CHAPTER 6

CHARACTERIZATION OF NONADDITIVE MEASURES OF AVERAGE CHARGE WITH ONE AND TWO PARAMETERS

6.1 Introduction

In this chapter a simple non-additive model based on λ-non-additivity is considered. We characterize average charge for heterogeneous questionnaires under λ-non-additivity and show that there are only two forms of the measure of charge for one and two parameter. Limiting and particular cases covers those results which are studied earlier. The average charge are shown to be bounded below by two non-additive mean value entropies, one is of order 1 and type α, and other is of order α and type β respectively, in case of one parameter also in case of two parameter these are bounded below by two non-additive mean value entropies, one is of order 1, type β and the other is of order α type β respectively.

Interesting application of information theory have been found in questionnaire theory (Picard [89]). Picard and Campbell [24] have shown the relationship between noiseless coding theory and questionnaire theory through a charging scheme based on the resolution of questions. The "only if" part of the Kraft's inequality has been generalized by Ducan [28] to heterogeneous questionnaires. Using the charging scheme based on $\log_2 d$ for a question of resolution d , an extended noiseless coding theorem has been proved by Ducan [28] for heterogeneous questionnaires, which states that average charge is bounded below by Shannon's [98] entropy.

Sharma and Garg [100] have introduced the charge of order t by using the additivity condition of measure of average charge and have shown that charge of order t is bounded below by Renyi's [95] entropy of order α . However Shannon entropy and Renyi entropy are both additive for independent distributions.

Measure of entropy which are non-additive have been studied by Havrda and Charvat [39] and later by Darcozy [27]. Some idea of the role of the non-additivity property of measures may be hold from examples that can be taken from biological and social sciences. Consider two drugs D_1 and D_2 that may cure at different times the ailments A_1 and A_2 from which a person suffers. But the drugs D_1 and D_2 together may not cure a person suffering from both the ailments A_1 and A_2. There can several examples of this type drawn from social systems. Therefore we need to consider non-additive models.

With a system we can associate a quantitative measure to bring out some of its characteristics. The measure that we associate may be additive, non-additive or both. If M is a non-additive measure, then the non-additive model should consider the difference

$$M(PQ) = M(P) - M(Q),$$

where P and Q are two independent system. The index 'λ' defined as

$$\lambda = \frac{M(PQ) - M(P) - M(Q)}{M(P)M(Q)},$$

may be taken to characterize the non-additive model. This index λ possesses two elegant properties, viz. symmetry in P and Q and reducing to zero for an additive model. This model has been considered by Harvda and Charvat [39] and others referred to above. There are other models but we confine our attention to this model.

By considering entropy to be mean value of non-additive self information, Sharma and Mittal [101] have obtained two measures of entropy which are non-additive and include Renyi's [95] entropy of order α as well as Havrda-Charvat , Darcozy entropy of type β as particular cases.

6.2 Notations and Definitions

Let there be a finite state space $\Theta = (\theta_1, \theta_2, ..., \theta_m)$ with probability distribution $P = (p_1, p_2, ..., p_m)$ such that the probability θ_i being the true state is denoted by p_i, $(i = 1, 2, ..., m)$ and

$$\sum_{i=1}^{n} p_i = 1 ; \qquad p_i \geq 0 \qquad (i = 1, 2, ..., m) .$$

Let Q denote a questionnaire defined on Θ and n_{id} represent the number of questions of resolution d required to determine the state θ_i. Now, if this questionnaire is heterogeneous, valid and uses n_{id} questions of resolution d $(d = 1, 2, ..., m)$, then (Ducan [28])

$$\sum_{i=1}^{m} \prod_{d=1}^{\infty} d^{-n_{id}} \leq 1,$$

is satisfied. If $C(Q)$ is the random charge when $\log_2 d$ is the charge for each question of resolution d, then the expected charge for Q is given by

$$E_p C(Q) = \sum_{i=1}^{m} \sum_{d=1}^{\infty} p_i n_{id} \log_2 d .$$

More generally, the random charge may be a function of $C(Q)$. If we take a continuous, strictly increasing function, viz., $\phi : [1, \infty[\to R$, the random charge for Q may be given by

$$C_\phi(Q) = \phi\left(\log_2 \prod_{d=1}^{\infty} d^{n_{id}} \right).$$

Consequently, the generalized average charge for Q may be taken as

$$E_p^\phi C_\phi(Q) = \phi^{-1}\left[\sum_{i=1}^{m} p_i \phi\left(\sum_{d=1}^{\infty} n_{id} \log_2 d \right) \right]. \qquad (6.2.\ 1)$$

It is interesting to see that for arbitrary P, Eq. (6.2.1) reduces to $E_p C(Q)$ in only two situations. The first arises when $n_{1d} = n_{2d} = ... = n_{md} = n_d$ (say), then

$$E_p^\phi C_\phi(Q) = \sum_{d=1}^{\infty} n_d \log_2 d = E_p C(Q).$$

Secondly, if we take ϕ to be a linear function, i.e.

$$\phi(x) = \phi_0(x) = ax + b; \qquad a \neq 0, \quad x \in [1, \infty[$$

Then, also

$$E_p{}^{\phi}C(Q)=\sum_{i=1}^{m}\sum_{d=1}^{\infty}p_i n_{id}\log_2 d=E_pC(Q).$$

Now consider two independent state space $\Theta=(\theta_1,\theta_2,...,\theta_J)$ and $\Theta^*=(\theta_1^*,\theta_2^*,...,\theta_k^*)$ with associated probability distributions $P=(p_1,p_2,...,p_j)$ and $U=(u_1,u_2,...,u_k)$ Such that

$$p_j\geq 0,\quad \sum_{j=1}^{J}p_j=1\ ,\ (j=1,\ 2,...,J)$$

and $u_k\geq 0$, $\sum_{k=1}^{K}p_k=1$, $(k=1,\ 2,...,K)$. Since Θ and Θ^* are independent the probability of the pair (θ_j,θ_k^*) is $p_ju_k(j=1,2,...,J;k=1,2,...,K)$.

Let us denote by PU the probability distribution on $\Theta\times\Theta^*\{p_1u_1,p_1u_2,...,p_1u_K,p_2u_1,p_2u_2,...,p_2u_K,...,p_Ju_1,p_Ju_2,...,p_Ju_K\}$ and let the valid heterogeneous questionnaire Q_1 and Q_2 exist on Θ and Θ^*, which use precisely $m_{jd}(j=1,2,...,J)$ and $n_{kd}(k=1,2,...,K)$ questions of resolution d respectively to determine θ_j and θ_k^*. A questionnaire say Q, may now be developed from the above two questionnaire on Θ and Θ^* in which $m_{jd}+n_{kd}$ $(j=1,2,...,J;\ k=1,2,...,K)$ questions of resolution d are required to determine the pair (θ_j,θ_k^*).

Now, because a questionnaire for (θ_j,θ_k^*) exists with $m_{jd}+n_{kd}$ questions of resolution d $(d=1,\ 2,...)$, we have the inequality

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\prod_{d=1}^{\infty}d^{-(m_{jd}+n_{kd})}\leq 1.$$

Further, if $E_{PU}^{\phi}C_{\phi}(Q)$ is a measure of average charge for Q, then we can have the following two situations. Firstly, it can be the sum of the average charges for Q_1 and Q_2 separately i.e.

$$E_{PU}^{\phi}C_{\phi}(Q)=E_P^{\phi}C_{\phi}(Q_1)+E_U^{\phi}C_{\phi}(Q_2)\,,\tag{6.2.2}$$

using this additivity condition Sharma and Garg [107] have characterized the average charge

$$E_p C(Q) = \sum_{i=1}^{m} \sum_{d=1}^{\infty} p_i n_{id} \log_2 d \tag{6.2.3}$$

and the charge of order t.

$$E_p{}^t C(Q) = \frac{1}{t} \log_2 \left(\sum_{i=1}^{m} p_i \prod_{d=1}^{\infty} d^{tn_{id}} \right); \qquad 0 < t < \infty . \tag{6.2.4}$$

Secondly, $E_{PU}^{\phi} C_{\phi}(Q)$ can be λ-non-additive measure of average charge, satisfying the relation,

$$E_{PU}^{\phi} C_{\phi}(Q) = E_P^{\phi} C_{\phi}(Q_1) + E_U^{\phi} C_{\phi}(Q_2) + \lambda E_P^{\phi} C_{\phi}(Q_1) E_U^{\phi} C_{\phi}(Q_2) \tag{6.2.5}$$

and the mean value property

$$E_p^{\phi} C_{\phi}(Q) = \phi^{-1} \left(\sum_{j=1}^{J} p_j \phi \left(c \left(\sum_{d=1}^{\infty} n_{jd} \log_2 d \right) \right) \right), \tag{6.2.6}$$

where $c\left(\sum_{j=1}^{J} n_{jd} \log_2 d \right)$ is λ-non-additive function of random charge based on a $\log_2 d$ charging scheme.

It is worthwhile to note the significance of the function $c(.)$ which is taken up for study in the next section. It gives the character of the charge for a given j, the function ϕ playing a role only in finding the average.

Two new measures that so result includes average charge given by Eq. (6.2.3) and charge of order t given in Eq. (6.2.4) as limiting cases.

6.3 Characterization of Non-Additive Measures of Average Charge with One Parameter.

In this section two new measures of average charge for heterogeneous questionnaires are characterized by using the non-additivity condition of measure of average charge. Particular and limiting cases of these measures are also given, lower bounds on these measures being obtained.

We take the mean value non-additive measures of charge given by Eq. (6.2.6) to satisfy the Eq. (6.2.3) (c.f. Sharma and Mittal [101]). First of all we shall determine the non-additive charge function c of the charging scheme based on $\log_2 d$ which uses precisely n_{jd} questions of resolution d to determine the state θ_j satisfying the non-additivity relation

$$c\left(\sum_{d=1}^{\infty} n_d \log_2 d + \sum_{d=1}^{\infty} m_d \log_2 d\right) = c\left(\sum_{d=1}^{\infty} n_d \log_2 d\right) + c\left(\sum_{d=1}^{\infty} m_d \log_2 d\right)$$

$$+ \lambda c\left(\sum_{d=1}^{\infty} n_d \log_2 d\right) c\left(\sum_{d=1}^{\infty} m_d \log_2 d\right), \qquad \lambda \neq 0. \qquad (6.3.1)$$

If we introduce the following notation

$$\sum_{d=1}^{\infty} n_d \log_2 d = n., \qquad \sum_{d=1}^{\infty} m_d \log_2 d = m.$$

$$\sum_{d=1}^{\infty} n_{id} \log_2 d = n_{i.}, \qquad \sum_{d=1}^{\infty} m_{kd} \log_2 d = m_{k.},$$

Then Eq. (6.3.1) takes the form

$$c(n.+m.) = c(n.) + c(m.) + \lambda c(n.) l(m.).$$

Taking $f(n) = 1 + \lambda c(n.)$, we get

$$f(n.+m.) = f(n.) f(m.). \qquad (6.3.2)$$

The most general non-zero solution of Eq. (6.3.2), may be given by

$$f(n) = 2^{\left(\frac{1-\alpha}{\alpha}\right) n.},$$

where $\alpha > 0, \neq 1$ is arbitrary constant.

Thus

$$c(n.) = \frac{2^{\left(\frac{1-\alpha}{\alpha}\right) n.} - 1}{\lambda}, \quad \lambda \neq 0 \qquad (6.3.3)$$

Now making proper choice of the constant λ, let us take it to be

$$\lambda = (2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1), \ \alpha \neq 1 \ \text{ as } \lambda \neq 0.$$

For this value of λ, we can see that when it tends to zero, i.e. when $\alpha \to 1$, the function $c(n)$ reduces to additive one , and in general

$$c\{n) = \frac{2^{\left(\frac{1-\alpha}{\alpha}\right) n.} - 1}{2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1}, \quad \alpha \neq 0, \neq 1. \qquad (6.3.4)$$

Next we proceed to determine $E_p{}^{\phi}C_{\phi}(Q)$ by first evaluating the values of ϕ. We put the value of $c(n)$ from Eq. (6.3.4) in Eq. (6.2.6) and then use the Eq. (6.2.5) with $\lambda = (2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1),\ \alpha \neq 1$, to get

$$\phi^{-1}\left(\sum_{j=1}^{J}\sum_{k=1}^{K} p_j u_k \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)(n_j+m_k)}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)$$

$$=\phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)+\phi^{-1}\left(\sum_{k=1}^{K} u_k \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)m_k}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)$$

$$+\left(2^{\left(\frac{1-\alpha}{\alpha}\right)}-1\right)\phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)\phi^{-1}\left(\sum_{k=1}^{K} u_k \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)m_k}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right). \qquad (6.3.5)$$

Take

$U = \{u\},\ m_{kd} = m_d\ (k = 1,2,...,K), P = (p_1, p_2,..., p_J),\ p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1$ such that $p_j > 0,\ \sum_{j=1}^{J} p_j = 1$ in Eq. (6.3.5) to get

$$\phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)(n_j+m)}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right) = \phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)2^{\left(\frac{1-\alpha}{\alpha}\right)m}+\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)m}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}$$

or

$$\psi_{m_d}{}^{-1}\left(\sum_{j=1}^{J} p_j \psi_{m_d}\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_i}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right) = \phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right)\right)2^{\left(\frac{1-\alpha}{\alpha}\right)m}+\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)m}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1},$$

(6.3.6)

where

$$\psi_{m_d}\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right) = \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)(n_j+m)}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right) \qquad (6.3.7)$$

Now, refer Hardy Littlewood and Polya [37], there must be a linear relation between ψ_{m_d} and ϕ, i.e.

$$\psi_{m_d}(n) = A(m)\phi(n) + B(m), \tag{6.3.8}$$

where $A(m)$ and $B(m)$ are independent of n.

Using Eq. (6.3.7) and Eq. (6.3.8), we have

$$g(n+m) = A(m)g(n) + B(m), \tag{6.3.9}$$

where

$$g\{n) = \phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n} - 1}{2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1}\right) \tag{6.3.10}$$

or

$$G(n+m) = A(m)G(n) + G(m), \tag{6.3.11}$$

where

$$G(n) = g(n) - a \tag{6.3.12}$$

and a is a constant.

From the symmetry of Eq. (6.3.11), we get

$$A(m)G(n) + G(m) = A(n)G(m) + G(n) \tag{6.3.13}$$

There are two cases, viz., $A(n) \equiv 1$ and $A(n) \neq 1$.

Case I. $A(n) \equiv 1$, the resulting form of Eq. (6.3.13) has the most general continuous solution

$$G(n) = bn, \tag{6.3.14}$$

where b is an arbitrary constant.

Using Eq. (6.3.12) and Eq. (6.3.10) in Eq. (6.3.14) we have

$$\phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n} - 1}{2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1}\right) = a + bn,$$

which implies

$$\phi(n_d \log_2 d) = a + \frac{b\alpha}{1-\alpha}\log_2\left(1 + (2^{\frac{1-\alpha}{\alpha}} - 1)\sum_{d=1}^{\infty} n_d \log_2 d\right) \tag{6.3.15}$$

Case II when $A(n) \neq 1$, by Hardy et. al [37]. The resulting equation has most general continuous solutions

$$A(n) = 0 \text{ (which we neglect) for all } n,$$

and

$$A(n) = 2^{tn}, \tag{6.3.16}$$

where t is an arbitrary nonzero constant.

Using Eq. (6.3.16) and Eq. (6.3.12), Eq. (6.3.10) gives

$$\phi\left(\frac{2^{\left(\frac{1-\alpha}{\alpha}\right)n}-1}{2^{\left(\frac{1-\alpha}{\alpha}\right)}-1}\right) = a + \frac{2^{tn}-1}{K}$$

which gives

$$\phi\left(\sum_{d=1}^{\infty} n_d \log_2 d\right) = a + \frac{\{(2^{\frac{1-\alpha}{\alpha}}-1)\sum_{d=1}^{\infty} n_d \log_2 d + 1\}^{t\alpha/(1-\alpha)} - 1}{K}. \qquad \alpha \neq 1,\ t \neq 0 \tag{6.3.17}$$

The values of ϕ given by Eq. (6.3.15) and Eq. (6.3.17) determine the following two non-additive measures of charge defined by

$$E_P^{(1,\alpha)}C(Q) = (2^{\frac{1-\alpha}{\alpha}}-1)^{-1}\left[2^{\frac{(1-\alpha)}{\alpha}\sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd}\log_2 d} - 1\right], \quad \alpha \neq 0, \alpha \neq 1, \tag{6.3.18}$$

$$E_P^{(t,\alpha)}C(Q) = (2^{\frac{1-\alpha}{\alpha}}-1)^{-1}\left[2^{\frac{(1-\alpha)}{\alpha t}\log_2 \sum_{j=1}^{J} p_j \prod_{d=1}^{\infty} d^{tn_{jd}}} - 1\right], \quad \alpha, t \neq 0, \alpha \neq 1. \tag{6.3.19}$$

These charges for charging scheme based on $\log_2 d$ denoted by $E_P^{(1,\alpha)}C(Q)$ and $E_P^{(t,\alpha)}C(Q)$ may be named as non-additive type α charges of order 1 and t respectively.

These results are contained in the following theorem.

Theorem 6.3.1 The average charges given by Eq. (6.2.4) of a questionnaire Q (which uses precisely n_{jd} questions of resolution d to determine the jth state) defined on the state space $\Theta = (\theta_1, \theta_2, ..., \theta_J)$ with probability distribution

$P=(p_1,p_2,...,p_J),\ p_j\geq 0, \sum_{J=1}^{J} p_j=1$ satisfying $\sum_{j=1}^{J}\prod_{d=1}^{\infty} d^{-n_{jd}}\leq 1$ and non-additivity relation can be only of one of the two forms given in Eq. (6.3.18) and Eq. (6.3.19).

6.3.1 Limiting and Particular Cases

It is immediate to see that

(1) $\lim_{t\to 0} E_P^{(t,\alpha)}C(Q)=E_P^{(1,\alpha)}C(Q)$. (6.3.20)

(2) $\lim_{\alpha\to 1} E_P^{(1,\alpha)}C(Q)=\sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd}\log_2 d = E_P C(Q)$. (6.3.21)

The ordinary average charge due to Ducan [28].

(3) $\lim_{\alpha\to 1} E_P^{(t,\alpha)}C(Q)=\frac{1}{t}\log_2\left(\sum_{j=1}^{J} p_j\prod_{d=1}^{\infty} d^{tn_{jd}}\right)=E_P{}^{t}C(Q)$. (6.3.22)

The charge of order t defined by Sharma and Garg [100].

(4) For $n_{1d}=n_{2d}=...=n_{nd}=n_d$ (say), both the expressions for charge given by Eq. (6.3.20) and Eq. (6.3.21) reduce to

$$\left(2^{\left(\frac{1-\alpha}{\alpha}\right)}-1\right)^{-1}\left[2^{\left(\frac{1-\alpha}{\alpha}\right)\sum_{d=1}^{\infty} n_d\log_2 d}-1\right],$$

which in the limiting case when α approaches unity, reduces to $\sum_{d=1}^{\infty} n_d\log_2 d$.

Thus we have shown that $E_P^{(1,\alpha)}C(Q)$ and $E_P^{(t,\alpha)}C(Q)$ are type α generalizations

$$\text{of } E_P C(Q)=\sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd}\log_2 d$$

and

$$E_P{}^{t}C(Q)\frac{1}{t}\log_2\left(\sum_{j=1}^{J} p_j\prod_{d=1}^{\infty} d^{tn_{jd}}\right)$$

respectively. We now prove the following:

Theorem 6.3.2 Let $\Theta=(\theta_1,\theta_2,...,\theta_m)$ be a finite state space and $P=(p_1,p_2,...,p_m)$ be a probability vector. If Q is a valid heterogeneous questionnaire and $C(Q)$ is the random charge based on $\log_2 d$ for each question of resolution d, then

(i) $E_P^{(1,\alpha)}C(Q) \geq H^*(P;1,\alpha)$, (6.3.23)

with equality iff $n_{id} = 0$ for all $d > m$. and $p_i = \prod_{d=2}^{m} d^{-n_{id}}$,

where

$$H^*(P;1,\alpha) = \frac{2^{(\frac{\alpha-1}{\alpha})\sum_{i=1}^{m} p_i \log_2 p_i} - 1}{2^{\frac{1-\alpha}{\alpha}} - 1}, \qquad \alpha \neq 0,\ \alpha \neq 1.$$

(ii) $E_P^{(t,\alpha)}C(Q) \geq H^*(P;1,\alpha)$, (6.3.24)

with equality iff

$$n_{id} = 0 \text{ for all } d > m$$

and

$$\frac{p_i^{\alpha}}{\sum_{j=1}^{m} p_j^{\alpha}} = \prod_{d=2}^{m} d^{-n_{id}} \qquad (i = 1, 2, ..., m),$$

where

$$\alpha = (1+t)^{-1}$$

and

$$H^*(P;\alpha) = \frac{\left(\sum_{i=1}^{m} p_i^{\alpha}\right)^{\frac{1}{\alpha}} - 1}{2^{\frac{1-\alpha}{\alpha}} - 1}, \qquad \alpha \neq 1,\ \alpha > 0.$$

Proof . (i) Ducan [28] has shown that

$$\sum_{i=1}^{m}\sum_{d=1}^{\infty} p_i n_{id} \log_2 d \geq -\sum_{i=1}^{m} p_i \log_2 p_i, \tag{6.3.25}$$

with equality iff

$$n_{id} = 0 \text{ for all } d > m \text{ and } p_i = \prod_{d=2}^{m} d^{-n_{id}}; \quad i = 1, 2, ..., m.$$

Now $(2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1) > 0$ or <0 according as $\alpha < 1$ or >1 respectively.

Therefore from the above after suitable manipulation, we get the inequality

$$\frac{2^{\frac{1-\alpha}{\alpha}\sum_{i=1}^{m}\sum_{d=1}^{\infty} p_i n_{id} \log_2 d}-1}{2^{\frac{1-\alpha}{\alpha}}-1} \geq \frac{2^{\frac{\alpha-1}{\alpha}\sum_{i=1}^{m} p_i \log_2 p_i}-1}{2^{\frac{1-\alpha}{\alpha}}-1}, \quad \alpha \neq 0, \neq 1$$

which is the Eq. (6.3.23), having the same equality conditions as for Eq. (6.3.25).

(ii) We now prove part (ii) .

If $t = 0$ and $\alpha = 1$, the result is the same proved in part (i). For other values, we use Holder's inequality

$$\left(\sum_{i=1}^{m} x_i^p\right)^{1/p} \cdot \left(\sum_{i=1}^{m} y_i^q\right) \leq \sum_{i=1}^{m} x_i y_i, \tag{6.3.26}$$

where $p^{-1} + q^{-1} = 1$ and $p < 1$.

Making the substitutions $p = -t,\ \ q = (1-\alpha),\ x_i = p_i^{-1/t} \prod_{d=1}^{\infty} d^{-n_{id}}$ and $y_i = p_i^{1/t}$ in

and raising the power to $(1-\alpha)/\alpha$ of both sides and using that $(2^{\left(\frac{1-\alpha}{\alpha}\right)} - 1) > 0$ or <0 according as $\alpha < 1$ or >1 we get Eq. (6.3.24) after simple manipulation. Hence the theorem is proved.

6.4 Characterization of Non-Additive Measures of Average Charge with Two Parameters.

The most general non-zero solution of Eq. (6.3.2), may also be given by

$$f(n) = 2^{\left(\frac{1-\beta}{\alpha}\right)n.},$$

where $\alpha, \beta \neq 1$ are arbitrary constants.

Thus

$$c(n.) = \frac{2^{\left(\frac{1-\beta}{\alpha}\right)n.} - 1}{\lambda}, \ \ \lambda \neq 0 \ . \tag{6.4.1}$$

Now making proper choice of the constant λ, let us take it to be

$$\lambda = (2^{\left(\frac{1-\beta}{\alpha}\right)} - 1),\ \beta \neq 1 \ \text{ as } \lambda \neq 0$$

For this value of λ, we can see that when it tends to zero, i.e. when $\beta \to 1$, the function $c(n)$ reduces to additive one , and in general

$$c\{n) = \frac{2^{\left(\frac{1-\beta}{\alpha}\right)n.} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}, \quad \beta \neq 1 \ . \tag{6.4.2}$$

Next we proceed to determine $E_p{}^{\phi}C_{\phi}(Q)$ by first evaluating the values of ϕ. We put the value of $c(n)$ from Eq. (6.4.2) in Eq. (6.2.6) and then use the Eq. (6.2.5) with $\lambda = (2^{\left(\frac{1-\beta}{\alpha}\right)} - 1),\ \beta \neq 1,$ to get

$$\phi^{-1}\left(\sum_{j=1}^{J}\sum_{k=1}^{K} p_j u_k \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)(n_j+m_k)} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right)$$

$$= \phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_j} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right) + \phi^{-1}\left(\sum_{k=1}^{K} u_k \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)m_k} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right)$$

$$+ \left(2^{\left(\frac{1-\beta}{\alpha}\right)} - 1\right)\phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_j} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right)\phi^{-1}\left(\sum_{k=1}^{K} u_k \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)m_k} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right). \tag{6.4.3}$$

Take

$U = \{u\},\ m_{kd} = m_d\ (k = 1, 2, ..., K), P = (p_1, p_2, ..., p_J),\ p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1$ such that $p_j > 0,\ \sum_{j=1}^{J} p_j = 1$ in Eq. (6.4.3) to get

$$\phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)(n_j+m)} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right) = \phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_j} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}\right)\right)2^{\left(\frac{1-\beta}{\alpha}\right)m} + \frac{2^{\left(\frac{1-\beta}{\alpha}\right)m} - 1}{2^{\left(\frac{1-\beta}{\alpha}\right)} - 1}$$

or

$$\psi_{m_d}^{-1}\left(\sum_{j=1}^{J} p_j \psi_{m_d}\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_i}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right)\right) = \phi^{-1}\left(\sum_{j=1}^{J} p_j \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right)\right) 2^{\left(\frac{1-\beta}{\alpha}\right)m} + \frac{2^{\left(\frac{1-\beta}{\alpha}\right)m}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}, \tag{6.4.4}$$

where

$$\psi_{m_d}\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n_j}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right) = \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)(n_j+m)}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right). \tag{6.4.5}$$

Now, refer Hardy Littlewood and Polya [37], there must be a linear relation between ψ_{m_d} and ϕ, i.e.

$$\psi_{m_d}(n) = A(m)\phi(n) + B(m), \tag{6.4.6}$$

where $A(m)$ and $B(m)$ are independent of n.

Using Eq. (6.4.5) and Eq. (6.4.6), we have

$$g(n+m) = A(m)g(n) + B(m), \tag{6.4.7}$$

where

$$g\{n) = \phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right) \tag{6.4.8}$$

or

$$G(n+m) = A(m)G(n) + G(m), \tag{6.4.9}$$

where

$$G(n) = g(n) - a \tag{6.4.10}$$

and a is a constant.

From the symmetry of Eq. (6.4.9), we get

$$A(m)G(n) + G(m) = A(n)G(m) + G(n). \tag{6.4.11}$$

There are two cases, viz., $A(n) \equiv 1$ and $A(n) \neq 1$.

Case I. $A(n) \equiv 1$, the resulting form of Eq. (6.4.11) has the most general continuous solution

$$G(n) = bn, \tag{6.4.12}$$

where b is an arbitrary constant.

Using Eq. (6.4.10) and Eq. (6.4.8), Eq. (6.4.12) gives

$$\phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right)=a+bn\,,$$

which implies

$$\phi(n_d \log_2 d) = a + \frac{b\alpha}{1-\beta}\log_2\left(1+(2^{\frac{1-\beta}{\alpha}}-1)\sum_{d=1}^{\infty} n_d \log_2 d\right). \quad (6.4.13)$$

Case II when $A(n) \neq 1$, by Hardy et. al [37] the resulting equation has most general continuous solutions

$A(n) = 0$ (which we neglect) for all n,

and $$A(n) = 2^{tn}\,, \quad (6.4.14)$$

where t is an arbitrary nonzero constant.

using Eq. (6.4.14) and Eq. (6.4.10) , Eq. (6.4.8) gives

$$\phi\left(\frac{2^{\left(\frac{1-\beta}{\alpha}\right)n}-1}{2^{\left(\frac{1-\beta}{\alpha}\right)}-1}\right)=a+\frac{2^{tn}-1}{K}$$

which gives

$$\phi\left(\sum_{d=1}^{\infty} n_d \log_2 d\right) = a + \frac{\{(2^{\frac{1-\beta}{\alpha}}-1)\sum_{d=1}^{\infty} n_d \log_2 d + 1\}^{t\alpha/(1-\beta)}-1}{K}\,. \quad (6.4.15)$$

$$\beta \neq 1,\ t \neq 0$$

The values of ϕ given by Eq. (6.4.13) and Eq. (6.4.15) determine the following two non-additive measures of charge defined by

$$E_P^{(1,\alpha,\beta)}C(Q) = (2^{\frac{1-\alpha}{\beta}}-1)^{-1}\left[2^{\frac{(1-\beta)}{\alpha}\sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd}\log_2 d}-1\right], \quad \alpha \neq 0, \beta \neq 1\,, \quad (6.4.16)$$

and

$$E_P^{(t,\alpha,\beta)}C(Q) = (2^{\frac{1-\beta}{\alpha}}-1)^{-1}\left[2^{\frac{(1-\beta)}{\alpha t}\log_2 \sum_{j=1}^{J} p_j \prod_{d=1}^{\infty} d^{tn_{jd}}}-1\right], \quad \alpha, t \neq 0, \beta \neq 1\,. \quad (6.4.17)$$

These charges for charging scheme based on $\log_2 d$ denoted by $E_P^{(1,\alpha,\beta)}C(Q)$ and $E_P^{(t,\alpha,\beta)}C(Q)$ may be named as non-additive type (α,β) charges of order 1 and t respectively.

These results are contained in the following theorem:

Theorem 6.4.1. The average charges given by Eq. (6.2.4) of a questionnaire Q (which uses precisely n_{jd} questions of resolution d to determine the jth state) defined on the state space $\Theta = (\theta_1, \theta_2, ..., \theta_J)$ with probability distribution $P = (p_1, p_2, ..., p_J),\ p_j \geq 0, \sum_{J=1}^{J} p_j = 1$ satisfying $\sum_{j=1}^{J}\prod_{d=1}^{\infty} d^{-n_{jd}} \leq 1$ and non-additivity relation can be only of one of the two forms given in Eq. (6.4.16) and Eq. (6.4.17).

6.4.1 Limiting and Particular Cases

It is immediate to see that

(1) $\lim_{t\to 0} E_P^{(t,\alpha,\beta)}C(Q) = E_P^{(1,\alpha,\beta)}C(Q)$. (6.4.18)

(2) $\lim_{\beta\to 1} E_P^{(1,\alpha,\beta)}C(Q) = \sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd} \log_2 d = E_P C(Q)$, (6.4.19)

which is the ordinary average charge due to Ducan [28].

(3) $\lim_{\beta\to 1} E_P^{(t,\alpha,\beta)}C(Q) = \frac{1}{t}\log_2\left(\sum_{j=1}^{J} p_j \prod_{d=1}^{\infty} d^{tn_{jd}}\right) = E_P{}^t C(Q)$, (6.4.20)

which is the charge of order t defined by Sharma and Garg [100].

(4) For $n_{1d} = n_{2d} = ... = n_{nd} = n_d$ (say), both the expressions for charge given by Eq. (6.4.18) and Eq. (6.4.19) reduce to

$$\left(2^{\left(\frac{1-\beta}{\alpha}\right)} - 1\right)^{-1}\left[2^{\left(\frac{1-\beta}{\alpha}\right)\sum_{d=1}^{\infty} n_d \log_2 d} - 1\right],$$

which in the limiting case when β approaches unity, reduces to $\sum_{d=1}^{\infty} n_d \log_2 d$.

Thus we have shown that $E_P^{(1,\alpha,\beta)}C(Q)$ and $E_P^{(t,\alpha,\beta)}C(Q)$ are type β generalizations of

$$E_P C(Q) = \sum_{j=1}^{J}\sum_{d=1}^{\infty} p_j n_{jd} \log_2 d \quad \text{and} \quad E_P{}^t C(Q) \frac{1}{t}\log_2\left(\sum_{j=1}^{J} p_j \prod_{d=1}^{\infty} d^{tn_{jd}}\right)$$

respectively.

We now prove the following:

Theorem 6.4.2. Let $\Theta = (\theta_1, \theta_2, ..., \theta_m)$ be a finite state space and $P = (p_1, p_2, ..., p_m)$ be a probability vector. If Q is a valid heterogeneous questionnaire and $C(Q)$ is the random charge based on $\log_2 d$ for each question of resolution d, then

(i) $$E_P^{(1,\alpha,\beta)} C(Q) \geq H^*(P;\alpha,\beta) \;, \tag{6.4.21}$$

with equality iff $n_{id} = 0$ for all $d > m$. and $p_i = \prod_{d=2}^{m} d^{-n_{id}}$,

where

$$H^*(P;\alpha,\beta) = \frac{2^{(\frac{\beta-1}{\alpha})\sum_{i=1}^{m} p_i \log_2 p_i} - 1}{2^{\frac{1-\beta}{\alpha}} - 1}, \qquad \alpha \neq 0,\ \beta \neq 1.$$

(ii) $$E_P^{(t,\beta)} C(Q) \geq H^*(P;\alpha,\beta), \tag{6.4.22}$$

with equality iff

$$n_{id} = 0 \text{ for all } d > m \quad \text{and} \quad \frac{p_i^{\alpha}}{\sum_{j=1}^{m} p_j^{\alpha}} = \prod_{d=2}^{m} d^{-n_{id}} \quad (i = 1, 2, ..., m),$$

where

$$\alpha = (1+t)^{-1}$$

and

$$H^*(P;\alpha,\beta) = \frac{\left(\sum_{i=1}^{m} p_i^{\alpha}\right)^{(\frac{\beta-1}{\alpha-1})} - 1}{2^{\frac{1-\beta}{\alpha}} - 1}, \qquad \alpha \neq 1,\ \beta \neq 1, \alpha > 0, \beta > 0.$$

Proof . (i) Ducan [28] has shown that

$$\sum_{i=1}^{m}\sum_{d=1}^{\infty} p_i n_{id} \log_2 d \geq -\sum_{i=1}^{m} p_i \log_2 p_i, \tag{6.4.23}$$

with equality iff

$$n_{id} = 0 \text{ for all } d > m \text{ and } p_i = \prod_{d=2}^{m} d^{-n_{id}}; \quad i = 1,2,...,m$$

Now, $(2^{\left(\frac{1-\beta}{\alpha}\right)} - 1) > 0$ or <0 according as $\beta < 1$ or >1 respectively.

Therefore from the above after suitable manipulation, we get the inequality

$$\frac{2^{\frac{1-\beta}{\alpha}\sum_{i=1}^{m}\sum_{d=1}^{\infty} p_i n_{id} \log_2 d} - 1}{2^{\frac{1-\beta}{\alpha}} - 1} \geq \frac{2^{\frac{\beta-1}{\alpha}\sum_{i=1}^{m} p_i \log_2 p_i} - 1}{2^{\frac{1-\beta}{\alpha}} - 1}, \quad \beta \neq 1$$

which is the Eq. (6.4.21), having the same equality conditions as for Eq. (6.4.23).

(ii) We now prove part (ii).

If $t = 0$ and $\alpha = 1$, the result is the same proved in part (i). For other values, we use Holder's inequality

$$\left(\sum_{i=1}^{m} x_i^p\right)^{1/p} \cdot \left(\sum_{i=1}^{m} y_i^q\right) \leq \sum_{i=1}^{m} x_i y_i, \tag{6.4.24}$$

where $p^{-1} + q^{-1} = 1$ and $p < 1$.

Making the substitutions $p = -t,\ q = (1-\alpha),\ x_i = p_i^{-1/t} \prod_{d=1}^{\infty} d^{-n_{id}}$ and $y_i = p_i^{1/t}$ in Eq. (6.4.24) and raising the power to $(1-\beta)/\alpha$ of both sides and using that $(2^{\left(\frac{1-\beta}{\alpha}\right)} - 1) > 0$ or <0 according as $\beta < 1$ or >1 we get Eq. (6.4.22) after simple manipulation.

Hence the theorem is proved.

6.5 Conclusions

In this chapter, by considering the average charge for heterogeneous questionnaires to be non-additive, we have introduced new measures of average charge which includes the average charge defined by Ducan [28] and the charge of order t and discussed as limiting and particular cases. Lower bounds on these new measures of charge have also been obtained.