CHAPTER 2

AXIOMATIC CHARACTERIZATION OF ENTROPY OF TYPE (α,β,γ)

2.1 Introduction

This chapter deals with characterization of a measure of information of type (α,β,γ) with taking certain axioms parallel to those considered earlier by Havrda and Charvat along with the recursive relation. Some properties of this measure are also studied. This measure includes Shannon information measure as a special case. In the section 2.4 we start with a triparametric self information function and triparametric entropy. Some familiar entropies are derived as particular cases. A measure called information deviation and some generalization of Kullback's information are obtained under some boundary conditions.

Shannon's measure of entropy for a discrete probability distribution

$$P=(p_1,...,p_n),\ \ p_i \geq 0,\ \ \sum_{i=1}^{n} p_i = 1,$$

given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ ,$$

has been characterized in several ways (see Aczel [1]). Out of the many ways of characterization the two elegant approaches are found in the work of

a) Faddeev [30], who uses branching property

$$H_n(p_1,...,p_n) = H_{n-1}(p_1+p_2,p_3,...,p_n)+(p_1+p_2)H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right), \tag{2.1.1}$$

n =3, 4, … for the above distribution P, as the basic postulate, and

b) Chaundy and McLeod [26], who studied the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \qquad for \;\; p_i \geq 0, q_j \geq 0. \quad (2.1.2)$$

Both the above mentioned approaches have been extensively exploited and generalized. The most general form of Eq. (2.1.2) has been studied by Sharma and Taneja [106], who considered the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i) g(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i) f(q_j), \qquad (2.1.3)$$

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1, \; p_i \geq 0, \; q_j \geq 0$$

We define the information measure as

$$H_n(p_1,...,p_n;\alpha,\beta,\gamma) = (2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}), \;\; \alpha \neq \gamma \neq \beta, \quad \alpha,\beta,\gamma > 0,$$

(2.1.4)

for a complete probability distribution

$$P = (p_1,..., p_n), \;\; p_i \geq 0, \; \sum_{i=1}^{n} p_i = 1 .$$

when $\alpha = \gamma = 1$(or $\beta = \gamma = 1$) then the measure defined by Eq. (2.1.4) reduces to entropy of type β(or α) given by

$$H_n(p_1,..., p_n; \beta) = (2^{1-\beta} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\beta} - 1 \right], \;\; \beta \neq 1, \; \beta > 0 . \quad (2.1.5)$$

When $\beta \to 1$, measure given by Eq. (2.1.5) reduces to Shannon's entropy [98]

$$H_n(p_1,..., p_2) = -\sum_{i=1}^{n} p_i \log_2 p_i . \qquad (2.1.6)$$

The measure defined by Eq. (2.1.5) was characterized by many authors by different approaches. Harvda and Charvat [39] characterized Eq. (2.1.5) by an axiomatic approach. Darcozy [27] studied Eq. (2.1.5) by a functional equation. A joint characterization of the measure given by Eq. (2.1.5) and Eq. (2.1.6) has been done by author in two different ways. Firstly by a generalized functional equation having four different functions and secondly by an axiomatic approach.

Functional measures of type β have also been obtained by Sharma and Taneja [106]. Later on C. Tsallis [129] gives the applications of Eq. (2.1.5) in physics.

In this chapter, we characterized the measure defined by Eq. (2.1.4) by taking certain axioms parallel to those considered earlier by Havrda and Charvat [39] along with the recursive relation defined by Eq. (2.1.7). Some properties of this measure are also studied.

The measure defined by Eq. (2.1.4) satisfies a recursive relation as follows:

$$H_n(p_1,\ldots,p_n;\alpha,\beta,\gamma)-H_{n-1}(p_1+p_2,p_3,\ldots,p_n;\alpha,\beta,\gamma)$$
$$=\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}(p_1+p_2)^{\alpha/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2};\alpha,\gamma\right)$$
$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}(p_1+p_2)^{\beta/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2};\gamma,\beta\right),$$
$$\alpha\neq\gamma\neq\beta,\quad \alpha,\beta,\gamma>0, \quad (2.1.7)$$

where

$$p_1+p_2>0,\quad A_{(\alpha,\gamma)}=(2^{\frac{\gamma-\alpha}{\gamma}}-1)\quad and\quad A_{(\beta,\gamma)}=(2^{\frac{\gamma-\beta}{\gamma}}-1)\ .$$
$$H(p_1,p_2,\ldots,p_n;\alpha,\gamma)=A^{-1}_{(\alpha,\gamma)}\left[\sum_{i=1}^{n}p_i^{\alpha/\gamma}-1\right];\ \alpha\neq\gamma;\alpha,\gamma>0\neq1,$$
$$H(p_1,p_2,\ldots,p_n;\gamma,\beta)=A^{-1}_{(\beta,\gamma)}\left[1-\sum_{i=1}^{n}p_i^{\beta/\gamma}\right];\ \beta\neq\gamma;\beta,\gamma>0\neq1.$$

Proof:

$$H_n(p_1,\ldots,p_n;\alpha,\beta,\gamma)-H_{n-1}(p_1+p_2,p_3,\ldots,p_n;\alpha,\beta,\gamma)$$
$$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1^{\alpha/\gamma}-p_1^{\beta/\gamma})+(p_2^{\alpha/\gamma}-p_2^{\beta/\gamma})+....+(p_n^{\alpha/\gamma}-p_n^{\beta/\gamma})\}$$
$$-(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1+p_2)^{\alpha/\gamma}-(p_1+p_2)^{\beta/\gamma}+(p_3^{\alpha/\gamma}-p_3^{\beta/\gamma})+....+(p_n^{\alpha/\gamma}-p_n^{\beta/\gamma})\}$$
$$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{p_1^{\alpha/\gamma}-p_1^{\beta/\gamma}+p_2^{\alpha/\gamma}-p_2^{\beta/\gamma}-(p_1+p_2)^{\alpha/\gamma}+(p_1+p_2)^{\beta/\gamma}\}$$
$$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{p_1^{\alpha/\gamma}+p_2^{\alpha/\gamma}-(p_1+p_2)^{\alpha/\gamma}\}$$
$$+(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1+p_2)^{\beta/\gamma}-p_1^{\beta/\gamma}-p_2^{\beta/\gamma}\}$$
$$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}(p_1+p_2)^{\alpha/\gamma}\left[\frac{p_1^{\alpha/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}+\frac{p_2^{\alpha/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}-1\right]$$

$$+(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}(p_1+p_2)^{\beta/\gamma}\left[1-\frac{p_1^{\beta/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}-\frac{p_2^{\beta/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}\right]$$

$$=\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}(p_1+p_2)^{\alpha/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2};\alpha,\gamma\right)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}(p_1+p_2)^{\beta/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2};\gamma,\beta\right).$$

This proves Eq. (2.1.7).

2.2 Set of Axioms

For characterizing a measure of information of type (α,β,γ) associated with a probability distribution

$$P=(p_1,\ldots,p_n),\ p_i\geq 0,\ \sum_{i=1}^{n}p_i=1\ ,$$

we introduce the following axioms:

(1) $H_n(p_1,\ldots,p_n;\alpha,\beta,\gamma)$ is continuous in the region

$$p_i\geq 0,\ \sum_{i=1}^{n}p_i=1,\ \alpha,\ \beta,\ \gamma>0.$$

(2) $H_2(1,0;\alpha,\beta,\gamma)=0$.

(3) $H_2(\frac{1}{2},\frac{1}{2};\alpha,\beta,\gamma)=1,\ \alpha,\beta,\gamma>0$.

(4) $H_n(p_1,\ldots,p_{i-1},0,\ p_{i+1},\ldots,p_n;\alpha,\beta,\gamma)=$

$H_{n-1}(p_1,\ldots,p_{i-1},\ p_{i+1},\ldots,p_n;\alpha,\beta,\gamma)$

for every i = 1,2,…,n.

(5) $H_{n+1}(p_1,\ldots,p_{i-1},v_{i_1},v_{i_2},p_{i+1},\ldots,p_n;\alpha,\beta,\gamma)$

$-H_n(p_1,\ldots,p_{i-1},p_i,p_{i+1},\ldots,p_n;\alpha,\beta,\gamma)$

$$=\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}p_i^{\alpha/\gamma}H_2(v_{i_1}/p_i,v_{i_2}/p_i;\alpha,\gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}p_i^{\beta/\gamma}H_2(v_{i_1}/p_i,v_{i_2}/p_i;\gamma,\beta),\ \alpha\neq\gamma\neq\beta,\quad \alpha,\beta,\gamma>0,$$

for every $v_{i_1} + v_{i_2} = p_i > 0,\ i = 1,2\dots,n$,

where, $A_{(\alpha,\gamma)} = (2^{\frac{\gamma-\alpha}{\gamma}} - 1)$ *and* $A_{(\beta,\gamma)} = (2^{\frac{\gamma-\beta}{\gamma}} - 1),\quad \alpha \neq \gamma \neq \beta$.

Theorem 2.2.1 If $\alpha \neq \beta \neq \gamma; \alpha,\beta,\gamma > 0$, then the axioms (1)-(5) determine a measure given by

$$H_n(p_1,\dots,p_n;\alpha,\beta,\gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}(p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}),\ \alpha \neq \gamma \neq \beta,\quad \alpha,\beta,\gamma > 0, \tag{2.2.1}$$

where $A_{(\alpha,\gamma)} = (2^{\frac{\gamma-\alpha}{\gamma}} - 1)$ *and* $A_{(\beta,\gamma)} = (2^{\frac{\gamma-\beta}{\gamma}} - 1)$.

Before proving the theorem we prove some intermediate results based on the above axioms.

Lemma 2.2.1 If $v_k \geq 0, k = 1,2,\dots,m; \sum_{k=1}^{m} v_k = p_i > 0$, then

$$H_{n+m-1}(p_1,\dots,p_{i-1,}v_1,\dots,v_m,\ p_{i+1},\dots,p_n;\alpha,\beta,\gamma)$$

$$= H_n(p_1,\dots,p_n;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} p_i^{\alpha/\gamma} H_m(v_1/p_i,\dots,v_m/p_i;\alpha,\gamma)$$

$$. \quad + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} p_i^{\beta/\gamma} H_m(v_1/p_i,\dots,v_m/p_i;\gamma,\beta). \tag{2.2.2}$$

Proof :- To prove the lemma, we proceed by induction. For m =2, the desired statement holds(cf. Axiom(4)) Let us suppose that the result is true for numbers less than or equal to m, we shall prove it for m+1.We have

$$H_{n+m}(p_1,\dots,p_{i-1,}v_1,\dots,v_{m+1},\ p_{i+1},\dots,p_n;\alpha,\beta,\gamma) = H_{n+1}(p_1,\dots,p_{i-1,}v_1,L,\ p_{i+1},\dots,p_n;\alpha,\beta,\gamma)$$

$$+ \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} L^{\alpha/\gamma} H_m(v_2/L,\dots,v_{m+1}/L;\alpha,\gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} L^{\beta/\gamma} H_m(v_2/L,\dots,v_{m+1}/L;\gamma,\beta)$$

where $L = v_2 + \dots + v_{m+1}$

$$= H_n(p_1,\dots,p_n;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} p_i^{\alpha/\gamma} H_2(v_1/p_i, L/p_i;\alpha,\gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}\, p_i^{\beta/\gamma} H_2(v_1/p_i\,,L/p_i\,;\,\gamma,\beta)$$

$$+\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}\, L^{\alpha/\gamma} H_m(v_2/L\,,...,v_{m+1}/L\,;\alpha,\gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}\, L^{\beta/\gamma} H_m(v_2/L\,,...,v_{m+1}/L\,;\,\gamma,\beta)$$

$$= H_n(p_1,\ldots,p_n;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}\{p_i^{\alpha/\gamma} H_2(v_1/p_i\,,L/p_i\,;\alpha,\gamma)$$

$$+L^{\alpha/\gamma} H_m(v_2/L,...,v_{m+1}/L;\alpha,\gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}\{p_i^{\beta/\gamma} H_2(v_1/p_i\,,L/p_i\,;\,\gamma,\beta) + L^{\beta/\gamma} H_m(v_2/L,...,v_{m+1}/L;\,\gamma,\beta)\}, \qquad (2.2.3)$$

where $p_i = v_1 + L > 0$

One more application of induction premise yields

$$H_{m+1}(v_1/p_i,\ldots,v_{m+1}/p_i\,;\alpha,\beta,\gamma)$$

$$= H_2(v_1/p_i\,,L/p_i;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}(L/p_i)^{\alpha/\gamma} H_m(v_2/L,...,v_{m+1}/L;\alpha,\gamma)$$

$$+\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}(L/p_i)^{\beta/\gamma} H_m(v_2/L,...,v_{m+1}/L\,;\gamma,\beta) \qquad (2.2.4)$$

For $\beta = \gamma$, Eq. (2.2.4) reduces to

$$H_{m+1}(v_1/p_i,\ldots,v_{m+1}/p_i\,;\alpha,\gamma) = H_2(v_1/p_i\,,L/p_i\,;\alpha,\gamma)$$
$$+(L/p_i)^{\alpha/\gamma} H_m(v_2/L,...,v_{m+1}/L\,;\alpha,\gamma). \qquad (2.2.5)$$

Similarly for $\alpha = \gamma$, Eq. (2.2.4) reduces to

$$H_{m+1}(v_1/p_i,\ldots,v_{m+1}/p_i\,;\gamma,\beta) = H_2(v_1/p_i\,,L/p_i\,;\gamma,\beta)$$
$$+(L/p_i)^{\beta/\gamma} H_m(v_2/L,...,v_{m+1}/L\,;\gamma,\ \beta). \qquad (2.2.6)$$

Eq. (2.2.2) together with Eq. (2.2.5) and Eq. (2.2.6) gives the desired result.

Lemma 2.2.2 If $v_{ij} \ge 0,\ j = 1,2,...,m_i, \sum_{j=1}^{m_i} v_{ij} = p_i > 0,\ i = 1,2,...,n,\ \sum_{i=1}^{n} p_i = 1$, then

$$H_{m_1+\ldots+m_n}(v_{11},v_{12},\ldots,v_{1m_1}:\ldots:v_{n1},v_{n2},\ldots,v_{nm_n};\alpha,\beta,\gamma)$$

$$= H_n(p_1, p_2, ..., p_n; \alpha, \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \sum_{i=1}^{n} p_i^{\alpha/\gamma} H_{m_i}(v_{i1}/p_i, ..., v_{im_i}/p_i; \alpha, \gamma) +$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} \sum_{i=1}^{n} p_i^{\beta/\gamma} H_{m_i}(v_{i1}/p_i, ..., v_{im_i}/p_i; \gamma, \beta) \quad (2.2.7)$$

Proof: Proof of this lemma is directly follow from lemma 2.2.1.

Lemma 2.2.3 If $F(n; \alpha, \beta, \gamma) = H_n(1/n, ..., 1/n; \alpha, \beta, \gamma)$, then

$$F(n; \alpha, \beta, \gamma) = \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} F(n; \alpha, \gamma) + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} F(n; \gamma, \beta), \quad (2.2.8)$$

where

$$F(n; \alpha, \gamma) = A_{(\alpha,\gamma)}^{-1}(n^{\frac{\gamma-\alpha}{\gamma}} - 1), \ \alpha \neq \gamma,$$

and

$$F(n; \gamma, \beta) = A_{(\beta,\gamma)}^{-1}(n^{\frac{\gamma-\beta}{\gamma}} - 1), \ \beta \neq \gamma. \quad (2.2.9)$$

Proof: Replacing in Lemma 2.2.2 m_i by m and putting $v_{ij} = 1/mn$, i=1,2,…,n; j=1,2,…,m, where m and n are positive integer, we have

$$F(mn; \alpha, \beta, \gamma) = F(m; \alpha, \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/m)^{\frac{\alpha-\gamma}{\gamma}} F(n; \alpha, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/m)^{\frac{\beta-\gamma}{\gamma}} F(n; \gamma, \beta) \quad (2.2.10)$$

$$F(mn; \alpha, \beta, \gamma) = F(n; \alpha, \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/n)^{\frac{\alpha-\gamma}{\gamma}} F(m; \alpha, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/n)^{\beta/\gamma - 1} F(m; \gamma, \beta), \quad (2.2.11)$$

Putting m = 1 in Eq. (2.2.10) and using $F(1; \alpha, \beta, \gamma) = 0$ (by axiom 2), we get

$$F(n; \alpha, \beta, \gamma) = \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\lambda)} - A_{(\beta,\gamma)}} F(n; \alpha, \gamma) + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} F(n; \gamma, \beta),$$

which is Eq. (2.2.8).

Comparing the right hand sides of Eq. (2.2.10) and Eq. (2.2.11), we get.

$$F(m; \alpha, \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/m)^{\frac{\alpha}{\alpha-\gamma}} F(n; \alpha, \gamma) + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/m)^{\frac{\beta}{\beta-\gamma}} F(n; \gamma, \beta)$$

$$= F(n;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/n)^{\frac{\alpha}{\alpha-\gamma}} F(m;\alpha,\gamma) + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/n)^{\frac{\beta}{\beta-\gamma}} F(m;\gamma,\beta) \tag{2.2.12}$$

Eq. (2.2.12) together with Eq. (2.2.8) gives

$$A_{(\alpha,\gamma)}\{[1-(1/n)^{\alpha/\gamma-1}]F(m;\alpha,\gamma)+[(1/m)^{\alpha/\gamma-1}-1]F(n;\alpha,\gamma)\}$$

$$= A_{(\beta,\gamma)}\{[1-(1/n)^{\beta/\gamma-1}]F(m;\gamma,\beta)+[(1/m)^{\beta/\gamma-1}-1]F(n;\gamma,\beta)\}. \tag{2.2.13}$$

Putting n = 2 in Eq. (2.2.13) and use $F(2,\alpha,\beta,\gamma) = H_2(\frac{1}{2},\frac{1}{2};\alpha,\beta,\gamma) = 1$, we get

$$A_{(\alpha,\gamma)}\{(1-2^{1-\alpha/\gamma})F(m;\alpha,\gamma)-(1-(1/m)^{\alpha/\gamma-1})\}$$

$$= A_{(\beta,\gamma)}\{(1-2^{1-\beta/\gamma})F(m;\gamma,\beta)-(1-(1/m)^{\beta/\gamma-1})\} = C \ (say),$$

i.e., $A_{(\alpha,\gamma)}\{(1-2^{1-\alpha/\gamma})F(m;\alpha,\gamma)-(1-(1/m)^{\alpha/\gamma-1})\} = \mathrm{C}$,

where C is an arbitrary constant.

For m = 1, we get C=0

Thus, we have

$$F(m;\alpha,\gamma) = \frac{1-m^{1-\alpha/\gamma}}{1-2^{1-\alpha/\gamma}} = A^{-1}_{(\alpha,\gamma)}(m^{1-\alpha/\gamma}-1), \quad \alpha \neq \gamma .$$

Similarly,

$$F(m;\gamma,\beta) = \frac{1-m^{1-\beta/\gamma}}{1-2^{1-\beta/\gamma}} = A^{-1}_{(\beta,\gamma)}(m^{1-\beta/\gamma}-1), \quad \beta \neq \gamma ,$$

which is Eq. (2.2.9).

Now Eq. (2.2.8) together with Eq. (2.2.9) gives

$$F(n;\alpha,\beta,\gamma) = \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} F(n;\alpha,\gamma) + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} F(n;\gamma,\beta),$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}(n^{1-\alpha/\gamma} - n^{1-\beta/\gamma}). \tag{2.2.14}$$

Proof of the theorem:

We prove the theorem for rationals and then the continuity axiom (1) extends the result for reals. For this let m and r_i's be positive integers such that $\sum_{i=1}^{n} r_i = m$ and if we put $p_i = r_i/m$, i=1,2,….,n then an application of lemma 2.2.2 gives

$$H_m(\underbrace{1/m,...,1/m}_{r_1},...,\underbrace{1/m,...,1/m}_{r_n};\alpha,\beta,\gamma)$$

$$= H_n(p_1, p_2,..., p_n;\alpha,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \sum_{i=1}^{n} p_i^{\alpha/\gamma} H_{r_i}(1/r_i,...,1/r_i;\alpha,\gamma)$$

i.e.,

$$H_n(p_1,..., p_n;\alpha,\beta,\gamma) = F(m;\alpha,\beta,\gamma) \tag{2.2.15}$$

Eq. (2.2.15) together with Eq. (2.2.9) and Eq. (2.2.12) gives

$$H_n(p_1,..., p_n;\alpha,\beta,\gamma) = \frac{1}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}),$$

$$\alpha \neq \gamma \neq \beta, \quad \alpha,\beta,\gamma > 0,$$

which is Eq. (2.2.1).

This completes the proof of the theorem.

2.3 Properties of Entropy of Type (α,β,γ)

The measure $H_n(P;\alpha,\beta,\gamma)$, where $P = (p_1,..., p_n)$, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$ is a probability distribution, as characterized in the preceding section, satisfies certain properties, which are given in the following theorems.

Theorem 2.3.1 The measure $H_n(P;\alpha,\beta,\gamma)$ is non-negative for $\alpha \neq \gamma \neq \beta, \quad \alpha,\beta,\gamma > 0$

Proof:

Case 1. $\alpha > \gamma$; $\beta < \gamma \quad \Rightarrow \frac{\alpha}{\gamma} > 1 \, and \, \frac{\beta}{\lambda} < 1$

$$\Rightarrow \quad \sum_{i=1}^{n} p_i^{\alpha/\gamma} < 1 \, and \sum_{i=1}^{n} p_i^{\beta/\gamma} > 1$$

$$\Rightarrow \quad \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}) < 0$$

Since, $\alpha > \gamma \; and \; \beta < \gamma$

$$\Rightarrow \quad 1 - \alpha/\gamma < 0 \, and \, 1 - \beta/\gamma > 0$$

$$\Rightarrow \quad 2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma} < 0$$

$$\Rightarrow \quad (2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma})^{-1} < 0$$

$$\Rightarrow \quad (2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}) > 0$$

Case 2. Similarly for $\alpha < \gamma \; and \; \beta > \gamma$, we get

$$(2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma})^{-1} \sum_{i=1}^{n} p_i^{\alpha/\gamma} - p_i^{\beta/\gamma} > 0$$

Therefore from case 1,case 2 and axiom 2, we get

$$H_n(P;\alpha, \beta, \gamma) \geq 0 .$$

This completes the proof of theorem.

Definition.2.3.1 We shall use the following definition of a convex function.

A function $f(.)$ over the points in a convex set R is convex $\cap$ if for all $r_1, r_2 \in R$ and $\mu \in (0, 1)$

$$\mu f(r_1) + (1-\mu) f(r_2) \leq f(\mu r_1 + (1-\mu) r_2). \quad (2.3.1)$$

The function $f(.)$ is convex $\cup$ if Eq. (2.3.1) holds with $\geq$ in place of $\leq$.

Theorem 2.3.2 The measure $H_n(P;\alpha, \beta, \gamma)$ is convex $\cap$ function of the probability distribution $P = (p_1,..., p_n), p_i \geq 0, \sum_{i=1}^{n} p_i = 1$, when either

$$\alpha > \gamma; \; \beta \leq \gamma \; or \; \beta > \gamma; \alpha \leq \gamma .$$

Proof:- Let there be r distributions

$$P_k(X) = \{p_k(x_1),..., p_k(x_n)\}, \quad \sum_{i=1}^{n} p_k(x_i) = 1, \; k = 1,2,...,r, \quad (2.3.2)$$

associated with the random variable $X = (x_1,..., x_n)$.

Consider r numbers $(a_1,..., a_r)$ such that $a_k \geq 0$ and $\sum_{k=1}^{r} a_k = 1$,

define

$$P_o(X) = \{p_o(x_1),..., p_o(x_n)\}, \quad (2.3.3)$$

where

$$p_o(x_i) = \sum_{k=1}^{r} a_k p_k(x_i), \; i = 1,2,...,n. \quad (2.3.4)$$

Obviously $\sum_{i=1}^{n} p_o(x_i) = 1$ and thus $P_o(x)$ is a bonafide distribution of X.

Let $\alpha > \gamma, 0 < \beta \le \gamma$, then we have

$$\sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - H_n(P_0(\alpha,\beta,\gamma)$$

$$= \sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}\left\{\left[\sum_{j=1}^{r} a_j p_j\right]^{\alpha/\gamma} - \left[\sum_{j=1}^{r} a_j p_j\right]^{\beta/\gamma}\right\} \quad (2.3.5)$$

$$\le \sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}(\sum_{j=1}^{r} a_j p_j^{\alpha/\gamma} - \sum_{j=1}^{r} a_j p_j^{\beta/\gamma}) = 0$$

(by Jensen inequality)

$$\Rightarrow \sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - H_n(P_o;\alpha,\beta,\gamma) \le 0$$

i.e. $\sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) \le H_n(P_o;\alpha,\beta,\gamma)$

for $\alpha > \gamma, 0 < \beta \le \gamma$.

By symmetry in α, β, γ the above result is true for

$\beta > \gamma, 0 < \alpha \le \gamma$.

Theorem 2.3.3 The measure $H_n(p;\alpha,\beta,\gamma)$ satisfies the following relations:

(i) Generalized –Additive:

$$H_{mn}(P*Q;\alpha,\beta,\gamma) = G_n(P;\alpha,\beta,\gamma)H_m(Q;\alpha,\beta,\gamma) + G_m(Q;\alpha,\beta,\gamma)H_n(P;\alpha,\beta,\gamma)$$

$$\alpha, \beta, \gamma > 0, \quad (2.3.6)$$

where

$$G_n(P;\alpha,\beta,\gamma) = \frac{1}{2}\sum_{i=1}^{n}(p_i^{\alpha/\gamma} + p_i^{\beta/\gamma}),\ \alpha, \beta, \gamma > 0. \quad (2.3.7)$$

(ii) Sub-Additive: For, $\alpha, \beta > \gamma$ he measure $H_n(p;\alpha,\beta,\gamma)$ is sub-additive

i.e.,

$$H_{nm}(P*Q;\alpha,\beta,\gamma) \le H_n(P;\alpha,\beta,\gamma) + H_m(Q;\alpha,\beta,\gamma), \quad (2.3.8)$$

where $P = (p_1,\ldots,p_n)$, $Q = (q_1,\ldots,q_m)$ and

$$P * Q = (p_1 q_1 ,..., p_1 q_m ,..., p_n q_1 ,..., p_n q_m) ,$$

are complete probability distributions.

Proof.(i) We have

$$H_{nm}(P*Q;\alpha,\beta,\gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [(p_i q_j)^{\alpha/\gamma} - (p_i q_j)^{\beta/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [(p_i q_j)^{\alpha/\gamma} - (p_i q_j)^{\beta/\gamma} + p_i^{\alpha/\gamma} q_j^{\beta/\gamma} - p_i^{\alpha/\gamma} q_j^{\beta/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [p_i^{\alpha/\gamma} q_j^{\alpha/\gamma} - p_i^{\beta/\gamma} q_j^{\beta/\gamma} + p_i^{\alpha/\lambda} q_j^{\beta/\gamma} - p_i^{\alpha/\gamma} q_j^{\beta/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [p_i^{\alpha/\gamma}(q_j^{\alpha/\gamma} + q_j^{\beta/\gamma}) - q_j^{\beta/\gamma}(p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} [\sum_{i=1}^{n} p_i^{\alpha/\gamma} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma}) - \sum_{j=1}^{m} q_j^{\beta/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})] \qquad (2.3.9)$$

Also

$$H_{nm}(P*Q;\alpha,\beta,\gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [(p_i q_j)^{\alpha/\gamma} - (p_i q_j)^{\beta/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [(p_i q_j)^{\alpha/\gamma} - (p_i q_j)^{\beta/\gamma} + p_i^{\beta/\gamma} q_j^{\alpha/\gamma} - p_i^{\beta/\gamma} q_j^{\alpha/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [p_i^{\alpha/\gamma} q_j^{\alpha/\gamma} - p_i^{\beta/\gamma} q_j^{\beta/\gamma} + p_i^{\beta/\gamma} q_j^{\alpha/\gamma} - p_i^{\beta/\gamma} q_j^{\alpha/\gamma}]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} [q_j^{\alpha/\gamma}(p_i^{\alpha/\gamma} + p_i^{\beta/\gamma}) - p_i^{\beta/\gamma}(q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})]$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} [\sum_{j=1}^{m} q_j^{\alpha/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma}) - \sum_{i=1}^{n} p_i^{\beta/\gamma} \sum_{i=1}^{n} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})] \qquad (2.3.10)$$

Adding Eq. (2.3.9) and Eq. (2.3.10), we get

$$2\, H_{nm}(P*Q;\alpha,\beta,\gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} [\sum_{i=1}^{n} p_i^{\alpha/\gamma} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})$$

$$- \sum_{j=1}^{m} q_j^{\beta/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})]$$

$$+ (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} [\sum_{j=1}^{m} q_j^{\alpha/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma}) - \sum_{i=1}^{n} p_i^{\beta/\gamma} \sum_{i=1}^{n} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})]$$

$$=\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{j=1}^{m}(q_j^{\alpha/\gamma}-q_j^{\beta/\gamma})$$

$$+\sum_{j=1}^{m}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}-p_i^{\beta/\gamma})$$

$$H_{nm}(P*Q;\alpha,\beta,\gamma)=\frac{1}{2}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{j=1}^{m}(q_j^{\alpha/\gamma}-q_j^{\beta/\gamma})$$

$$+\frac{1}{2}\sum_{j=1}^{m}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}-p_i^{\beta/\gamma})$$

Using Eq. (2.3.7)

$$H_{nm}(P*Q;\alpha,\beta,s)=G_n(P;\alpha,\ \beta,\gamma)H_m(Q;\alpha,\ \beta,\gamma)+G_m(Q;\alpha,\ \beta,\gamma)H_n(P;\alpha,\ \beta,\gamma)$$

Which is Eq. (2.3.6). This completes the proof of part (i).

Proof (ii):- From part (i), we have

$$H_{nm}(P*Q;\alpha,\beta,\gamma)=G_n(P;\alpha,\ \beta,\gamma)H_m(Q;\alpha,\ \beta,\gamma)+G_m(Q;\alpha,\ \beta,\gamma)H_n(P;\alpha,\ \beta,\gamma)\ ,$$

as $G_n(P;\alpha,\ \beta,\gamma)=\frac{1}{2}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})\le 1\ for\ \alpha,\beta\ge\gamma$

$$\therefore\qquad H_{nm}(P*Q;\alpha,\beta,\gamma)\le H_m(Q;\alpha,\ \beta,\gamma)+H_n(P;\alpha,\ \beta,\gamma)$$

This proves the sub-additivity.

2.4 Triparametric Self Information Function and Entropy

Shannon [98] first introduced the idea of self-information function in the form

$$f(x)=-\log_2 x,\quad 0<x\le 1. \tag{2.4.1}$$

We use the method of averaging self-information introduced by Shannon. Like Shannon, we introduce a triparametric self-information function defined by

$$f_3(x;\alpha,\beta,\gamma)=\frac{k(x^{\alpha/\gamma}-x^{\beta/\gamma})}{x},\quad 0<x\le 1,\ \alpha\ge 0,\beta\ge 0,\ \gamma\ge 0,\ \alpha\ne\beta\ne\gamma, \tag{2.4.2}$$

where k is a constant, depending upon the real valued parameters α,β,γ and k is ascertained by a suitable pair $(x,\ f_3)$. We apply the following conditions on f_3:

i) $f_3\to\infty$ as $x\to 0$, when $\gamma >$ at least one of $\alpha,\ \beta$.

ii) $f_3 = 0$, when $x = 1$; or $f_3 \to 0$, when $x \to 0$ for $\alpha, \beta > \gamma$.

iii) $f_3 = 1$, $x = \frac{1}{2}$. then $k = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}} \right)^{-1}$.

The function shows the following particular behaviors:

a) If α, β are fixed, then for $x < \frac{1}{2}$, $f_3 \to \infty$ as $\gamma \to \infty$; and for $x > \frac{1}{2}$, $f_3 \to 0$ as $\gamma \to \infty$.

b) For any fixed γ, $f_3 \to -(2x)^{\frac{\alpha-\gamma}{\gamma}} \log_2 x$, as $\alpha \to \beta$.

c) If $\beta = \gamma$ and $\alpha \to \gamma (\alpha < \gamma)$, then $f_3 \to -\log_2 x$.

Self-information function is different from information function. Different authors, namely Darcozy [27], Aczel [1], Chaundy and Mcleod [26], Havrda and Charvat [39], Kannapan [49], Sharma and Taneja [106], Mittal [81] and some others have solved some typical functional equations and have used their solutions as entropy, inaccuracy, directed divergence, etc., in the capacity of finite measures only in complete probability distributions. The method of averaging self-informations includes the case of generalized probability distributions. Further since it is uncertain and difficult to choose an arbitrary functional equation and to find its suitable solutions to be used as information measures, it becomes easier if we choose any suitable parametric self-information function that can satisfy a number of effective boundary conditions. We have given a most simple and general choice in Eq. (2.4.2).

2.5 Triparametric Entropy

Let $P = (p_1, p_2, ..., p_n)$ be a finite discrete probability distribution, where $0 < p_i \le 1$, $\sum_{i=1}^{n} p_i \le 1$. Then averaging the function $f_3(p_i; \alpha, \beta, \gamma)$ with respect to P, we define the triparametric entropy as

$$H(P; \alpha, \beta, \gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}} \right)^{-1} \left[\sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}) \right] \Big/ \sum_{i=1}^{n} p_i \,, \quad (2.5.1)$$

where $\alpha \geq 0, \beta \geq 0, \ \gamma \geq 0, \alpha \neq \beta \neq \gamma.$

When P is complete, we have

$$H(P;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\right], \qquad (2.5.2)$$

$\alpha \geq 0, \beta \geq 0, \ \gamma \geq 0, \alpha \neq \beta \neq \gamma.$

2.6 Some Familiar Entropies

From Eq. (2.5.2), we get the following entropies as particular cases :

(i) $\gamma = 1$ gives Sharma and Taneja's [107] entropy of type (α, β) in the form

$$H(P;\alpha,\beta) = \left(2^{1-\alpha} - 2^{1-\beta}\right)^{-1} \left[\sum_{i=1}^{n} (p_i^{\alpha} - p_i^{\beta})\right], \quad \alpha \neq \beta \qquad (2.6.1)$$

and

$$\underset{\alpha \to \beta}{Lt} H(P;\alpha,\beta) = \left(\sum_{i=1}^{n} p_i^{\beta} \log_2 \frac{1}{p_i}\right) 2^{\beta-1}.$$

(ii) Putting $\alpha = \gamma = 1$, we get Darcozy's [27] entropy of type β as

$$H(P;\beta) = \left(2^{1-\beta} - 1\right)^{-1} \left[\sum_{i=1}^{n} (p_i^{\beta} - 1)\right], \quad \beta > 0, \ \beta \neq 1 \qquad (2.6.2)$$

(iii)When $\beta = \gamma$ and $\alpha \to \gamma (\alpha < \gamma)$, then Eq. (2.5.2) reduces to

$$H(P) = \sum_{i=1}^{n} p_i \log_2 \frac{1}{p_i} \quad , \qquad (2.6.3)$$

which is Shannon's [98] entropy.

(iv)When $n > 2$, then $H \to \infty$ as $\gamma \to \infty$ for any fixed $\alpha, \beta < \gamma$; when $n = 1$, then $H = 0, \ p_1 = 1;$ and when $n = 2$, then $H = 1$.

2.7 Application of the Entropy Defined by Eq. (2.5.2)

2.7.1 Joint Entropy

For joint probability distribution, a relation similar to Eq. (2.5.2) also holds in the form

$$H(PQ;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{k=1}^{n} \sum_{j=1}^{m} (p_{kj}^{\alpha/\gamma} - p_{kj}^{\beta/\gamma})\right],$$

$$0 < p_{kj} \le 1,\ \sum_{k=1}^{n}\sum_{j=1}^{m} p_{kj} = 1;\ \alpha \ge 0,\ \beta \ge 0,\ \gamma \ge 0,\ \alpha \ne \beta. \tag{2.7.1}$$

Theorem 2.7.1 If $P = (p_1, p_2, ..., p_n)$ be the distribution of input symbols of a source, $Q = (q_1, q_2, ..., q_n)$ be that of output symbols, and

$$PQ = (p_{k1}, p_{k2}, ..., p_{km};\ k = 1,2,...,n)$$

be the joint distribution of input and output symbols, also

$$R_k = \left(\frac{p_{k1}}{p_k}, \frac{p_{k2}}{p_k}, ..., \frac{p_{km}}{p_k}\right)$$

be the condition distribution of output symbols and

$$R_j = \left(\frac{p_{1j}}{q_j}, \frac{p_{2j}}{q_j}, ..., \frac{p_{nj}}{q_j}\right)$$

be the condition distribution of input symbols, where

$$p_{kj}/p_k = p_{j|k}, (j = 1,2,...,m);\ p_{kj}/q_j = p_{k|j}, (k = 1,2,...,n);$$

$$\sum_{j=1}^{m} p_{kj} = p_k \text{ and } \sum_{k=1}^{n} p_{kj} = q_j,$$

then,

$$H(PQ;\alpha,\beta,\gamma) = \sum_{k=1}^{n} p_k^{\frac{\beta}{\gamma}} H(R_k;\alpha,\beta,\gamma) + \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\beta-\gamma}{\gamma}}\right)^{-1}\left[\sum_{k=1}^{n}\left(p_k^{\frac{\alpha}{\gamma}} - p_k^{\frac{\beta}{\gamma}}\right)\sum_{j=1}^{m} p_{j|k}^{\frac{\alpha}{\gamma}}\right]. \tag{2.7.2}$$

Putting $\alpha = \gamma = 1$ and using $\sum_{j=1}^{m} p_{j|k} = 1$ in Eq. (2.7.2), we have

$$H(PQ;\ \beta) = \sum_{k=1}^{n} p_k^{\beta} H_1(R_k;\beta) + H_1(P;\beta). \tag{2.7.3}$$

Theorem 2.7.2 If $p_{kj} = p_k q_j$, then

$$H(PQ;\ \alpha,\beta,\gamma) = \sum_{k=1}^{n} p_k^{\frac{\alpha}{\gamma}} H(R_k;\alpha,\beta,\gamma) + \sum_{j=1}^{m} q_j^{\frac{\beta}{\gamma}} H(R_j;\alpha,\beta,\gamma)$$

$$= \sum_{k=1}^{n} p_k^{\frac{\alpha}{\gamma}} H(Q;\alpha,\beta,\gamma) + \sum_{j=1}^{m} q_j^{\frac{\beta}{\gamma}} H(P;\alpha,\beta,\gamma). \tag{2.7.4}$$

2.7.2 Triparametric Information Function

With the help of Eq. (2.5.2) we define a triparametric information function in the form

$$F_3(x) = F(x;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}} \right)^{-1} (x^{\alpha/\gamma} - x^{\beta/\gamma}), \tag{2.7.5}$$

$$\alpha \geq 0, \beta \geq 0,\ \gamma \geq 0,\ \alpha \neq \beta \neq \gamma$$

$$0 < x \leq 1,$$

where $F(0) = 0$ not always but $F(1) = 0$, $F(\frac{1}{2}) = \frac{1}{2}$ always.

Thus

$$H(P;\alpha,\beta,\gamma) = \sum_{k=1}^{n} F(p_k),\ \ 0 < p_k \leq 1,\ \ \ \sum_{ki=1}^{n} p_k = 1.$$

Putting $a = \alpha/\gamma,\ \ b = \beta/\gamma$ in Eq. (2.7.5), we have

$$F_3(x) = F(x;\alpha,\beta,\gamma) = \left(2^{1-a} - 2^{1-b}\right)^{-1} (x^a - x^b),\ \ -\infty < a,b < \infty,\ \ a \neq b. \tag{2.7.6}$$

Now, from practical point of view, as far as an inaccuracy in a measure is concerned, a measure is associated with at least two probability distributions, corresponding to which at least two variables u and v are needed. This suggests the choice of at least four parameters a, b, c and d.

2.7.3 Generalized Information Function

Concerning an association of two variables u, v and four parameters a, b, c, d, an information measure similar to Eq. (2.7.6) is introduced by

$$F_4(u,v) = F(u,v;a,b,c,d) = G[u^a v^b - u^c v^d],$$

$$0 < u,v \leq 1,\ \ -\infty < a,b,c,d < \infty,\ \ a \neq b. \neq c \neq d \tag{2.7.7}$$

as the generalized information function, which possesses the characteristic of becoming both bounded and unbounded.

2.7.3.1 Boundary Conditions

(i) At $u = 1,\ v = \frac{1}{2},\ F_4\left(1, \frac{1}{2}\right) = \frac{1}{2}$ so that $G = (2^{1-b} - 2^{1-d})^{-1}$, where $b \neq d$.

If $a+b=c+d$, where $a \neq c$, then $F_4\left(\frac{1}{2},\frac{1}{2}\right)=0$,

similarly at $u=\frac{1}{2}, v=1,\ F_4\left(\frac{1}{2},1\right)=\frac{1}{2}$ so that $G=(2^{1-a}-2^{1-c})^{-1}$, where $a \neq c$.

(ii) At $u=1, v=\frac{1}{2},\ F_4\left(1,\frac{1}{2}\right)=\frac{1}{2}$, so that $G=(2^{-b}-2^{-d})^{-1}$, where $b \neq d$.

At $u=\frac{1}{2}, v=1,\ F_4\left(\frac{1}{2},1\right)=1$, so that $G=(2^{-a}-2^{-c})^{-1}$, where $a \neq c$.

2.7.3.2 Generalized Inaccuracy

Let $P=(p_1,p_2,...,p_n)$ and $Q=(q_1,q_2,...,q_n)$ be two discrete probability distribution concerned with Eq. (2.7.7), where

$$0<p_i \leq 1, 0<q_i \leq 1,\ \ \sum_{i=1}^{n} p_i=1, \sum_{i=1}^{n} q_i=1,\ (u,v)=(p_i,q_i) \text{ or } (q_i,p_i),\ i=1,2,...,n .$$

We may then define the generalized inaccuracies by

$$I_4(P\|Q)=\sum_{i=1}^{n} F_4(p_i,q_i)=(2^{1-b}-2^{1-d})^{-1}\left[\sum_{i=1}^{n} p_i^a q_i^b-\sum_{i=1}^{n} p_i^c q_i^d\right],\ \ b \neq d, \tag{2.7.8}$$

and

$$I_4(Q\|P)=\sum_{i=1}^{n} F_4(q_i,p_i)=(2^{1-b}-2^{1-d})^{-1}\left[\sum_{i=1}^{n} q_i^a p_i^b-\sum_{i=1}^{n} q_i^c p_i^d\right],\ \ b \neq d, \tag{2.7.9}$$

which follows from Eq. (2.2.7) and boundary condition (i).

Given P and Q we see that

(i) $I_4(P\|Q) \to +\infty$ or $-\infty$, according as $a \to -\infty$ or $c \to -\infty$ for $b<d$; or as $c \to -\infty$ or $a \to -\infty$ for $b>d$.

(ii) If $d=1, c=0$ then $I_4(P\|Q) \to (1-2^{1-b})^{-1}$ as $a \to \infty$.

(iii) If $d=1, c=0$ then $I_4(P\|Q) \to 1$ as $b \to \infty$.

It is noted that, when $d=1, c=0$, then

$$I_4(Q\|P)=(2^{1-b}-1)^{-1}\left[\sum_{i=1}^{n} p_i^a q_i^b-1\right]. \tag{2.7.10}$$

2.7.3.3 Information Deviations

If $d=1, c=0, a+b=1$, then we introduce the quantities:

$$D(Q\|P\|Q) = I_4(P\|Q) = H(Q) - H(Q\|P) \tag{2.7.11}$$

and

$$D(P\|Q\|P) = I_4(Q\|P) = H(P) - H(P\|Q), \tag{2.7.12}$$

as information deviations of Q from P and P from Q respectively, where

$$H(P) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}, \quad H(Q) = \sum_{k=1}^{n} q_k \log_2 \frac{1}{q_k}$$

are Shannon's entropies [98] and

$$H(Q\|P) = \sum_{k=1}^{n} q_k \log_2 \frac{1}{p_k}, \quad H(P\|Q) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{q_k}$$

are Kerridge's[66] inaccuracies. Thus,

$$D(Q\|P\|Q) = \sum_{k=1}^{n} q_k \log_2 \frac{p_k}{q_k}, \quad D(P\|Q\|P) = \sum_{k=1}^{n} p_k \log_2 \frac{q_k}{p_k}. \tag{2.7.13}$$

2.7.3.4 Kullback's Information and its Generalizations

If we take the boundary conditions (ii), then

$$I_4(P\|Q) = \frac{1}{2} I_4^*(P\|Q),$$

where

$$I_4^*(P\|Q) = (2^{-b} - 2^{-d})^{-1} \left[\sum_{i=1}^{n} p_i^a q_i^b - \sum_{i=1}^{n} p_i^c q_i^d \right], \ b \neq d.$$

Now If $d=1, c=0, a+b=1$, then

$$\underset{b\to 0}{Lt}\, I_4(P\|Q) = \tfrac{1}{2} I(P\|P\|Q), \quad \underset{b\to 0}{Lt}\, I_4(Q\|P) = \tfrac{1}{2} I(Q\|Q\|P), \tag{2.7.14}$$

where

$$D(P\|P\|Q) = \sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k} = H(P\|Q) - H(P) \tag{2.7.15}$$

and

$$D(Q\|Q\|P)=\sum_{k=1}^{n} q_k \log_2 \frac{q_k}{p_k} = H(Q\|P)-H(Q), \tag{2.7.16}$$

represents kullback's [67] informations.

Information deviations and Kullback's informations are equal and opposite measures. The fact follows from

$$D(Q\|P\|Q)+I(Q\|Q\|P)=0, D(P\|Q\|P)+I(P\|P\|Q)=0. \tag{2.7.17}$$

It may be noted that information deviations and Kullback's informations become zero, if $p_k = q_k$ for $k = 1,2,...,n.$

2.7.3.5 Generalized Boundary Conditions

We shall now show that so far as our generalized inaccuracies given by Eq. (2.7.8) and Eq. (2.7.9) are concerned, there exist certain boundary conditions for which certain limiting functions of Eq. (2.7.8) and Eq. (2.7.9) may be taken as the generalized forms of Kullback's informations. For this, we generalized the boundary conditions in the following ways and get the results:

(i) Let $u=1,\ v=\frac{1}{2},\ F_4\left(1,\frac{1}{2}\right)=\frac{1}{2^m}$, where m is a real number ≥ 0

then for $d=1,\ c=0,\ a+b=1$ we have

$$I^{(1)}(P,Q,m)=\underset{b\to 0}{Lt}\, I_4(P\|Q)=2^{-m}\sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k}, \quad m\geq 0 \tag{2.7.18}$$

and is called first generalized Kullback information. Put $m=0$ in Eq. (2.7.18), we get Kullback's information. The information given by Eq. (2.7.18) decreases as m increases.

(ii) Let $u=1,\ v=\frac{1}{2},\ F_4\left(1,\frac{1}{2}\right)=\frac{1}{2^m}$, where m is a real number ≥ 0.

Also let $d=0,\ c=1+m,\ a+b=1+m$, then we have

$$I^{(2)}(P,Q,m)=\underset{b\to 0}{Lt}\, I_4(P\|Q)=2^{-m}\sum_{k=1}^{n} p_k^{m+1} \log_2 \frac{p_k}{q_k}, \quad m\geq 0, \tag{2.7.19}$$

to be called the second generalized Kullback's information. It is observed that

$$I^{(2)}(P,Q,m)\leq I^{(1)}(P,Q,m).$$

Put $m=0$ in Eq. (2.7.19), we get Kullback's information.

(iii) Let $u = 1, v = \frac{1}{2}, F_4\left(1, \frac{1}{2}\right) = 2^{-1/m}$, where m is a real number ≥ 0. Then the values $d = 0, c = 1 + 1/m, a + b = 1 + 1/m$, lead to the information

$$I^{(3)}(P, Q, m) = \underset{b \to 0}{Lt}\, I_4(P \| Q) = 2^{-1/m} \sum_{k=1}^{n} p_k^{1/m+1} \log_2 \frac{p_k}{q_k} .$$

This may be called the third generalized Kullback's information. In this case

$$\underset{m \to 0}{Lt}\, I^{(3)}(P, Q, m) = 0 \quad \text{and} \quad \underset{m \to \infty}{Lt}\, I^{(3)}(P, Q, m) = I(P \| P \| Q).$$

2.8 Conclusions

In addition to well known information measure of Shannon [98], Renyi's [95], Havrda-Charvat [39], Vajda [131], Darcozy [27] and Sharma and Taneja [107] we have considered a measure which we call (α, β, γ) information measure. We have given some basic axioms and properties with recursive relation. The Shannon measure included in the (α, β, γ) information measure for the limiting case $\alpha = \gamma = 1$ and $\beta \to 1$ or $\beta = \gamma = 1$ and $\alpha \to 1$. As a conclusion we remarked that (α, β, γ) information measure is a new measure for which it is worth while to consider further properties.