Chapter 6

On a Weighted Retro Banach Frames
For Discrete Signal Spaces
CHAPTER-6

ON A WEIGHTED RETRO BANACH FRAMES FOR DISCRETE SIGNAL SPACES

6.1 Introduction

In this chapter, we introduce a weighted retro Banach frame for discrete signal space. Necessary and sufficient condition for the existence of weighted retro Banach frames is obtained. Construction of weighted retro Banach frames from bounded linear operator is discussed. A Paley-Wiener type perturbation result for weighted retro Banach frame in Banach spaces setting is given.

In fact, in Section 6.2, we discuss about significance of retro Banach frame for discrete signal space. In Section 6.3, we define weighted retro Banach frame for discrete signal space and give an example. In Section 6.4, an application of weighted retro Banach frame discussed through example. In Section 6.5, we give necessary and sufficient condition for a weighted retro Banach frame to be an exact. In particular, we give necessary and sufficient condition for a sequence to be a weighted retro Banach frame. It is also given necessary and sufficient condition of a weighted retro Banach frame for discrete signal space. In Section 6.6, construction of weighted retro Banach frame for discrete signal space from bounded linear operator is discussed. Give necessary and sufficient condition for construction of weighted retro Banach for discrete signal space. Also we give better Bessel bound for the sum of two weighted retro Banach frames. Finally, we give a Lemma, if an analysis operator associated with weighted Bessel sequence is coercive, then respective weighted retro Bessel sequence constitute a weighted retro Banach frame for discrete signal space. In Section 6.7, relationship between weighted retro Bessel bounds and weighted retro Banach frame bounds is discussed. We give interesting result on relation between weighted retro Bessel bounds and weighted retro Banach frame bounds such that after overlapping the resultant system constitute a weighted Banach frame for discrete signal space. In Section 6.8, we give a Paley-Wiener type
perturbation for weighted retro Banach frames.

6.2 Significance of Retro Banach frame for Discrete Signal Space

If a signal is transmitted in the space, then there is an error which is hidden in the coefficients given by the reconstruction system. One of such reconstruction system is a frame which can reconstruction the underlying space by mean of an infinite series (or by an operator). Consider a discrete signal space \( \mathbb{H} = L^2(\Omega) \), where \( \Omega = \mathbb{N} \) with counting measure. Let \( \{ \Phi_n \} \subset \mathbb{H} \) be a Hilbert frame for \( \mathbb{H} \). Let \( \Phi \) be a non zero signal in \( \mathbb{H} \). Then, \( \Phi \) can be recovered by reconstruction formula (0.2) which is given in Chapter-0 as

\[
\Phi = SS^{-1}\Phi = \sum_{n=1}^{\infty} \langle S^{-1}\Phi, \Phi_n \rangle \Phi_n
\]

where \( S \) is the frame operator which is positive continuous invertible operator from \( \mathbb{H} \) onto \( \mathbb{H} \). If we transmit the signal \( \Phi \) into space, then it is in the form of frame coefficients \( \{ \langle \Phi, \Phi_n \rangle \} \). An error is always expected, let it be \( \varepsilon = \{ \xi_n \} \). Therefore, the signal received by the receiver is \( \{ \langle \Phi, \Phi_n \rangle + \xi_n \} \). So, the original signal, in general, cannot be written a linear combination of \( \{ \Phi_n \} \) over \( \{ \langle \Phi, \Phi_n \rangle + \xi_n \} \) over \( \mathbb{H} \) over \( \mathbb{H} \). On the other hand a retro Banach frame reconstructs the signal by a retro pre-frame operator.

Let \( \{ e_n \} \) be an orthonormal basis for the discrete signal space \( \mathbb{H} \). Choose \( \Phi_n = e_n \), for all \( n \in \mathbb{N} \). Then, \( \{ \Phi_n \} \) is an exact Hilbert frame for \( \mathbb{H} \). Let \( S \) be the frame operator for \( \{ \Phi_n \} \). Fix \( 0 \neq \Phi \in \mathbb{H} \).

Consider the error \( \varepsilon = \{ \langle \Phi, -\Phi_1 + \Phi_{n+1} \rangle \} \). This makes sense, because errors are not constant. Then, not all elements of \( \mathbb{H} \) can be reconstructed by \( \{ \Phi_n \} \) over \( \{ \langle \Phi, \Phi_n \rangle + \xi_n \} \). In particular, it is easy to check that \( \Phi_1 \) cannot be written as a
linear combination of \( \{ \Phi_n \} \) over \( \{ \langle \Phi, \Phi_n \rangle + \xi_n \} \). The family of vectors corresponding to the information given \( \{ \langle \Phi, \Phi_n \rangle + \xi_n \} \) (of the disturbed signal) is \( \{ \Psi_n \equiv \Phi_n + (\Phi_1 + \Phi_{n+1}) \} \). Note that there exists no reconstruction operator \( \Theta_0 \) such that \( \mathcal{F}_0 \equiv (\{ \Psi_n \}, \Theta_0) \) a retro Banach frame for \( \mathbb{H}^+ \).

Indeed, if \( a_0 \) and \( b_0 \) be retro frame bounds for \( \mathcal{F}_0 \), then
\[
0 < a_0 \left\| \Phi^* \right\|_d \leq \left\| \Phi^* (\Psi_n) \right\|_{\mathcal{F}_d} \leq b_0 \left\| \Phi^* \right\|_d, \quad \text{for each } \Phi^* \in \mathbb{H}^+.
\]
(6.1)

If \( \Phi_0^* \in \mathbb{H}^+ \) is given by
\[
\Phi_0^* (\{ \beta_j \}) = \left\{ \begin{array}{cc}
\beta_1, & j = 1 \\
\beta_{j+1} - \beta_j, & j > 1, \quad j \in \mathbb{N}, \forall \{ \beta_j \} \in \mathbb{H},
\end{array} \right.
\]
then \( \Phi_0^* \) is a non zero functional in \( \mathbb{H}^+ \). Put \( \Phi^* = \Phi_0^* \) in retro frame inequality (6.1), we obtain \( \Phi_0^* = 0 \). This is a contradiction.

By giving suitable weight to the vectors obtained from the disturbed frame for the signal space \( \mathbb{H} \), we can make the reconstruction operator such that each vector (signal) of \( \mathbb{H} \) can be reconstructed by the said reconstruction operator (see Example 6.4.1.).

### 6.3 A Weighted Retro Banach Frames.

In this section, we define weighted retro Banach frame for discrete signal space and existence of weighted retro Banach frames give through an example. we give the following definitions.

**Definition 6.3.1.** Let \( \omega = \{ \omega_n \} \) be a weight. A pair \( \mathcal{F} \equiv \{ \{ \omega_n \Phi_n \}, \Theta \} \) \( \{ \Phi_n \} \subset \mathbb{H}, \Theta: E_d \rightarrow \mathbb{H}^+ \) is called a **weighted retro Banach frame** (or **\( \omega \)-retro Banach frame**) for \( \mathbb{H}^+ \) with respect to \( E_d \) if

(i) \( \{ \Phi^* (\omega_n \Phi_n) \} \in E_d, \quad \text{for each } \Phi^* \in \mathbb{H}^+ \)

(ii) there exist positive constants \( a_0 \) and \( b_0 \ (0 < a_0 \leq b_0 < \infty) \) such that
\[ a_0 \|\Phi^*\|_{E_d^{\ast}} \leq \|\Phi^*(\omega_n \Phi_n)\|_{E_d} \leq b_0 \|\Phi^*\|_{\mathbb{H}^*}, \text{ for each } \Phi^* \in \mathbb{H}^* \quad (6.2) \]

(iii) \( \Theta \) is a bounded linear operator such that
\[ \Theta\{\Phi^*(\omega_n \Phi_n)\} = \Phi^*, \quad \text{for all } \Phi^* \in \mathbb{H}^*. \]

The positive constants \( a_0 \) and \( b_0 \), respectively are called lower and upper frame bounds for the weighted retro Banach frame \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \) and collectively known as weighted retro frame bounds (or simply bounds) for \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \).

The operator \( \Theta : E_d \to \mathbb{H}^* \) is called reconstruction operator (or pre-frame operator). The inequality (6.2) is called retro frame inequality for the weighted retro Banach frame \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \).

**Definition 6.3.2.** A weighted retro Banach frame \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \)
\[ \{\Phi_n\} \subset \mathbb{H}, \quad \Theta : E_d \to \mathbb{H}^* \]
for \( \mathbb{H}^* \) with respect to \( E_d \) and with bounds \( a_0, b_0 \) is said to be

(i). **Tight**, if it is possible to choose \( a_0 = b_0 \)

(ii). **Parseval**, if \( a_0 = b_0 = 1 \)

(iii). **Exact**, if there exists no reconstruction operator \( \Theta_0 \) such that
\[ \{\omega_n \Phi_n\}_{n \in \mathbb{N}}, \Theta_0 \quad (j \in \mathbb{N}) \]
a weighted retro Banach frame for \( \mathbb{H}^* \).

Regarding existence of weighted retro Banach frames. We have following example.

**Example 6.3.3.** Let \( E = \mathbb{H} \) and let \( \Phi_n = e_n \), where \( \{e_n\} \) is the canonical basis for \( \mathbb{H} \). Put \( \omega_n = \frac{1}{n^2}, n \in \mathbb{N} \). Then, \( \{\omega_n\} \) is a weight.

Put \( \mathcal{A}_d = \{\Phi^*(\omega_n \Phi_n)\} : \Phi^* \in \mathbb{H}^* \}. \) Then \( \mathcal{A}_d \) is a Banach space with norm given by
\[ \|\Phi^*(\omega_n \Phi_n)\|_{\mathcal{A}_d} = \|\Phi^*\|_{\mathbb{H}^*} \]
Therefore, \( \Theta : \{ \Phi^* (\omega_n \Phi_n) \} \to \Phi^* \) is a bounded linear operator from \( \mathcal{A}_d \) onto \( \mathbb{H}^* \) such that \( \mathcal{F} = \{ (\omega_n \Phi_n), \Theta \} \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{A}_d \).

### 6.4 Application of Weighted Retro Banach Frame

The following example gives an application of a weighted retro Banach frame for a discrete signal space.

**Example 6.4.1.** Consider a retro Banach frame \( \{ \Psi_n \}, \Theta \) for a discrete signal space \( \mathbb{H}^* \) given in Section 6.2. Let us give suitable weight to \( \mathcal{F}_0 = \{ \Psi_n, \Theta \} \), say

\[ \omega = \{ \omega_k \} = \{ 2, 1, 1, 1, \ldots \}. \]

Choose \( \mathcal{Z}_{d_0} = \{ \{ \Phi^* (\omega_n \Phi_n + (-\Phi_1 + \Phi_{n+1})) \} : \Phi^* \in \mathbb{H}^* \} \). Then \( \mathcal{Z}_{d_0} \) is a Banach space with the norm given by

\[ \left\| \{ \Phi^* (\omega_n \Phi_n + (-\Phi_1 + \Phi_{n+1})) \} \right\|_{\mathcal{Z}_{d_0}} = \| \Phi^* \|_{\mathbb{H}^*}. \]

Define \( \hat{\Theta}_0 : \mathcal{Z}_{d_0} \to \mathbb{H}^* \) by

\[ \hat{\Theta}_0 \left( \{ \Phi^* (\omega_n \Phi_n + (-\Phi_1 + \Phi_{n+1})) \} \right) = \Phi^*, \quad \Phi^* \in \mathbb{H}^*. \]

Then, \( \hat{\Theta}_0 \in \mathcal{B} \left( \mathcal{Z}_{d_0}, \mathbb{H}^* \right) \). Hence, \( \{ (\omega_n \Phi_n + (-\Phi_1 + \Phi_{n+1})) \}, \hat{\Theta}_0 \) is a weighted retro Banach frame for \( \mathbb{H}^* \).

Thus, by giving suitable weight to a given Hilbert frame and knowing the nature of vectors which appear in the error, we can find a reconstruction operator which can reconstruct each vector of the underlying signal space. This reflects the importance of weighted retro Banach frames.
6.5 Existence of Weighted Retro Banach Frame

In this section, we give the necessary and sufficient condition for a weighted retro Banach frame to be an exact. Also, we give necessary condition for a weighted retro Banach frame to be an exact. In particular, we give necessary and sufficient condition for a sequence to be a weighted retro Banach frame. It is also given necessary and sufficient condition for existence of weighted retro Banach frame in discrete signal space. The following proposition provides necessary and sufficient conditions for exactness of a weighted retro Banach frame.

**Proposition 6.5.1.** Let \( \omega = \{ \omega_n \} \) be a weight and let \( \mathcal{F} = \{ \{ \omega_n \Phi_n \} , \Theta \} \) be a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_j \). Then \( \mathcal{F} \) is an exact if and only if \( \omega_n \Phi_n \notin [\omega_i \Phi_i]_{x_n} \), for all \( n \in \mathbb{N} \).

**Proof.** Suppose that \( \{ \omega_n \Phi_n \} , \Theta \) is an exact retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_j \). Then, there exists no reconstruction operator \( \Theta_0 \) such that \( \{ \omega_i \Phi_i \} , \Theta_0 \) is a retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_{d_0} \), where \( j \in \mathbb{N} \) be an arbitrary but fixed and \( Z_{d_0} \) is some associated Banach space of scalar valued sequences indexed by \( \mathbb{N} \).

Claim: \( [\omega_i \Phi_i]_{x_n} \neq \mathbb{H}^* \), for all \( n \in \mathbb{N} \).

Let, if possible, \( [\omega_i \Phi_i]_{x_n} = \mathbb{H}^* \), for some \( n \in \mathbb{N} \).

Define a sequence \( \{ \psi_i \} \subset \mathbb{H}^* \) by

\[
\psi_i = \pi(\omega_i \Phi_i), \quad \forall i \in \mathbb{N}, \ i \neq n.
\]

(where \( \pi \) is the canonical isomorphism of \( \mathbb{H} \) into \( \mathbb{H}^* \)).

Then, \( \{ \psi_i \} \) is total over \( \mathbb{H}^* \).

Indeed, let \( \Phi^* \neq 0 \in \mathbb{H}^* \) such that

\[
\psi_i(\Phi^*) = 0, \quad \text{for all } i \in \mathbb{N}, \ i \neq n.
\]
Then
\[
(\pi(\omega_i \Phi_j))(\Phi^*) = 0, \quad \text{for all } i \in \mathbb{N}, \; i \neq n.
\]
\[
\Rightarrow \Phi^*(\omega_i \Phi_j) = 0, \quad \text{for all } i \in \mathbb{N}, \; i \neq n.
\]
Since \([\omega_i \Phi_j]_{i \neq n} = \mathbb{H}^\perp\), we have \(\Phi^* = 0\). We get a contradiction.

Then, \(\{\psi_i\}_{i \neq n}\) is total over \(\mathbb{H}^\perp\). Therefore, by Lemma 1.4.11, there exists an associated Banach space
\[
\mathcal{Z}_{d_i} = \left\{\left(\psi_j(\Phi^*)\right)_{j \neq i} : \Phi^* \in \mathbb{H}^\perp\right\} = \left\{\left(\Phi^*(\omega_j \Phi_i)\right)_{j \neq i} : \Phi^* \in \mathbb{H}^\perp\right\}
\]
of scalar valued sequences with the norm given by
\[
\left\|\left(\Phi^*(\omega_j \Phi_i)\right)\right\|_{\mathcal{Z}_{d_i}} = \left\|\Phi^*\right\|_{\mathbb{H}^\perp}, \quad \Phi^* \in \mathbb{H}^\perp.
\]
Define \(\Theta_i : \mathcal{Z}_{d_i} \rightarrow \mathbb{H}^\perp\) by
\[
\Theta_i\left(\left(\Phi^*(\omega_j \Phi_i)\right)\right) = \Phi^*, \quad \Phi^* \in \mathbb{H}^\perp.
\]
Then, \(\Theta_i\) is a bounded linear operator such that \(\{\omega_i \Phi_j\}_{i \neq n}, \Theta\) is a retro Banach frame \(\mathbb{H}^\perp\) with respect to \(\mathcal{Z}_{d_i}\). This is a contradiction.

Hence, we have \([\omega_i \Phi_j]_{i \neq n} \neq \mathbb{H}^\perp\).

Also, by retro frame inequality for a retro Banach space \(\{\omega_i \Phi_j\}, \Theta\), we have
\[
[\omega_i \Phi_j] = \mathbb{H}.
\]
Therefore, we have
\[
\omega_i \Phi_j \notin [\omega_i \Phi_j]_{i \neq n}, \quad \text{for all } n \in \mathbb{N}.
\]
Conversely, let \(\omega_i \Phi_j \notin [\omega_i \Phi_j]_{i \neq n}\), for all \(n \in \mathbb{N}\).

In order to show that \(\{\omega_i \Phi_j\}, \Theta\) is exact. Let, if possible, \(\{\omega_i \Phi_j\}, \Theta\) be not exact. Then, there exists a reconstruction \(\Theta_2 : \mathcal{Z}_{d_2} \rightarrow \mathbb{H}^\perp\), such that
\[
\{\omega_i \Phi_j\}_{i \neq k}, \Theta_2\] is a retro Banach frame for \(\mathbb{H}^\perp\) with respect to \(\mathcal{Z}_{d_2}\), where \(k\) is some positive integer and \(\mathcal{Z}_{d_2}\) is associated Banach space of scalar valued
sequences. Let \( A, B \) be a choice of bounds for \( \left\{ \omega_i \Phi_i \right\}_{i \neq k} \). Then
\[
A \left\| \Phi^* \right\|_{\mathbb{H}^*} \leq \left\| \left( \Phi^* \left( \omega_i \Phi_i \right) \right)_{i \neq k} \right\|_{\mathbb{H}^*} \leq B \left\| \Phi^* \right\|_{\mathbb{H}^*}, \quad \Phi^* \in \mathbb{H}^*.
\] (6.3)

If for some \( \Phi^* \neq 0 \in \mathbb{H}^* \),
\[
\Phi^* \left( \omega_i \Phi_i \right) = 0, \text{ for all } i \in \mathbb{N}, i \neq k.
\]
Then, by frame inequality (6.3), we get \( \Phi^* = 0 \).

Hence, \( [\omega_i \Phi_i]_{i \neq k} = \mathbb{H} \), which is a contradiction.

Therefore, \( \left( \{ \omega_i \Phi_i \}, \Theta \right) \) is an exact weighted retro Banach frame for \( \mathbb{H}^* \).

Now, we give the necessary condition for a weighted retro Banach frame to be an exact.

**Proposition 6.5.2.** If \( \mathcal{F} = \left( \{ \omega_n \Phi_n \}, \Theta \right) \) is an exact weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_j \). Then
\[
\lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i = 0 \Rightarrow \lim_{n \to \infty} \alpha_j^{(n)} = 0 \left( j \in \mathbb{N} \right).
\]

**Proof.** Since \( \mathcal{F} = \left( \{ \omega_n \Phi_n \}, \Theta \right) \) is an exact. Therefore, by Hahn Banach theorem, there exists \( \{ \Psi_n^* \} \in \mathbb{H}^* \) such that
\[
\Psi_n^* \left( \omega_m \Phi_m \right) = \delta_{nm}, \text{ for all } n, m \in \mathbb{N}.
\]

Also by retro frame inequality for a retro Banach frame \( \mathcal{F} = \left( \{ \omega_n \Phi_n \}, \Theta \right) \), we have
\[
[\omega_n \Phi_n] = \mathbb{H}^*.
\]

So
\[
\lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i \in \mathbb{H}^*.
\]

Suppose that
\[
\lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i = 0.
\]
Then,
\[
\Psi_j^* \left( \lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_j \right) = 0, \text{ for all } j \in \mathbb{N}.
\]

\[
\Rightarrow \left( \lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \Psi_j^* (\omega_i \Phi_j) \right) = 0, \text{ for all } j \in \mathbb{N}.
\]

\[
\Rightarrow \lim_{n \to \infty} \left[ \alpha_j^{(n)} \Psi_j^* (\omega_j \Phi_j) + \ldots + \alpha_j^{(n)} \Psi_j^* (\omega_j \Phi_j) + \ldots + \alpha_j^{(n)} \Psi_j^* (\omega_{m_j} \Phi_{m_j}) \right] = 0,
\]

\[\forall \ j \in \mathbb{N}.
\]

\[
\Rightarrow \lim_{n \to \infty} \alpha_j^{(n)} = 0, \text{ for all } j \in \mathbb{N}.
\]

**Remark 6.5.3.** If \( \mathcal{F} \equiv \{ \omega_n \Phi_n, \Theta \} \) and \( \{ \Psi_n^* \} \in \mathbb{H}^* \) be the same in the proposition.

6.5.2. Then in general, \( [\Psi_n^*] \neq \mathbb{H}^* \). If \( [\Psi_n^*] = \mathbb{H}^* \), then

\[
\lim_{n \to \infty} \alpha_j^{(n)} = 0, \text{ for all } j \in \mathbb{N} \Rightarrow \lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i = 0.
\]

Indeed, for a fixed \( j \in \mathbb{N} \), we have

\[
0 = \lim_{n \to \infty} \alpha_j^{(n)}
= \Psi_j^* \left( \lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i \right)
= \Psi_j^* (\Phi_0), \text{ where } \Phi_0 = \lim_{n \to \infty} \sum_{i=1}^{m} \alpha_i^{(n)} \omega_i \Phi_i.
\]

Thus, \( \Psi_j^* (\Phi_0) = 0 \), for all \( j \in \mathbb{N} \).

Therefore, by using the fact that \( [\Psi_n^*] = \mathbb{H}^* \), we obtain \( \Phi_0 = 0 \).

The following theorem gives necessary and sufficient condition for a given sequence \( \{ \Phi_n \} \) in \( \mathbb{H} \) to be a weighted retro Banach frame for discrete signal space.

**Theorem 6.5.4.** Let \( \{ \omega_n \} \) be a weight and let \( \{ \Phi_n \} \subset \mathbb{H} \). A system \( \mathcal{F} \equiv \{ \omega_n \Phi_n, \Theta \} \) is a weighted retro Banach frame for \( \mathbb{H}^* \) if and only if

\[
\text{dist}(\Phi, L_n) \to 0 \text{ as } n \to \infty, \text{ for all } \Phi \in \mathbb{H}, \text{ where } L_n = [\omega_1 \Phi_1, \omega_2 \Phi_2, \ldots, \omega_n \Phi_n], \text{ for all } n \in \mathbb{N}.
\]
Proof. Suppose first that $\mathcal{F} = \left\{ \{\omega_n \Phi_n\}, \Theta \right\}$ is a weighted retro Banach frame for $\mathbb{H}^\ast$. Let $A_0$ and $B_0$ be the choice of bounds for $\mathcal{F}$. Then

$$A_0 \|\Phi^\ast\|_{\mathbb{H}^\ast} \leq \|\Phi^\ast (\omega_n \Phi_n)\|_{\mathbb{H}^\ast} \leq B_0 \|\Phi^\ast\|_{\mathbb{H}^\ast}, \text{ for all } \Phi^\ast \in \mathbb{H}^\ast. \quad (6.4)$$

Suppose that the condition $\text{dist} (\Phi, L_n) \to 0$ as $n \to \infty$, for all $\Phi \in \mathbb{H}$, is not satisfied.

Then there exists a non-zero functional $\Phi_0 \in \mathbb{H}$ such that

$$\lim_{n \to \infty} \text{dist} (\Phi, L_n) \neq 0.$$ 

Note that $\text{dist} (\Phi, L_n) \geq \text{dist} (\Phi, L_{n+1})$ for all $n \in \mathbb{N}$.

This is because $\left\{ L_n \right\}$ is a nested system of subspaces and by definition of distance of a point from a set. Now $\left\{ \text{dist} (\Phi_0, L_n) \right\}$ is coercive monotonic decreasing sequence.

So, $\left\{ \text{dist} (\Phi_0, L_n) \right\}$ is convergent and its limit is a positive real number, since otherwise $\text{dist} (\Phi, L_n) \to 0$ as $n \to \infty$ (which is not possible).

Let

$$\lim_{n \to \infty} \text{dist} (\Phi, L_n) = \xi > 0.$$ 

Choose $D = \bigcup_n L_n$. Then, by using the fact that

$$\text{dist} (\Phi_0, D) = \inf \left\{ \text{dist} (\Phi_0, L_n) \right\},$$

We obtain,

$$\text{dist} (\Phi_0, D) \geq \xi > 0. \quad (6.5)$$

Now we show that $\Phi_0 \notin \overline{D}$. Let, if possible, $\Phi_0 \in \overline{D}$. Then, we can find a sequence $\{ \zeta_n \} \subset D$ such that

$$\text{dist} (\zeta_n, \Phi_0) \to 0 \text{ as } n \to \infty.$$ 

By using (6.5), we have $\text{dist} (\Phi_0, D) \geq \xi$. 

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Therefore,
\[ \text{dist} \left( \zeta_n, \Phi_0 \right) \geq \xi > 0. \]
This is a contradiction to the fact that \( \text{dist} \left( \zeta_n, \Phi_0 \right) \to 0 \) as \( n \to \infty \).
Hence \( \Phi_0 \notin \overline{D} \).
Thus, by using Hahn-Banach theorem, there exists a non zero functional \( \Phi_0^* \in \mathbb{H}^* \) such that \( \Phi_0^* \left( \omega, \Phi_n \right) = 0 \), for all \( \Phi \in \mathbb{H} \).

Therefore, by retro frame inequality (6.4), we have \( \Phi_0 = 0 \), a contradiction.
Hence, \( \text{dist} \left( \Phi, L_n \right) \to 0 \) as \( n \to \infty \), for all \( \Phi \in \mathbb{H} \).

To prove the converse part, assume that there exists a system \( \{ \Phi_n \} \subset \mathbb{H} \) is such that \( \text{dist} \left( \Phi, L_n \right) \to 0 \) as \( n \to \infty \), for all \( \Phi \in \mathbb{H} \).

Then, in particular, for each \( \epsilon > 0 \) and for each \( \Phi \in \mathbb{H} \), we can find a \( \omega_j \Phi_j \) from some \( L_k \) such that
\[ \left\| \Phi - \omega_j \Phi_j \right\| < \epsilon, \]
which gives \( \{ \omega_j \Phi_j \} \) is complete in \( \mathbb{H} \).

Therefore, by using Lemma 1.4.11, \( Z = \left\{ \Phi^* \left( \omega_n \Phi_n \right) : \Phi^* \in \mathbb{H}^* \right\} \) is a Banach space of sequences of scalars with norm given by
\[ \left\| \Phi^* \left( \omega_n \Phi_n \right) \right\|_Z = \left\| \Phi^* \right\|_{\mathbb{H}^*}, \quad \forall \Phi^* \in \mathbb{H}^*. \]
Define \( \Theta : Z \to \mathbb{H}^* \) by
\[ \Theta \left( \Phi^* \left( \omega_n \Phi_n \right) \right) = \Phi^*, \quad \forall \Phi^* \in \mathbb{H}^*. \]
Then, \( \Theta \) is the bounded linear operator such that \( \left\{ \omega_n \Phi_n \right\}, \Theta \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z \).
The following theorem gives necessary and sufficient condition for the existence of a weighted retro Banach frame for discrete signal space with respect to a given associated Banach space of scalar valued sequences $Z_d$.

**Theorem 6.5.5.** A system $\mathcal{F} \equiv \{\omega_n \Phi_n, \Theta\}$ is a weighted retro Banach frame for $H^*$ with respect to $Z_d$ which is generated by $\{\Phi^*(\omega_n \Phi_n)\}$ if and only if $H^*$ is isomorphic to a closed subspace of $Z_d$.

**Proof.** Assume first that $\mathcal{F} \equiv \{\omega_n \Phi_n, \Theta\}$ is a weighted retro Banach frame for $H^*$ with respect to $Z_d$. Then, there exists positive constants $A_0, B_0$ such that

$$A_0 \|\Phi^*\|_{H^*} \leq \|\Phi^*(\omega_n \Phi_n)\|_{Z_d} \leq B_0 \|\Phi^*\|_{H^*},$$

for all $\Phi^* \in H^*$.

(6.6)

Also, let the analysis operator $T : H^* \to Z_d$ of $\{\omega_n \Phi_n\}$ be given by

$$T(\Phi^*) = \{\Phi^*(\omega_n \Phi_n)\},$$

for all $\Phi^* \in H^*$.

(6.7)

By using lower frame inequality in (6.6) and (6.7),

$$\|T(\Phi^*)\|_{Z_d} \geq A_0 \|\Phi^*\|_{H^*},$$

for all $\Phi^* \in H^*$.

Then, analysis operator $T$ of $\{\omega_n \Phi_n\}$ is coercive. Thus, $T$ is injective and has closed range. From the Inverse Mapping theorem, $H^*$ is isomorphic to the range $T(H^*)$, which is a subspace of $Z_d$.

For the converse part, assume that $M$ is a closed subspace of $Z_d$ and $U$ is an isomorphic from $H^*$ onto $M$. Let $\{P_i\}$ be the sequence of coordinate operators on $Z_d$, then

$$P_i(\{y_j\}) = y_i, \text{ for all } i \in \mathbb{N}.$$  

Choose, $\Phi^*(\omega_n \Phi_n) = P_n U(\Phi^*), \forall n \in \mathbb{N}.$ Then, for all $\Phi^* \in H^*$, we have

$$\|\Phi^*\|_{H^*} = \|U^{-1}U(\Phi^*)\|_{H^*} \leq \|U^{-1}\| \|U(\Phi^*)\|_{Z_d}.$$
Therefore,
\[
\frac{\|\Phi^*\|_{\mathcal{H}^*}}{\|U^{-1}\|} \leq \|\{\Phi^*(\omega_n \Phi_n)\}\|_{\mathcal{Z}_d} = \|\{P_n U(\Phi^*)\}\|_{\mathcal{Z}_d} = \|U(\Phi^*)\|_{\mathcal{Z}_d} \leq \|U\|\|\Phi^*\|_{\mathcal{H}^*}, \ \forall \Phi^* \in \mathbb{H}^*.
\]

Thus, we obtain
\[
\frac{\|\Phi^*\|_{\mathcal{H}^*}}{\|U^{-1}\|} \leq \|\{\Phi^*(\omega_n \Phi_n)\}\|_{\mathcal{Z}_d} \leq \|U\|\|\Phi^*\|_{\mathcal{H}^*}, \ \forall \Phi^* \in \mathbb{H}^*.
\]

Hence, \(\{\omega_n \Phi_n\}\) is a weighted retro Banach frame for \(\mathbb{H}^*\) with respect to \(\mathcal{Z}_d\).

### 6.6 Construction of Weighted Retro Banach Frames for \(\mathbb{H}^*\) From Bounded Linear Operators on \(\mathcal{Z}_d\).

Let \(\mathcal{F} \equiv \{\{\omega_n \Phi_n\}, \Theta\}\) be a weighted retro Banach frame for \(\mathbb{H}^*\) with respect to \(\mathcal{Z}_d\) and let \(\{\Psi_k\} \subset \mathbb{H}\). Let \(\hat{\Theta}\) be a bounded linear operator on \(\mathcal{Z}_d\) such that
\[
\hat{\Theta}\{\Phi^* (\omega_k \Phi_k)\} = \{\Phi^* (\omega_k \Psi_k)\}, \ \Phi^* \in \mathbb{H}^*.
\]

Then, in general, there exists no reconstruction operator \(\Theta_0\) such that \(\{\omega_k \Psi_k\}, \Theta_0\) is a retro Banach frame for \(\mathbb{H}^*\) with respect to \(\mathcal{Z}_d\). The following theorem provides necessary and sufficient condition for the construction of a weighted retro Banach frame for \(\mathbb{H}^*\) with respect to \(\mathcal{Z}_d\).

**Theorem 6.6.1.** Let \(\mathcal{F} \equiv \{\{\omega_n \Phi_n\}, \Theta\}\) be a weighted retro Banach frame for \(\mathbb{H}^*\) with respect to \(\mathcal{Z}_d\) and let \(\{\Psi_k\}\) be a sequence in \(\mathbb{H}\) such that \(\{\Phi^* (\omega_k \Psi_k)\} \in \mathcal{Z}_d, \ \Phi^* \in \mathbb{H}^*\). Assume that \(\hat{\Theta}\) is a bounded linear operator on \(\mathcal{Z}_d\).
such that
\[ \hat{\Theta}\left(\{\Phi^*(\omega_k \Phi_k)\}\right) = \{\Phi^*(\omega_k \Psi_k)\}, \Phi^* \in \mathbb{H}^* , \]
where \( \{\Psi_k\} \subset \mathbb{H} \). Then, there exists a reconstruction operator \( \Theta_0 \) such that
\( \left(\{\omega_k, \Psi_k\}, \Theta_0\right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_d \) if and only if
\[ \left\|\hat{\Theta}\left(\{\Phi^*(\omega_k \Phi_k)\}\right)\right\|_{Z_d} \geq \gamma \left\|Q\left(\{\Phi^*(\omega_k \Psi_k)\}\right)\right\|_{Z_d}, \]
where \( \gamma \) is a positive constant and \( Q \) is a bounded linear operator on \( Z_d \) such that
\[ Q\left(\{\Phi^*(\omega_k \Psi_k)\}\right) = \{\Phi^*(\omega_k \Phi_k)\}, \Phi^* \in \mathbb{H}^* . \]

**Proof.** Since \( \left(\{\omega_k, \Phi_k\}, \Theta\right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_d \) with bounds \( A, B \). Then, we have
\[ A\|\Phi^*\|_{\mathbb{H}^*} \leq \left\|\left\{\Phi^*(\omega_k \Phi_k)\}\right\|_{Z_d} \leq B\|\Phi^*\|_{\mathbb{H}^*}, \text{ for all } \Phi^* \in \mathbb{H}^*. \] (6.8)

Suppose that \( \left(\{\omega_k, \Psi_k\}, \Theta_0\right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( Z_d \) and with bounds \( a_0, b_0 \). Then, we have
\[ a_0\|\Phi^*\|_{\mathbb{H}^*} \leq \left\|\left\{\Phi^*(\omega_k \Psi_k)\}\right\|_{Z_d} \leq b_0\|\Phi^*\|_{\mathbb{H}^*}, \text{ for all } \Phi^* \in \mathbb{H}^*. \] (6.9)

Now using (6.8) and (6.9), we get
\[ \left\|\hat{\Theta}\left(\{\Phi^*(\omega_k \Phi_k)\}\right)\right\|_{Z_d} = \left\|\left\{\Phi^*(\omega_k \Psi_k)\}\right\|_{Z_d} \]
\[ \geq a_0\|\Phi^*\|_{\mathbb{H}^*} \]
\[ = \frac{a_0B}{B} \|\Phi^*\|_{\mathbb{H}^*} \]
\[ \geq \frac{a_0B}{B} \|Q\left(\{\Phi^*(\omega_k \Psi_k)\}\right)\|_{Z_d}. \]
Put $\gamma = \frac{a_0}{B}$ then, we get

$$\left\| \hat{\Theta} \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d} \geq \gamma \left\| Q \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d}$$

Conversely, let

$$\left\| \hat{\Theta} \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d} \geq \gamma \left\| Q \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d}$$  \hspace{1cm} (6.10)

By using (6.8) and (6.10), we have

$$\left\| \{ \Phi^* (\omega_k \Phi_k) \} \right\|_{\mathbb{Z}_d} = \left\| \hat{\Theta} \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d} \geq \gamma \left\| Q \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d} = \gamma \left\| \Phi^* \right\|_{\mathbb{Z}_d} = \gamma A \left\| \Phi^* \right\|$$

Thus, we have

$$\left\| \{ \Phi^* (\omega_k \Phi_k) \} \right\|_{\mathbb{Z}_d} \geq \gamma A \left\| \Phi^* \right\|_{\mathbb{H}^*}$$ \hspace{1cm} (6.11)

Also we have

$$\left\| \{ \Phi^* (\omega_k \Phi_k) \} \right\|_{\mathbb{Z}_d} = \left\| \hat{\Theta} \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) \right\|_{\mathbb{Z}_d} \leq \left\| \hat{\Theta} \right\| B \left\| \Phi^* \right\|_{\mathbb{H}^*}$$

Thus, we have
\[
\left\| \{ \Phi^* (\omega_k \Psi_k) \} \right\|_{\mathcal{H}_d} \leq \left\| \hat{\Theta} B \right\|_{\mathcal{H}^*},
\tag{6.12}
\]

From (6.11) and (6.12), we get
\[
\gamma A \left\| \Phi^* \right\|_{\mathcal{H}^*} \leq \left\| \{ \Phi^* (\omega_k \Psi_k) \} \right\|_{\mathcal{H}_d} \leq \left\| \hat{\Theta} B \right\|_{\mathcal{H}^*}, \text{ for all } \Phi^* \in \mathbb{H}^*.
\]

Put \( \Theta_0 = \Theta Q \). Then \( \Theta_0 : \mathcal{H}_d \rightarrow \mathbb{H}^* \) is bounded linear operator such that
\[
\Theta_0 \left( \{ \Phi^* (\omega_k \Psi_k) \} \right) = \Phi^*, \text{ for all } \Phi^* \in \mathbb{H}^*.
\]
Hence, \( \{ \omega_k \Psi_k, \Theta_0 \} \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{H}_d \).

The following theorem gives the better Bessel bound for the sum of two weighted retro Banach frames.

**Theorem 6.6.2.** Let \( \mathcal{F} = \{ \omega_n \phi_n \}, \Theta \) and \( \mathcal{G} = \{ \omega_n \phi_n \}, \Theta_0 \) be weighted retro Banach frames for \( \mathbb{H}^* \) with respect to \( \mathcal{H}_d \) and let \( T \) be a bounded linear invertible operator on \( \mathcal{H}_d \) such that
\[
T \left( \{ \Phi^* (\omega_k \phi_k) \} \right) = \{ \Phi^* (\omega_k \psi_k) \}, \Phi^* \in \mathbb{H}^*.
\]
Then, \( \{ \omega_k \phi_k + \omega_k \psi_k \} \) is a weighted retro Bessel sequence with bound
\[
\alpha = \min \left\{ \left\| \hat{\Theta} \right\| \left\| I + T \right\|, \left\| \hat{\Theta}_0 \right\| \left\| I + T^{-1} \right\| \right\},
\]
where \( \hat{\Theta}, \hat{\Theta}_0 \) are the analysis operators associated with \( \mathcal{F} \) and \( \mathcal{G} \), respectively and \( I \) is the identity operator on \( \mathcal{H}_d \).

**Proof.** Given \( \hat{\Theta}, \hat{\Theta}_0 \) are the analysis operators associated with \( \mathcal{F} \) and \( \mathcal{G} \), respectively. Then, we have
\[
\hat{\Theta} \left( \Phi^* \right) = \left\{ \Phi^* (\omega_k \phi_k) \right\}, \forall \Phi^* \in \mathbb{H}^*.
\]
and

\[ \hat{\Theta}_0(\Phi^*) = \{ \Phi^*(\omega_k \Psi_k) \}, \ \forall \Phi^* \in \mathbb{H}^*. \]

Also, \( I \) is the identity operator on \( Z_d \). Then

\[ I \{ \Phi^* (\omega_k \Phi_k) \} = \{ \Phi^*(\omega_k \Phi_k) \} \ \forall \Phi^* \in \mathbb{H}^* \]

and

\[ I \{ \Phi^*(\omega_k \Psi_k) \} = \{ \Phi^*(\omega_k \Psi_k) \} \ \forall \Phi^* \in \mathbb{H}^*. \]

Given \( T \) is the invertible operator on \( Z_d \) such that

\[ T \{ \Phi^*(\omega_k \Psi_k) \} = \{ \Phi^*(\omega_k \Psi_k) \} \ \forall \Phi^* \in \mathbb{H}^*. \]

By hypothesis, for all \( \Phi^* \in \mathbb{H}^* \), we have

\[ \left\| \{ \Phi^*(\omega_k \Phi_k + \omega_k \Psi_k) \} \right\|_{Z_d} = \left\| \{ \Phi^*(\omega_k \Phi_k) \} + \{ \Phi^*(\omega_k \Psi_k) \} \right\|_{Z_d} \]

\[ = \left\| (I + T) \{ \Phi^*(\omega_k \Phi_k) \} \right\|_{Z_d} \]

\[ \leq \left\| I + T \right\| \left\| \Phi^*(\omega_k \Phi_k) \right\|_{Z_d} \]

\[ = \left\| I + T \right\| \left\| \hat{\Theta}(\Phi^*) \right\|_{Z_d} \]

\[ \leq \left\| I + T \right\| \left\| \hat{\Theta} \right\| \left\| \Phi^* \right\|_{\mathbb{H}^*}. \]

Therefore,

\[ \left\| \{ \Phi^*(\omega_k \Phi_k + \omega_k \Psi_k) \} \right\|_{Z_d} \leq \left\| I + T \right\| \left\| \hat{\Theta} \right\| \left\| \Phi^* \right\|_{\mathbb{H}^*}, \ \forall \Phi^* \in \mathbb{H}^*. \quad (6.13) \]

Similarly, we can easily obtain that

\[ \left\| \{ \Phi^*(\omega_k \Phi_k + \omega_k \Psi_k) \} \right\|_{Z_d} \leq \left\| I + T^{-1} \right\| \left\| \hat{\Theta}_0 \right\| \left\| \Phi^* \right\|_{\mathbb{H}^*}, \ \forall \Phi^* \in \mathbb{H}^*. \quad (6.14) \]

Using (6.13) and (6.14), it follows that \( \{ \omega_k \Phi_k + \omega_k \Psi_k \} \) is a weighted retro Bessel sequence for \( \mathbb{H}^* \) with respect to \( Z_d \) and with bound

\[ \alpha = \min \left\{ \left\| \hat{\Theta} \right\| \left\| I + T \right\|, \left\| \hat{\Theta}_0 \right\| \left\| I + T^{-1} \right\| \right\}. \]
Remark 6.6.3. For the weighted retro Bessel sequence \( \{ \omega_k \Phi_k + \omega_k \Psi_k \} \) in Theorem 6.6.2, in general, there exists no reconstruction operator \( U \) such that \( (\{ \omega_k \Phi_k + \omega_k \Psi_k \}, U) \) is a weighted retro Banach frame for \( \mathbb{H}^+ \). If the analysis operator associated with weighted Bessel sequence is coercive, then the respective weighted retro Bessel sequence constitute a weighted retro Banach frame for the discrete signal space. This is summarized in the following lemma.

Lemma 6.6.4. Let \( \{ \gamma_k \} \subseteq \mathbb{H} \) be a weighted retro Bessel sequence for \( \mathbb{H}^+ \), then there exists a reconstruction operator \( U \) such that \( (\{ \gamma_k \}, U) \) is a weighted retro Banach frame for \( \mathbb{H}^+ \) if and only if its analysis operator is coercive.

Proof. Let \( \{ \gamma_k \} \subseteq \mathbb{H} \) be a weighted retro Bessel sequence for \( \mathbb{H}^+ \) with respect to \( \mathcal{Z}_d \). Then, there exists constant \( b_0 > 0 \), such that

\[
\left\| \Phi^* (\omega_k \gamma_k) \right\|_{\mathcal{Z}_d} \leq b_0 \left\| \Phi^* \right\|_{\mathcal{H}_d}, \text{ for all } \Phi^* \in \mathbb{H}^+.
\]  

(4.15)

Suppose that \( (\{ \omega_k \gamma_k \}, U) \) is a weighted retro Banach frame for \( \mathbb{H}^+ \) with respect to \( \mathcal{Z}_d \) and with bounds \( a_0, b_0 \). Then, we have

\[
a_0 \left\| \Phi^* \right\|_{\mathcal{Z}_d} \leq \left\| \Phi^* (\omega_k \gamma_k) \right\|_{\mathcal{Z}_d} \leq b_0 \left\| \Phi^* \right\|_{\mathcal{H}_d}, \text{ for all } \Phi^* \in \mathbb{H}^+.
\]  

(6.16)

To prove that, the analysis operator \( T : \mathbb{H}^+ \rightarrow \mathcal{Z}_d \) of \( \{ \gamma_k \} \) is given by

\[
T(\Phi^*) = \{ \Phi^* (\omega_k \gamma_k) \}, \text{ for all } \Phi^* \in \mathbb{H}^+.
\]  

(6.17)

is coercive (i.e. bounded below).

By using lower frame inequality of (6.16) and (6.17), we have

\[
\left\| T(\Phi^*) \right\|_{\mathcal{Z}_d} = \left\| \Phi^* (\omega_k \gamma_k) \right\|_{\mathcal{Z}_d} \geq a_0 \left\| \Phi^* \right\|_{\mathcal{H}_d}.
\]
Therefore,

\[ \left\| T\left( \Phi^* \right) \right\|_{\mathcal{Z}_d} \geq a_0 \left\| \Phi^* \right\|_{\mathcal{H}^*}, \forall \Phi^* \in \mathbb{H}^*. \]

Hence, this shows that \( T \) is coercive.

Conversely, let us assume that \( T \) is coercive, i.e.

\[ \left\| T\left( \Phi^* \right) \right\|_{\mathcal{Z}_d} \geq a_0 \left\| \Phi^* \right\|_{\mathcal{H}^*} \]

or

\[ \left\| \left( \Phi^* \left( \omega_k Y_k \right) \right) \right\|_{\mathcal{Z}_d} \geq a_0 \left\| \Phi^* \right\|_{\mathcal{H}^*}, \forall \Phi^* \in \mathbb{H}^*. \]  \hspace{1cm} (6.18)

By using (6.15) and (6.18), we get

\[ a_0 \left\| \Phi^* \right\|_{\mathcal{H}^*} \leq \left\| \left( \Phi^* \left( \omega_k Y_k \right) \right) \right\|_{\mathcal{Z}_d} \leq b_0 \left\| \Phi^* \right\|_{\mathcal{H}^*}, \text{ for all } \Phi^* \in \mathbb{H}^*. \]

Choose \( \mathcal{Z}_{d_i} = \left\{ \left\{ \Phi^* \left( \omega_k Y_k \right) \right\} : \Phi^* \in \mathbb{H}^* \right\} \). Then \( \mathcal{Z}_{d_i} \) is a Banach space with the norm is given by

\[ \left\| \left( \Phi^* \left( \omega_k Y_k \right) \right) \right\|_{\mathcal{Z}_{d_i}} = \left\| \Phi^* \right\|_{\mathcal{H}^*}, \text{ for all } \Phi^* \in \mathbb{H}^*. \]

Define \( U : \mathcal{Z}_{d_i} \rightarrow \mathbb{H}^* \), such that

\[ U \left( \left\{ \Phi^* \left( \omega_k Y_k \right) \right\} \right) = \Phi^*, \text{ for all } \Phi^* \in \mathbb{H}^*. \]

Then, \( U \) is a bounded linear operator such that \( \left\{ \left\{ \omega_k Y_k \right\}, U \right\} \) is a weighted retro Banach frame for \( \mathbb{H}^* \).
6.7 Relation Between Weighted Retro Bessel Bounds and Weighted Retro Banach Frame Bounds.

In signal processing there is an overlapping between the signal under consideration and other signals of high frequency (or energy). Here, we consider sum or difference (algebraic overlapping) of a weighted retro Banach frame with weighted retro Bessel sequence in discrete signal space. It is interesting to know the relation between weighted retro Bessel bounds and weighted retro Banach frame bounds such that after overlapping (algebraic operations) the resultant system constitute a weighted retro Banach frame for the discrete signal space. In this direction, the following proposition provides a result in Banach space setting.

**Proposition 6.7.1.** Let \( \mathcal{F} \equiv \{ \omega_n \Phi_n \}, \Theta \) be a weighted retro Banach frames for \( \mathbb{H}^* \) with respect to \( Z_d \) and with bounds \( A, B \) and let \( \{ \omega_k \Psi_k \} \) be a weighted retro Bessel sequence for \( \mathbb{H}^- \) with respect to \( Z_d \) with bound \( M \). Then, \( \{ \omega_k (\Phi_k \pm \Psi_k) \} \) is a weighted retro Banach frames for \( \mathbb{H}^* \), provided \( M < A \).

**Proof.** Since \( \mathcal{F} \equiv \{ \omega_n \Phi_n \}, \Theta \) is a weighted retro Banach frame for \( \mathbb{H}^- \) with respect to \( Z_d \) with bounds \( A, B \). Then, we have

\[
A \left\| \Phi \right\|_{\mathcal{F}} \leq \left\| \Phi^* \left( \omega_k \Phi_k \right) \right\|_{Z_d} \leq B \left\| \Phi^* \right\|_{\mathcal{F}} , \text{ for all } \Phi^* \in \mathbb{H}^*.
\]

Also \( \{ \omega_k \Psi_k \} \) be a weighted retro Bessel sequence for \( \mathbb{H}^- \) with respect to \( Z_d \) with bound \( M \). Then, we have

\[
\left\| \Phi^* \left( \omega_k \Psi_k \right) \right\|_{Z_d} \leq M \left\| \Phi^* \right\|_{\mathbb{H}^-} , \text{ for all } \Phi^* \in \mathbb{H}^-.
\]

Let \( T \) and \( R \) be the analysis operators associated with \( \{ \omega_k \Phi_k \} \) and \( \{ \omega_k \Psi_k \} \), respectively. Then, we have

\[
T \left( \Phi^* \right) = \{ \Phi^* \left( \omega_k \Phi_k \right) \}, \forall \Phi^* \in \mathbb{H}^*
\]
and
\[ R\left( \Phi^* \right) = \left\{ \Phi^* \left( \omega_k \Psi_k \right) \right\}, \quad \forall \Phi^* \in \mathbb{H}^*. \]

Now, we compute
\[
\left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{E}_d} = \left\| \Phi^* \left( \omega_k \Phi_k \right) \pm \left\{ \Phi^* \left( \omega_k \Psi_k \right) \right\} \right\|_{\mathcal{E}_d}
= \left\| T\left( \Phi^* \right) \pm R\left( \Phi^* \right) \right\|_{\mathcal{E}_d}
\geq \left\| T\left( \Phi^* \right) \right\|_{\mathcal{E}_d} - \left\| R\left( \Phi^* \right) \right\|_{\mathcal{E}_d}
\geq A \left\| \Phi^* \right\|_{\mathcal{H}}^2 - M \left\| \Phi^* \right\|_{\mathcal{H}}
= \left( A - M \right) \left\| \Phi^* \right\|_{\mathcal{H}}.
\]

Therefore,
\[
\left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{E}_d} \geq \left( A - M \right) \left\| \Phi^* \right\|_{\mathcal{H}}, \quad \forall \Phi^* \in \mathbb{H}^*. \]

Similarly, we have
\[
\left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{E}_d} = \left\| \Phi^* \left( \omega_k \Phi_k \right) \pm \left\{ \Phi^* \left( \omega_k \Psi_k \right) \right\} \right\|_{\mathcal{E}_d}
= \left\| T\left( \Phi^* \right) \pm R\left( \Phi^* \right) \right\|_{\mathcal{E}_d}
\leq \left\| T\left( \Phi^* \right) \right\|_{\mathcal{E}_d} + \left\| R\left( \Phi^* \right) \right\|_{\mathcal{E}_d}
\leq B \left\| \Phi^* \right\|_{\mathcal{H}} + M \left\| \Phi^* \right\|_{\mathcal{H}}
= \left( B + M \right) \left\| \Phi^* \right\|_{\mathcal{H}}.
\]

Therefore,
\[
\left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{E}_d} \leq \left( B + M \right) \left\| \Phi^* \right\|_{\mathcal{H}}, \quad \forall \Phi^* \in \mathbb{H}^*. \]

Hence, we obtain
\[
\left( A - M \right) \left\| \Phi^* \right\|_{\mathcal{H}} \leq \left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{E}_d} \leq \left( B + M \right) \left\| \Phi^* \right\|_{\mathcal{H}}, \quad \forall \Phi^* \in \mathbb{H}^*. \]

Choose \( \mathcal{Y}_d = \left\{ \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\}; \Phi^* \in \mathbb{H}^* \). Then, \( \mathcal{Y}_d \) is a Banach space with the norm given by
\[
\left\| \Phi^* \left( \omega_k \Phi_k \pm \omega_k \Psi_k \right) \right\|_{\mathcal{Y}_d} = \left\| \Phi^* \right\|_{\mathcal{H}}, \quad \forall \Phi^* \in \mathbb{H}^*. \]
Define \( \Theta_0 : \mathcal{Y}_d \rightarrow \mathbb{H}^* \) by
\[
\Theta_0 \left( \{ \Phi^* (\omega_k \Phi_k \pm \omega_k \Psi_k) \} \right) = \Phi^*, \ \forall \Phi^* \in \mathbb{H}^*.
\]
Then, \( \Theta_0 \) is bounded linear operator such that \( \left( \{ \omega_k (\Phi_k \pm \Psi_k) \}, \Theta_0 \right) \) is a weighted retro Banach frames for \( \mathbb{H}^* \) with respect to \( \mathcal{Y}_d \). The proposition is proved.

Let \( \mathcal{F} \equiv \left( \{ \omega_n \Phi_n \}, \Theta \right) \) be a weighted retro Banach frames for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \) and let \( \{ \Psi_k \} \subset \mathbb{H} \). The following proposition give an estimate of a weighted retro Bessel bound for \( \{ \omega_k \Phi_k + \omega_k \Psi_k \} \) such that \( \left( \{ \Psi_k \}, \Theta_0 \right) \) is a weighted retro Banach frames for \( \mathbb{H}^* \).

**Proposition 6.7.2.** Assume that \( \mathcal{F} \equiv \left( \{ \omega_n \Phi_n \}, \Theta \right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \) and let \( \{ \Psi_k \} \subset \mathbb{H} \). If \( \{ \omega_k \Phi_k + \omega_k \Psi_k \} \) is a weighted retro Bessel sequence for \( \mathbb{H}^* \) with Bessel bound \( \delta < \|\Theta\|^{-1} \). Then, there exists a reconstruction operator \( \Theta_0 \) such that \( \left( \{ \Psi_k \}, \Theta_0 \right) \) is a weighted retro Banach frames for \( \mathbb{H}^* \).

**Proof.** Assume that \( \mathcal{F} \equiv \left( \{ \omega_n \Phi_n \}, \Theta \right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \). Then, there exist positive constants \( a_0 \) and \( b_0 \) such that
\[
ad_0 \|\Phi^*\|_{\mathbb{H}^*} \leq \left\| \left( \Phi^* \left( \omega_k \Phi_k \right) \right) \right\|_{\mathcal{Z}_d} \leq b_0 \|\Phi^*\|_{\mathbb{H}^*}, \ \text{for all} \ \Phi^* \in \mathbb{H}^*.
\] (6.19)
and bounded linear operator \( \Theta : \mathcal{Z}_d \rightarrow \mathbb{H}^* \) such that
\[
\Theta \left( \{ \Phi^* (\omega_k \Phi_k) \} \right) = \Phi^*, \ \forall \Phi^* \in \mathbb{H}^*.
\] (6.20)
By hypothesis \( \{ \omega_k \Phi_k + \omega_k \Psi_k \} \) is a weighted retro Bessel sequence for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \) and with bound \( \delta < \|\Theta\|^{-1} \). Then, we have
\[
\left\| \left( \Phi^* (\omega_k \Phi_k + \omega_k \Psi_k) \right) \right\|_{\mathcal{Z}_d} \leq \delta \|\Phi^*\|_{\mathbb{H}^*}, \ \forall \Phi^* \in \mathbb{H}^*.
\] (6.21)
Now, we have

\[
\|\left(\Theta^{-1} - \delta\right)\Phi^*\|_{\mathbb{H}^+} = \|\Theta^{-1}\|_{\mathbb{H}^+} - \delta\|\Phi^*\|_{\mathbb{H}^+} \\
\leq \|\Phi^* (\omega_k \Phi_k)\|_{\mathbb{L}_d^2} - \|\Phi^* (\omega_k \Phi_k + \omega_k \Psi_k)\|_{\mathbb{L}_d^2} \quad \text{(by (6.20) and (6.21))} \\
\leq \|\Phi^* (\omega_k \Phi_k)\|_{\mathbb{L}_d^2} - \|\Phi^* (\omega_k \Phi_k + \omega_k \Psi_k)\|_{\mathbb{L}_d^2} \\
= \|\Phi^* (\omega_k \Phi_k)\|_{\mathbb{L}_d^2} \\
= \|\Phi^* (\omega_k \Phi_k + \omega_k \Psi_k)\|_{\mathbb{L}_d^2} + \|\Phi^* (\omega_k \Phi_k)\|_{\mathbb{L}_d^2} \\
= \delta\|\Phi^*\|_{\mathbb{H}^+} + b_0\|\Phi^*\|_{\mathbb{H}^+} \quad \text{(by (6.19) and (6.21))} \\
= (\delta + b_0)\|\Phi^*\|_{\mathbb{H}^+}.
\]

Therefore,

\[
\|\left(\Theta^{-1} - \delta\right)\Phi^*\|_{\mathbb{H}^+} \leq \|\Phi^* (\omega_k \Psi_k)\|_{\mathbb{L}_d^2} \leq (\delta + b_0)\|\Phi^*\|_{\mathbb{H}^+} \quad \text{for all } \Phi^* \in \mathbb{H}^+.
\]

Choose \( \mathcal{Y}_d = \left\{ \{\Phi^* (\omega_k \Psi_k)\} : \Phi^* \in \mathbb{H}^+ \right\} \). Then, \( \mathcal{Y}_d \) is a Banach space with the norm given by

\[
\|\{\Phi^* (\omega_k \Psi_k)\}\|_{\mathbb{L}_d^2} = \|\Phi^*\|_{\mathbb{H}^+}, \quad \forall \Phi^* \in \mathbb{H}^+.
\]

Define \( \Theta_0 : \mathcal{Y}_d \to \mathbb{H}^+ \) by

\[
\Theta_0 \left( \{\Phi^* (\omega_k \Psi_k)\} \right) = \Phi^*, \quad \forall \Phi^* \in \mathbb{H}^+.
\]

Then, \( \Theta_0 \) is a bounded linear operator such that \( (\{\omega_k \Psi_k\}, \Theta_0) \) is a weighted retro Banach frames for \( \mathbb{H}^+ \) with respect to \( \mathcal{Y}_d \).

**6.8 Perturbation of Weighted Retro Banach Frames.**

Perturbation theory is a very important tool in various areas of applied mathematics [14, 24, 36, 40, and 52]. In frame theory, it began with the fundamental perturbation result of Paley and Wiener [73]. The basic of Paley and Wiener was that a system that is sufficient close to an orthonormal system (basis) in a Hilbert space is also form an orthonormal system (basis). Since then, a number of variations
and generalization of this perturbation to the setting of Banach space [37, 49] and then to perturbation of the atomic decompositions, frames (Hilbert) and Banach frames, the reconstruction property in Banach spaces [15, 32, 80, and 83]. The following theorem gives a Paley-Wiener type perturbation (in Banach space setting) for weighted retro Banach frames.

**Theorem 6.8.1.** Let \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \) and with bounds \( A, B \) and let \( \{\Psi_k\} \subset \mathbb{H} \). Assume that \( \lambda, \mu, \nu \geq 0 \) are non-negative real numbers such that
\[
\max\left(\frac{\lambda + \nu}{A}, \mu\right) < 1
\]
and
\[
\left\| (T - R)(\Phi^*) \right\|_{\mathcal{Z}_d} \leq \lambda \left\| T(\Phi^*) \right\|_{\mathcal{Z}_d} + \mu \left\| R(\Phi^*) \right\|_{\mathcal{Z}_d} + \nu \left\| \Phi^* \right\|_{\mathcal{Z}_d}, \quad \text{for all } \Phi^* \in \mathbb{H}^* \quad (6.22)
\]
where \( T \) and \( R \) the analysis operators associated with \( \{\omega_n \Phi_n\} \) and \( \{\omega_k \Psi_k\} \), respectively. Then, there exists a reconstruction operator \( \Theta_0 \) such that \( \left( \{\omega_k \Psi_k\}, \Theta_0 \right) \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with bounds
\[
\left( \frac{1 - \lambda}{1 + \mu} A - \nu \right) \quad \text{and} \quad \left( \frac{1 + \lambda}{1 - \mu} B + \nu \right)
\]
with respect to a Banach space generated by \( \left\{ R(\Phi^*) \right\} : \Phi^* \in \mathbb{H}^* \).

**Proof.** Given \( \mathcal{F} \equiv \{\omega_n \Phi_n\}, \Theta \) is a weighted retro Banach frame for \( \mathbb{H}^* \) with respect to \( \mathcal{Z}_d \) and with bounds \( A, B \). Then, we have
\[
A \left\| \Phi^* \right\|_{\mathcal{Z}_d} \leq \left\| (\Phi^* \left( \omega_n \Phi_n \right) \right\|_{\mathcal{Z}_d} \leq B \left\| \Phi^* \right\|_{\mathcal{Z}_d}, \quad \text{for all } \Phi^* \in \mathbb{H}^*. \quad (6.23)
\]
Also \( T \) and \( R \) the analysis operators associated with \( \{\omega_n \Phi_n\} \) and \( \{\omega_k \Psi_k\} \), respectively. Then, we have
\[
T(\Phi^*) = \{\Phi^* \left( \omega_n \Phi_n \right) \}, \quad \forall \Phi^* \in \mathbb{H}^* \quad (6.24)
\]
and
\[ R(\Phi^*) = \{ \Phi^*(\omega \Psi_\lambda) \}, \forall \Phi^* \in \mathbb{H}^*. \] (6.25)

Since,
\[
\left\| (T - R)(\Phi^*) \right\|_2 \leq \left\| T(\Phi^*) - R(\Phi^*) \right\|_2 \\
= \left\| R(\Phi^*) - T(\Phi^*) \right\|_2 \\
\geq \left\| R(\Phi^*) \right\|_2 - \left\| T(\Phi^*) \right\|_2.
\]

Therefore,
\[
\left\| (T - R)(\Phi^*) \right\|_2 \geq \left\| R(\Phi^*) \right\|_2 - \left\| T(\Phi^*) \right\|_2.
\]

Or
\[
\left\| R(\Phi^*) \right\|_2 - \left\| T(\Phi^*) \right\|_2 \leq \left\| (T - R)(\Phi^*) \right\|_2. \tag{6.26}
\]

By using (6.21) and (6.25), we get
\[
\left\| R(\Phi^*) \right\|_2 - \left\| T(\Phi^*) \right\|_2 \leq \lambda \left\| T(\Phi^*) \right\|_2 + \mu \left\| R(\Phi^*) \right\|_2 + \nu \left\| \Phi^* \right\|_2\] \\
\[
(1 - \mu) \left\| R(\Phi^*) \right\|_2 \leq (1 + \lambda) \left\| T(\Phi^*) \right\|_2 + \nu \left\| \Phi^* \right\|_2.
\]

Therefore,
\[
\left\| R(\Phi^*) \right\|_2 \leq \left\| \frac{1 + \lambda}{1 - \mu} \right\| T(\Phi^*) \|_2 + \frac{\nu}{1 - \mu} \left\| \Phi^* \right\|_2\] \\
\[
\leq \left\| \frac{1 + \lambda}{1 - \mu} \right\| B \left\| \Phi^* \right\|_2 + \frac{\nu}{1 - \mu} \left\| \Phi^* \right\|_2 \quad \text{(by using (6.23) and (6.24))}
\]
\[
= \left( \frac{(1 + \lambda)B + \nu}{1 - \mu} \right) \left\| \Phi^* \right\|_2.
\]

Therefore,
\[
\left\| R(\Phi^*) \right\|_2 \leq \left( \frac{(1 + \lambda)B + \nu}{1 - \mu} \right) \left\| \Phi^* \right\|_2, \forall \Phi^* \in \mathbb{H}^*. \tag{6.27}
\]

Again,
\[
\left\| (T - R)(\Phi^*) \right\|_2 = \left\| T(\Phi^*) - R(\Phi^*) \right\|_2 \\
\geq \left\| T(\Phi^*) \right\|_2 - \left\| R(\Phi^*) \right\|_2.
\]

Therefore,
\[
\left\| (T - R)(\Phi^*) \right\|_2 \geq \left\| T(\Phi^*) \right\|_2 - \left\| R(\Phi^*) \right\|_2. \tag{6.28}
\]

By using (6.27) and (2.28), we get
\[ \lambda \left\| T(\Phi^*) \right\|_{L^q} + \mu \left\| R(\Phi^*) \right\|_{L^q} + \nu \left\| \Phi^* \right\|_{L^q} \geq \left\| T(\Phi^*) \right\|_{L^q} - \left\| R(\Phi^*) \right\|_{L^q}, \]

\[ (1 + \mu) \left\| R(\Phi^*) \right\|_{L^q} \geq (1 - \lambda) \left\| T(\Phi^*) \right\|_{L^q} - \nu \left\| \Phi^* \right\|_{L^q}, \]

\[ \left\| R(\Phi^*) \right\|_{L^q} \geq \left( \frac{1 - \lambda}{1 + \mu} \right) \left\| T(\Phi^*) \right\|_{L^q} - \frac{\nu}{(1 + \mu)} \left\| \Phi^* \right\|_{L^q}, \quad \text{(by (6.23) and (6.24))} \]

\[ = \left( \frac{(1 - \lambda) A - \nu}{1 + \mu} \right) \left\| \Phi^* \right\|_{L^q}. \]

Therefore, we obtain

\[ \left\| R(\Phi^*) \right\|_{L^q} \geq \left( \frac{(1 - \lambda) A - \nu}{1 + \mu} \right) \left\| \Phi^* \right\|_{L^q}, \quad \forall \Phi^* \in \mathbb{H}^+. \quad (6.29) \]

By (6.27) and (6.29), we get

\[ \left\{ \left( \frac{(1 - \lambda) A - \nu}{1 + \mu} \right) \left\| \Phi^* \right\|_{L^q} \leq \left\| R(\Phi^*) \right\|_{L^q} \leq \left( \frac{(1 + \lambda) B + \nu}{1 - \mu} \right) \left\| \Phi^* \right\|_{L^q}, \quad \forall \Phi^* \in \mathbb{H}^+. \]

Choose \( Q_d = \left\{ \left\{ R(\Phi^*) \right\}; \Phi^* \in \mathbb{H}^+ \right\}. \) Then, \( Q_d \) is a Banach space with the norm is given by

\[ \left\| \left\{ R(\Phi^*) \right\} \right\|_{Q_d} = \left\| \Phi^* \right\|_{L^q}, \quad \forall \Phi^* \in \mathbb{H}^+. \]

Define \( \Theta_0 : Q_d \to \mathbb{H}^+ \) by

\[ \Theta_0 \left( \left\{ R(\Phi^*) \right\} \right) = \Phi^*, \quad \forall \Phi^* \in \mathbb{H}^+. \]

Then, \( \Theta_0 \) is bounded linear operator such that \( \left\{ \{\omega_k, \Psi_k\}, \Theta_0 \right\} \) is a weighted retro Banach frames for \( \mathbb{H}^+ \) with respect to \( Q_d \) and with bounds \( \left( \frac{(1 - \lambda) A - \nu}{1 + \mu} \right) \) and \( \left( \frac{(1 + \lambda) B + \nu}{1 - \mu} \right), \) respectively.