_j q_j^{\beta}}\right)^j-1\right]$$

$$=\frac{n^{(\alpha-1)j}}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)^j\left(\frac{\sum_{i=1}^{m}v_j q_j^{\alpha+\beta-1}}{\sum_{i=1}^{m}v_j q_j^{\beta}}\right)^j-1\right]$$

$$=\frac{n^{(\alpha-1)j}}{1-\alpha}\left[\left\{\frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,\beta,j}(P;U)+1\right\}\left\{\frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,\beta,j}(Q;V)+1\right\}-1\right]$$

$$=\frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,\beta,j}(P;U)H_{\alpha,\beta,j}(Q;V)+H_{\alpha,\beta,j}(P;U)+H_{\alpha,\beta,j}(Q;V).$$

This is a well known functional equation and that measure defined by Eq. (3.4.4) is non additive.

If $\beta=1$, then Eq. (3.21) reduces to

$$H_{\alpha,j}(P;U)=\frac{\left(n^{\alpha-1}\right)^j}{1-\alpha}\left[\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha}}{\sum_{i=1}^{n}u_i p_i}\right)^j-1\right]. \tag{3.4.6}$$

For $j=1$ in Eq. (3.4.4), we get

$$H_{\alpha,\beta,1}(P;U)=\frac{n^{\alpha-1}}{1-\alpha}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}-1\right]. \tag{3.4.7}$$

When $\alpha\to 1$ the measures given by Eq. (3.4.1), Eq. (3.4.4) and Eq. (3.4.7) reduces to

$$H_{1,\beta,\phi}(P;U)=-\phi'(1)\left(\frac{\sum_{i=1}^{n}u_i p_i^{\beta}\log p_i}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right), \tag{3.4.8}$$

$$H_{1,\beta,j}(P;U) = -j\frac{\sum_{i=1}^{n} u_i p_i^{\beta} \log p_i}{\sum_{i=1}^{n} u_i p_i^{\beta}} \tag{3.4.9}$$

and
$$H_{1,\beta,1}(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i^{\beta} \log p_i}{\sum_{i=1}^{n} u_i p_i^{\beta}} . \tag{3.4.10}$$

If we take $\phi(x) = \log x$ in Eq. (3.4.1), we get

$$H_{\alpha,\beta,\phi}(P;U) = \frac{1}{1-\alpha}\left[\log\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) - \log(n^{\alpha-1})\right]$$

$$= \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right). \tag{3.4.11}$$

If $\beta = 1$, then Eq. (3.4.11) reduces to

$$H_{\alpha,\phi}(P;U) = \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right), \tag{3.4.12}$$

which is Bhaker and Hooda's measure of useful information of order α .

If $u_i = 1$ for each i, from Eq. (3.4.12) we have

$$H_{\alpha,\phi}(P;U) = \frac{1}{1-\alpha}\log\sum_{i=1}^{n} p_i^{\alpha} ,$$

which is Renyi's entropy.

If $u_i = 1$ for each i, from Eq. (3.4.11) we have

$$H_{\alpha,\beta,\phi}(P) = \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta}}\right). \tag{3.4.13}$$

It becomes Kapur [51] and Aczel and Darcozy [2] entropy.

If we take $\phi(x) = x\log x$, we get

$$H_{\alpha,\beta,\phi}(P;U)=\frac{1}{1-\alpha}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}n^{\alpha-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\log\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}n^{\alpha-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)-n^{\alpha-1}\log(n^{\alpha-1})\right]$$

$$=\frac{1}{1-\alpha}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\log\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)\right]. \qquad (3.4.14)$$

In case $\alpha\to 1$, Eq. (3.4.14) reduces to $H_{\alpha,\beta,\phi}(P;U)=-\frac{\sum_{i=1}^{n}u_i p_i^{\beta}\log p_i}{\sum_{i=1}^{n}u_i p_i^{\beta}}$, which is a 'useful' information measure of type β.

When utilities are ignored in Eq. (3.4.14), we have

$$H_{\alpha,\beta,\phi}(P)=\frac{1}{1-\alpha}\left[\frac{\sum_{i=1}^{n}p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}p_i^{\beta}}\log\left(\frac{\sum_{i=1}^{n}p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}p_i^{\beta}}\right)\right]. \qquad (3.4.15)$$

When $\alpha\to 1$, the Eq. (3.4.15) reduces to

$$H_{1,\beta,\phi}(P)=-\frac{\sum_{i=1}^{n}p_i^{\beta}\log p_i}{\sum_{i=1}^{n}p_i^{\beta}}. \qquad (3.4.16)$$

If we put β=1, then Eq. (3.4.16) reduces to Shannon's entropy [98].

3.5 Relationship Between Measure of Entropy and Directed Divergence

Now we deduce the various measures of entropy from the above mentioned measures of directed divergence by taking the second distribution as uniform distribution $Q=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$.

3.5.1 Behra and Chawla's Measure

From Eq. (3.3.1), we have

$$D_{\alpha,\beta,j}(P:Q;U)=\frac{1}{1-\alpha}\left[1-n^{(\alpha-1)j}\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)^j\right]$$

$$=\frac{n^{(\alpha-1)j}(n^{(\alpha-1)j}-1)}{1-\alpha}-\frac{n^{(\alpha-1)j}}{\alpha-1}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)^j\right]$$

$$=H_{\alpha,j}(Q;U)-H_{\alpha,\beta,j}(P;U),$$

where

$$H_{\alpha,\beta,j}(P;U)=\frac{n^{(\alpha-1)j}}{1-\alpha}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)^j\right] \quad (3.5.1)$$

and

$$H_{\alpha,j}(Q;U)=\frac{n^{(\alpha-1)j}(n^{(\alpha-1)j}-1)}{1-\alpha};\ n=2,3,...\ \alpha>0(\neq 1). \quad (3.5.2)$$

The fact that measure defined by Eq. (3.5.1) is maximal if all the probabilities are equal and if $p_i=\frac{1}{n}$ for all $i=1,2,....,n$, then the measures given by Eq. (3.5.1) and Eq. (3.5.2) are equal.

If utilities are ignored or $u_i=1$ for each i and β=1, then Eq. (3.5.1) becomes

$$H_{\alpha,1,j}(P;U)=\frac{n^{(\alpha-1)j}}{1-\alpha}\left(1-\sum_{i=1}^{n}p_i^{\alpha}\right)^j. \quad (3.5.3)$$

Which differs from the Behra and Chawla [14] entropy through only by a multiple constant.

3.5.2 Kapur's Measure

From Eq. (3.3.2), we have

$$D_{(\alpha,\beta)}(P:Q;U)=\frac{1}{f(\alpha)}\left[n^{(\alpha-1)}\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)-1\right]$$

$$=\frac{n^{\alpha-1}(1-n^{\alpha-1})}{f(\alpha)}-\frac{n^{\alpha-1}}{f(\alpha)}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)\right]$$

$$=H_{\alpha}(Q;U)-H_{\alpha,\beta}(P;U), \qquad (3.5.4)$$

where

$$H_{\alpha,\beta}(P;U)=\frac{n^{\alpha-1}}{f(\alpha)}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}u_i p_i^{\beta}}\right)\right];\ \alpha\neq 1,\ f(1)=0,\ f'(1)=1 \quad (3.5.5)$$

and

$$H_{\alpha}(Q;U)=\frac{n^{\alpha-1}(1-n^{1-\alpha})}{f(\alpha)};\ n=2,3,...\quad \alpha>0(\neq 1)\,. \qquad (3.5.6)$$

It is also fact that measure defined by Eq. (3.5.5) is maximal if all the probabilities are equal and if $p_i=\frac{1}{n}$ for all $i=1,2,....,n$, then the measures given by Eq. (3.5.5) and Eq. (3.5.6) are equal.

If utilities are ignored or $u_i=1$ for each i, then Eq. (3.5.5) becomes

$$H_{\alpha,\beta}(P)=\frac{n^{\alpha-1}}{f(\alpha)}\left[1-\left(\frac{\sum_{i=1}^{n}p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n}p_i^{\beta}}\right)\right];\ \alpha\neq 1,\ \beta\geq 1, f(1)=0,\ f'(1)=1, \quad (3.5.7)$$

which is slightly modified form of Kapur's [51] entropy.

If $\beta=1$, then Eq. (3.5.7) reduces to

$$H_{\alpha,\beta}(P)=\frac{n^{\alpha-1}}{f(\alpha)}\left[1-\left(\frac{\sum_{i=1}^{n}p_i^{\alpha}}{\sum_{i=1}^{n}p_i}\right)\right];\ \alpha\neq 1,\ f(1)=0,\ f'(1)=1, \quad (3.5.8)$$

it is slightly modified form of Kapur's [51] non-additive measure for incomplete probability distribution.

Evidently for complete probability distribution, Eq. (3.5.8) becomes

$$H_\alpha(P) = \frac{n^{\alpha-1}}{f(\alpha)}\left[1 - \sum_{i=1}^{n} p_i^\alpha\right]; \ \alpha > 0, \ f(1) = 0, \ f'(1) = 1, \qquad (3.5.9)$$

which is Kapur's [53] entropy of order α.

3.6 Conclusions

We have shown in each of the above cases that the directed divergence from the uniform distribution is equal to the excess of entropy of the uniform distribution over the entropy of given distribution. In many cases we have to slightly modify the definitions of entropies. Taking into consideration, some other well defined ‘useful’ measures of entropy, ‘useful’ measures of directed divergence and similar relationship can be obtained.