Chapter-4

On $X_{\Phi}$-Frames in Banach Spaces
CHAPTER-4
ON $X_\Phi$-FRAMES IN BANACH SPACES

4.1 Introduction

The concept of the model space $X_\Phi$ of sequences have been introduced and studied in Chapter-3. In this chapter, we have introduced and discussed $X_\Phi$-Bessel sequence, $X_\Phi$-frame, $X_\Phi$-Banach frame, dual of $X_\Phi$-frames and $X_\Phi$-Riesz bases for Banach spaces. In fact, in Section 4.2, we give definitions which are used throughout in this Chapter. In Section 4.3, we give example and counter–examples to exhibits relation among various types of $X_\Phi$-frames. In Section 4.4, we have studied some characterizations for $X_\Phi$-Bessel sequences and $X_\Phi$-frames. In Section 4.5, we have characterized the $X_\Phi$-frame for Banach space under bounded linear operators. In Section 4.6, we have given the necessary and sufficient condition for sequence of operators to be a $X_\Phi$-Banach frames with respect to model space $X_\Phi$. In Section 4.7, we will study and discuss about dual of $X_\Phi$-frame for Banach space and defined an independent $X_\Phi$-frame for Banach space. A characterization of an independent $X_\Phi$-frame in terms of $X_\Phi$-Riesz bases is obtained. It is proved that an independent $X_\Phi$-frame must have a dual frame. In Section 4.8, $X_\Phi$-Riesz bases for Banach space will be defined and it is proved that an independent $X_\Phi$-frame for Banach spaces is a $X_\Phi$-Riesz bases. Finally, we give the necessary and sufficient condition for a Banach space to have $X_\Phi$-Riesz bases.

4.2 Basic Definitions

We begin this section with the following definitions which are frequently used throughout the Chapter.
**Definition 4.2.1.** Let $E$ be a Banach space over $\mathbb{F}$ and $X_\Phi$ be a model space induced by $\{G_n\}$. For every $n \in \mathbb{N}$, $\{v_n\}$ be a sequence of bounded linear operator in $L(E,G_n)$. We say that the family $T = \{v_n\}$ of bounded linear operator is a $X_\Phi$-Bessel sequence for $E$ with respect to $X_\Phi$ if there exists a positive constant $B$ such that

$$\|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \text{ for all } x \in E.$$ 

Define

$$B_T = \inf \left\{ B > 0 : \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \forall x \in E \right\}.$$ 

We call $B_T$ the Bessel bound of $T$.

For any $T = \{v_n\}$, define $R_T : E \to X_\Phi$ such that

$$R_T(x) = \{v_n(x)\}, \text{ for all } x \in E.$$ 

Then, we call $R_T$ is the analysis operator of $T$. Clearly, from above definition we have $R_T \in L(E,X_\Phi)$.

**Definition 4.2.2.** Let $\Phi = \{G_n\}$ be a sequence of non-trivial subspaces of a Banach space $E$ and $\{v_n : v_n \in L(E,G_n), \forall n \in \mathbb{N}\}$ be a sequence of linear operators (not necessarily projections). Let $X_\Phi$ be a model space associated with $E$. Then, we say that $(\{G_n\}, \{v_n\})$ is a $X_\Phi$-frame for $E$ with respect to $X_\Phi$ if

(a) $\{v_n(x)\} \in X_\Phi$, for all $x \in E$

(b) there exist constants $A, B$ with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{X_\Phi} \leq B\|x\|_E, \text{ for all } x \in E.$$ 

The positive constants $A$ and $B$, respectively, are called lower and upper bounds for the $X_\Phi$-frame $(\{G_n\}, \{v_n\})$. 

Put
\[ A_T = \sup \left\{ A > 0 : A \|x\|_E \leq \left\| \left\{ v_n(x) \right\}_{x \in E} \right\|_{X_{\phi}}, \forall x \in E \right\} \]

\[ B_T = \inf \left\{ B > 0 : \left\| \left\{ v_n(x) \right\}_{x \in E} \right\|_{X_{\phi}} \leq B \|x\|_E, \forall x \in E \right\}. \]

These constants \( A_T, B_T \) are called lower and upper optimum bounds of \( T = \{v_n\} \).

**Definition 4.2.3.** A \( X_\phi \)-frame for \( E \) with respect to \( X_\phi \), is said to be a tight frame if \( A_T = B_T \).

In particular, if \( A_T = B_T = 1 \), then we say that \( T \) is a Parseval (or Normalized tight) \( X_\phi \)-frame for \( E \) with respect to \( X_\phi \).

**Definition 4.2.4.** Let \( T = \{v_n\} \) be a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) induced by \( \{G_n\} \). If there exists a bounded linear operator \( S : X_\phi \rightarrow E \) such that

\[ S \left( \left\{ v_n(x) \right\} \right) = x, \text{ for all } x \in E. \]

Then, we say that \( \left( \{v_n\}, S \right) \) is \( X_\phi \)-Banach frame for \( E \) with respect to \( X_\phi \), where the operator \( S \) is called a reconstruction operator for \( T \).

**Definition 4.2.5.** Let \( E \) and \( \{G_n\} \) be the Banach spaces, \( v_n \in L(E, G_n), \forall n \in \mathbb{N} \).

A family of operators \( T = \{v_n\} \) is called total if \( \left\{ x \in E : v_n(x) = 0, \forall n \in \mathbb{N} \right\} = \{0\} \).

**4.3 Examples and Counter-examples.**

In this section, we give examples and counter-examples to exhibits relation among various types of \( X_\phi \)-frames.

**Example 4.3.1.** Let \( E \) be a Banach space, \( E_d \) be a associated Banach space of scalar valued sequences indexed by \( \mathbb{N} \) and let \( \{f_n\} \subset E^\ast \) be a \( E_d \)-frame for \( E \).
with respect to $E_d$. Assume that

$$A\|x\|_E \leq \\| \{ f_n(x) \} \|_X \leq B\|x\|_E, \quad \text{for all } x \in E.$$  

Define functionals $\{ v_n \}$ as follows

$$v_n(x) = f_n(x), \quad \forall x \in E.$$  

Then, we have

$$A\|x\|_E \leq \| \{ v_n(x) \} \|_{X_{\phi}} = \| \{ f_n(x) \} \|_{X_{\phi}} \leq B\|x\|_E, \quad \text{for all } x \in E.$$  

Thus, $\{ v_n \}$ is a $X_{\phi}$-frame for $E$ with respect to $E_d$.

**Example 4.3.2.** Let $E$ be a Banach space defined as

$$E = l^2(\chi) = \left( \{ x_n \} : x_n \in \chi; \sum_{n=1}^{\infty} \| x_n \|_{\chi}^2 < \infty \right),$$

where $(\chi, \| . \|)$ is a Banach space, equipped with the norm given by

$$\| (x_n) \|_E = \left( \sum_{n=1}^{\infty} \| x_n \|_{\chi}^2 \right)^{\frac{1}{2}}.$$

We define for $n \in \mathbb{N}$,

$$G_n = \left\{ \delta_{i}^{x} + \delta_{n+1}^{x} : x \in \chi \right\}$$

and

$$v_n(x) = \delta_{i}^{\delta_{n+1}^{x} + \delta_{n+1}^{x}}, \quad x = \{ x_n \} \in E,$$

where $\delta_{i}^{x} = (0,0,\ldots,x_{\delta_{n+1}^{x}},0,0,\ldots)$ for all $n \in \mathbb{N}$ and $x \in \chi$.

But, since for any $0 \neq x \in \chi$, $\delta_{2}^{x} = (0,x,0,\ldots) \in E$ is such that

$$v_n(\delta_{2}^{x}) = 0, \quad \forall n \in \mathbb{N},$$

there exist no associated model space $X$ such that $\left( \{ G_n \}, \{ v_n \} \right)$ is a $X_{\phi}$-frame for $E$ with respect to $X$.  

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4.4 Some Characterizations on $X_{\phi}$-Bessel Sequences and $X_{\phi}$-Frames for Banach Spaces.

In this section, regarding consequence of definitions given in section 4.2, we give some characterizations on $X_{\phi}$-Bessel sequences and $X_{\phi}$-frames for Banach spaces. Also, give the necessary and sufficient condition for a $X_{\phi}$-Bessel sequence to be a $X_{\phi}$-frame.

**Theorem 4.4.1.** Let $T = \{v_n\}$ be a $X_{\phi}$-Bessel sequence for $E$ with respect to a model space $X_{\phi}$ induced by $\{G_n\}$. Then, $T = \{v_n\}$ is a $X_{\phi}$-frame if and only if its analysis operator $R_T$ is bounded below.

**Proof.** Given $T = \{v_n\}$ is a $X_{\phi}$-Bessel sequence for $E$ with respect to a model space $X_{\phi}$. Then, there exists $B > 0$, such that

$$\left\| \{v_n(x)\} \right\|_{X_{\phi}} \leq B \left\| x \right\|_E, \quad \forall x \in E. \quad (4.1)$$

Let us assume that $T = \{v_n\}$ is a $X_{\phi}$-frame for $E$ with respect to a model space $X_{\phi}$ and with bounds $A, B$. Then, we have

$$A \left\| x \right\|_E \leq \left\| \{v_n(x)\} \right\|_{X_{\phi}} \leq B \left\| x \right\|_E, \quad \forall x \in E. \quad (4.2)$$

Let $R_{T}: E \rightarrow X_{\phi}$ be the analysis operator such that

$$R_{T}(x) = \{v_n(x)\}, \quad \forall x \in E.$$

Now

$$\left\| R_{T}(x) \right\|_{X_{\phi}} = \left\| \{v_n(x)\} \right\|_{X_{\phi}} \geq A \left\| x \right\|_E, \quad \text{by (4.2)}$$

Therefore,

$$\left\| R_{T}(x) \right\|_{X_{\phi}} \geq A \left\| x \right\|_E, \quad \forall x \in E.$$

Thus, $R_{T}$ is bounded below.
Conversely, let us assume that $R_T$ is bounded below, i.e. there exists $A > 0$ such that

$$
\left\| R_T(x) \right\|_{X_\phi} \geq A \left\| x \right\|_E, \forall \, x \in E.
$$

or

$$
\left\{ \left\| v_n(x) \right\| \right\}_{X_\phi} \geq A \left\| x \right\|_E, \forall \, x \in E. \tag{4.3}
$$

By using (4.1) and (4.3), we have

$$
A \left\| x \right\|_E \leq \left\{ v_n(x) \right\}_{X_\phi} \leq B \left\| x \right\|_E, \forall \, x \in E.
$$

Thus, $T = \{v_n\}$ is a $X_\phi$-frame for $E$ with respect to a model space $X_\phi$.

The following theorem, gives the necessary and sufficient condition for a Banach space to have a $X_\phi$-frame.

**Theorem 4.4.2.** A Banach space $E$ has a $X_\phi$-frame with respect to a model space $X_\phi$ induced by $\{G_n\}$ if and only if $E$ is an isomorphic to a subspace $M$ of $X_\phi$.

**Proof:** Assume that $T = \{v_n\}$ be a $X_\phi$-Bessel sequence for $E$ with respect to a model space $X_\phi$ induced by $\{G_n\}$. Then,

(a) By Theorem 4.4.1, the analysis operator $R_T$ of $\{v_n\}$ is bounded below.

(b) Now we check $R_T$ is one-to-one.

Let $x \in \ker(R_T)$

$$
\Rightarrow R_T(x) = 0, \ x \in E.
$$

$$
\Rightarrow \{v_n(x)\} = 0, \ x \in E.
$$

Therefore, by using $X_\phi$-frame inequality, we get

$$
x = 0.
$$

It means that, $\ker(R_T) = \{0\} \Rightarrow R_T$ is one-to-one.

(c) $R_T$ has a closed range, i.e., $\text{ran}(R_T)$ is closed.

Let $\{u_n(x)\} \subset \text{ran}(R_T)$ be the sequence conversing to $u \in X_\phi$, i.e.,
\[
\lim_{n \to \infty} u_n(x) = u, \quad \forall n \in \mathbb{N}.
\]
Let \( \{x_n\} \subset E \) be the sequence such that
\[
R_r(x_n) = u_n(x), \quad \forall n \in \mathbb{N}.
\]
Then, \( \{R_r(x_n)\} \) is a Cauchy sequence. Since \( R_r^{-1} \) is continuous, \( \{x_n\} \) is Cauchy sequence in \( E \).
Therefore, there is a \( x \in E \), such that
\[
\lim_{n \to \infty} x_n = x.
\]
Since \( R_r \) is continuous, we have
\[
\lim_{n \to \infty} R_r(x_n) = R_r(x) \in \text{ran}(R_r).
\]
This gives,
\[
\lim_{n \to \infty} u_n(x) = u - R_r(x) \in \text{ran}(R_r).
\]
\[
\Rightarrow \{u_n(x)\} \rightarrow u \in \text{ran}(R_r).
\]
Hence, \( \text{ran}(R_r) \) is a closed subspace of \( X_\phi \).
From the Inverse Mapping theorem, we know that \( E \) is an isomorphic to \( \text{ran}(R_r) \), which is the subspace of \( X_\phi \).

Conversely, assume that \( M \) is the subspace of \( X_\phi \) and \( U \) is an isomorphism from \( E \) onto \( M \).
Let \( \{P_n\} \) be the sequence of coordinate operators on \( X_\phi \), then
\[
P_n \{y_m\} = y_n, \quad \text{for all} \ n \in \mathbb{N}.
\]
Define, \( v_n = P_nU, \forall n \in \mathbb{N} \), then \( v_n : E \to G_n \) is linear and bounded for every \( n \in \mathbb{N} \), such that
\[
\{v_n(x)\} = \{P_nU(x)\} \in X_\phi, \quad \text{for all} \ x \in E.
\]
Then, for all \( x \in E \), we have
\[
\|x\| = \|U^{-1}Ux\| \leq \|U^{-1}\|\|Ux\|
\]
Therefore,
Thus, \( \{v_n\} \) is a \( X_\phi \)-frame for \( E \) with respect to \( X_\phi \).

### 4.5 Characterization of \( X_\Phi \)-Frames Under Bounded Linear Operators

Similar to frames and \( g \)-frames [82], the following proposition show that the image of \( X_\phi \)-frame under bounded linear operator is also a \( X_\phi \)-frame.

**Proposition 4.5.1.** Let \( T = \{v_n\} \) be a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) induced by \( \{G_n\} \) and with bounds \( A_T, B_T \). Let \( W \) be a bounded invertible operator on \( E \) and \( u_n = v_n W \), for all \( n \in \mathbb{N} \). Then, \( \{u_n\} \) is a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) and with bounds \( A_T \|W^{-1}\|^{-1} \) and \( B_T \|W\| \).

**Proof.** Let \( T = \{v_n\} \) is a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) induced by \( \{G_n\} \) and with bounds \( A_T, B_T \). Then, we have

\[
A_T \|x\|_E \leq \left\| \{v_n(x)\} \right\|_{X_\phi} \leq B_T \|x\|_E, \quad \forall x \in E.
\] (4.4)

and \( W : E \to E \) is a bounded operator such that

\[
W(x) = x, \quad \forall x \in E.
\]

Now replace \( x \) by \( Wx \) in (4.4), we get

\[
A_T \|Wx\|_E \leq \left\| \{v_n W(x)\} \right\|_{X_\phi} \leq B_T \|Wx\|_E, \quad \forall x \in E.
\]

or

\[
A_T \|Wx\|_E \leq \left\| \{u_n(x)\} \right\|_{X_\phi} \leq B_T \|Wx\|_E, \quad \forall x \in E.
\]

Since \( W \) is invertible, then

\[
A_T \|W^{-1}\|^{-1} \|x\|_E \leq \left\| \{u_n(x)\} \right\|_{X_\phi} \leq B_T \|W\|\|x\|_E, \quad \forall x \in E.
\]

Hence, \( \{u_n\} \) is a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) and with
bounds $A_r \| W^{-1} \|^{-1}$ and $B_r \| W \|$. 

**Corollary 4.5.2.** Let $T = \{ v_n \}$ be a $X_\phi$-frame for $E$ with respect to a model space $X_\phi$ induced by $\{ G_n \}$ and with bounds $A_r, B_r$. Let $W : E \to E$ be an isometry and $u_n = v_n W$, for all $n \in \mathbb{N}$. Then, $\{ u_n \}$ is a $X_\phi$-frame for $E$ with respect to a model space $X_\phi$ and with bounds $A_r \| W^{-1} \|^{-1}$ and $B_r \| W \|$. 

The following theorem give the necessary and sufficient condition for the image of $X_\phi$-frame under bounded linear operator is also a $X_\phi$-frame.

**Theorem 4.5.3.** Let a Banach space $E$ has a $X_\phi$-frame with respect to a model space $X_\phi$ induced by $\{ G_n \}$ and $W : E \to E$ be a bounded operator. Then, $\{ v_n W \}$ is a $X_\phi$-frame for $E$ if and only if $W$ is a bounded below.

**Proof.** Let $T = \{ v_n \}$ be a $X_\phi$-frame for $E$ with respect to a model space $X_\phi$ induced by $\{ G_n \}$ and with bounds $A, B$. Then, we have

$$A \| x \|_E \leq \| (v_n(x)) \|_{X_\phi} \leq B \| x \|_E, \quad \forall x \in E. \quad (4.5)$$

and $W : E \to E$ is a bounded operator such that

$$W(x) = x, \quad \forall x \in E.$$

Let us assume that $\{ v_n W \}$ is a $X_\phi$-frame for $E$ with respect to a model space $X_\phi$ and with bounds $m, n$. Then, we have

$$m \| x \|_E \leq \| (v_n W(x)) \|_{X_\phi} \leq n \| x \|_E, \quad \forall x \in E. \quad (4.6)$$

Now replace $x$ by $W(x)$ in (4.5), we get

$$A \| W(x) \|_E \leq \| (v_n W(x)) \|_{X_\phi} \leq B \| W(x) \|_E, \quad \forall x \in E. \quad (4.7)$$

By using (4.6) and (4.7), we have
$$m\|x\|_E \leq B\|W(x)\|_E, \quad \forall x \in E.$$  

or  

$$\|W(x)\|_E \geq \frac{m}{B}\|x\|_E, \quad \forall x \in E.$$  

Put \( \frac{m}{B} = \delta \).  

Thus for each \( x \in E \), we have  

$$\|W(x)\|_E \geq \delta\|x\|_E.$$  

Therefore, \( W \) is bounded below.  

Conversely, suppose \( W \) is bounded below, there exists \( \delta > 0 \), such that for each \( x \in E \), we have  

$$\|W(x)\|_E \geq \delta\|x\|_E \quad (4.8)$$  

Now,  

$$A\delta\|x\|_E \leq A\|W(x)\|_E \quad \text{by (4.8)}$$  

$$\leq \left\| \{v_n W(x)\} \right\|_{X_\phi} \quad \text{by (4.7)}$$  

$$\leq B\|W(x)\|_E$$  

$$\leq B\|W\|\|x\|_E.$$  

Therefore,  

$$A\delta\|x\|_E \leq A\left\| \{v_n W(x)\} \right\|_{X_\phi} \leq B\|W\|\|x\|_E, \quad x \in E.$$  

Hence, \( \{v_n W\} \) is a \( X_\phi \)-frame for \( E \) with respect to a model space \( X_\phi \) and with bounds \( A\delta \) and \( B\|W\| \), respectively.
4.6 Characterization of $X_\phi$-Banach Frame for Banach Space.

In this section, we shall give a necessary and sufficient condition for a Banach space $E$ which possessing $X_\phi$-Banach frames.

**Theorem 4.6.1.** A Banach space $E$ has a $X_\phi$-Banach frame with respect to a model space $X_\phi$ if and only if $E$ is an isomorphic to a complemented subspace of $X_\phi$.

**Proof.** Assume that $(\{v_n\},S)$ is $X_\phi$-Banach frame for $E$ with respect to model space $X_\phi$ and $R_T$ is the analysis operator of $T = \{v_n\}$. For any $x \in E$, we have $SR_T x = x$, i.e., $SR_T = I_E$.

Let $P = R_T S$, then we see that

$$P^* = P^\ast = P^T P = (R_T S) (R_T S) = R_T (SR_T) S = R_T (I_S) S = R_T S = P,$$

and

$$\text{ran} (P) = \text{ran} (R_T).$$

So $P$ is the projection from $X_\phi$ to $\text{ran} (R_T)$.

This implies that $R_T : E \to \text{ran} (R_T)$ is an isomorphism and $\text{ran} (R_T)$ is a complemented subspace of $X_\phi$.

Conversely, assume that $U : E \to M$ is an isomorphism, where $M$ is a complemented subspace of $X_\phi$ and $\{P_n\}$ be the sequence of coordinate operators on $X_\phi$.

Then,

$$\{P_n (y)\} \in X_\phi, \forall y \in M.$$
From the proof of the converse part of Theorem 4.4.2, the formula
\[ v_n = PU_n, \quad \forall n \in \mathbb{N} \]
define a \( X_{\Phi} \)-frame \( \{v_n\} \) for \( E \) with respect to \( X_{\Phi} \).

Since \( M \) is complemented, it follows that there exists a projection \( P \) from \( X_{\Phi} \) onto \( M \).

Put \( S = U^{-1}P \), then \( S \) is the bounded linear operator from \( X_{\Phi} \) to \( E \) satisfying
\[ S\left(\{v_n(x)\}\right) = U^{-1}P\left(\{PU_n(x)\}\right) = U^{-1}PU_n(x) = x, \forall x \in E. \]

This show that \( \{v_n\}, S \) is \( X_{\Phi} \)-Banach frame for \( E \) with respect to the model space \( X_{\Phi} \).

### 4.7 Dual of \( X_{\Phi} \)-Frames

In this section, we will study and discuss about dual of \( X_{\Phi} \)-frames for Banach space \( E \) with respect to \( X_{\Phi} \). Let \( X_{\Phi} \) be a model space induced by \( \{G_n\} \).

Consider the sequence space
\[ T\left(\{G_n^*\}\right) = \left\{ \{y_n^*\} \in \prod_{n=1}^{\infty} G_n^*: \sum_{n=1}^{\infty} y_n^*(y_n) \text{ converges, } \forall \{y_n\} \in X_{\Phi} \right\}. \]

This is a Banach space equipped with the following norm
\[ \left\|\{y_n^*\}\right\| = \sup_{\|y_n\| \leq 1} \sup_{1 \leq i < \infty} \left\{ \sum_{n=1}^{\infty} \langle y_n^*, y_i \rangle \right\}, \quad \forall \{y_n^*\} \in T\left(\{G_n^*\}\right). \]

Then, \( T\left(\{G_n^*\}\right) \) is an isomorphic to the dual of \( X_{\Phi} \) under the mapping
\[ \{y_n^*\} \rightarrow h, \quad \text{where} \]
\[ h\left(\{y_n\}\right) = \sum_{n=1}^{\infty} y_n^* y_n, \quad \{y_n\} \in X_{\Phi}, \quad h \in X_{\Phi}^*. \]

Thus, the space \( X_{\Phi}^* \) is also a generalized \( BK \)-space.

Hence, for any \( \{y_n\} \in X_{\Phi} \) and \( \{y_n^*\} \in X_{\Phi}^* \), we have \( \left\{ \{y_n^*\}, \{y_n\}\right\} = \sum_{n=1}^{\infty} y_n^* y_n. \)
Definition 4.7.1. Let $T = \{v_n\}$ be a $X_\Phi$-frame for $E$ with respect to $X_\Phi$ and $Q = \{u_n\}$ be a $X_\Phi^*$-frame for $E^*$ with respect to $X_\Phi^*$. If these two $X_\Phi$-frames satisfy the following conditions:

$$x = \sum_{n=1}^{\infty} u_n^* v_n x, \quad \forall x \in E. \quad (4.9)$$

$$x^* = \sum_{n=1}^{\infty} v_n^* u_n x^*, \quad \forall x^* \in E^*. \quad (4.10)$$

Then, we call $(T, Q)$ is a pair of dual frames for $E$. Here, one of them is called a dual frame of others.

Let $T = \{v_n\}$ be a $X_\Phi$-Bessel sequence for $E$ with respect to $X_\Phi$ and $R_T$ be an analysis operator of $T = \{v_n\}$. Then, we have

$$R_T (x) = \{v_n (x)\}, \quad \forall x \in E.$$  

It is easy to see that adjoint operator $R_T^*$ of $R_T$ is defined as follows;

$$R_T^* : X_\Phi^* \rightarrow E^*, \text{ such that}$$

$$R_T^* \left( \left\{ y_n^* \right\} \right) = \sum_{n=1}^{\infty} v_n^* y_n, \quad \forall \left\{ y_n^* \right\} \in X_\Phi^*.$$  

Indeed, suppose that $R_T$ and $R_Q$ are the analysis operators of $T = \{v_n\}$ and $Q = \{u_n\}$ respectively, then from (4.9) and (4.10) we find that

$$R_Q^* R_T x = R_Q^* \left( \left\{ v_n (x) \right\} \right)$$

$$= \sum_{n=1}^{\infty} u_n^* v_n x$$

$$= x$$

$$= I_E x, \quad \forall x \in E.$$  

$$\Rightarrow \quad R_Q^* R_T = I_E.$$  

Similarly, we find that
\[ R^*_f R^*_Q x = R^*_f \left( \{ u_n(x) \} \right) \]
\[ = \sum_{n=1}^{\infty} v^*_n u_n x^* \]
\[ = x^* \]
\[ = I_{E^*} x^*, \quad \forall x^* \in E^*. \]
\[ \Rightarrow R^*_f R^*_Q = I_{E^*}. \]

**Definition 4.7.2.** A family \( T = \{ v^*_n \} \) of operators, where \( v^*_n \in L(E, G_n) \ \forall n \in \mathbb{N} \), is said to be an independent, if the following condition is satisfied:

\[ \sum_{n=1}^{\infty} v^*_n x^*_n = 0, \quad \{ x^*_n \} \in X^*_\phi \Rightarrow x^*_n = 0, \quad \forall n \in \mathbb{N}. \]

Here, we give the necessary and sufficient condition for a \( X^*_\phi \)-Bessel sequence to be an independent \( X^*_\phi \)-frame.

**Theorem 4.7.3.** Let \( T = \{ v^*_n \} \) be a \( X^*_\phi \)-Bessel sequence for \( E \) with respect to \( X^*_\phi \). Then, \( \{ v^*_n \} \) is an independent \( X^*_\phi \)-frame if and only if its analysis operator \( R^*_f \) is invertible.

**Proof.** Assume that \( \{ v^*_n \} \) is an independent \( X^*_\phi \)-frame. Then, we have

\[ \sum_{n=1}^{\infty} v^*_n x^*_n = 0, \quad \{ x^*_n \} \in X^*_\phi \Rightarrow x^*_n = 0, \quad \forall n \in \mathbb{N}. \]

In order to show that \( R^*_f \) is invertible. We first to show that \( R^*_f \) is an injective. So let

\[ \{ x^*_n \} \in \ker \left( R^*_f \right) \]
\[ \Rightarrow R^*_f \left( \{ x^*_n \} \right) = 0. \]
\[ \Rightarrow \sum_{n=1}^{\infty} v^*_n x^*_n = 0. \quad \text{(By definition of } R^*_f \text{)} \]
\[ \Rightarrow x^*_n = 0, \quad \forall n \in \mathbb{N}. \]

Thus,
\[ \ker(R^*_f) = \{0\}, \]
\[ \Rightarrow R^*_f \text{ is injective.} \]

Now, we see that
\[ \overline{\text{ran}(R_f)} = \left( \ker(R^*_f) \right)^\perp = \{0\}^\perp = X_\varphi. \]

Therefore, range of \( R_f \) is dense in \( X_\varphi \).

In addition, from the definition of \( X_\varphi \)-frames, we know that \( R_f \) is bounded below and \( \text{ran}(R_f) \) is closed. (It is proved in the Theorem 4.4.2).

Hence \( R_f \) is invertible.

Conversely, let us assume that \( R_f \) is invertible. Then \( R_f \) is bounded below. Thus by Theorem 4.4.1, \( \{v_n\} \) is a \( X_\varphi \)-frame.

In order to prove that \( \{v_n\} \) is an independent \( X_\varphi \)-frame.

Let, if possible, \( \{v_n\} \) is not an independent \( X_\varphi \)-frame. Then, there exists a non-zero sequence \( \{y_n^*\} \subseteq X_\varphi^* \) (assume \( n_0 \in \mathbb{N}, y^*_{n_0} \neq 0 \)) such that
\[ v^*_{n_0} y^*_{n_0} + \sum_{n \neq n_0} v^*_{n} y^*_{n} = 0. \]
\[ v^*_{n_0} y^*_{n_0} = -\sum_{n \neq n_0} v^*_{n} y^*_{n}. \quad (4.11) \]

Since \( y^*_{n_0} \neq 0 \), there exists \( y_{n_0} \in G_{n_0} \) such that
\[ y^*_{n_0} \left( y_{n_0} \right) \neq 0. \quad (4.12) \]

Again, since \( R_f \) is invertible, there is \( x \in E \) such that
\[ R_f(x) = \{\delta_{n_0}, y_{n_0}\}, \text{ i.e.,} \]
\[ v_n(x) = \delta_{n_0}, \quad \forall n \in \mathbb{N}. \quad (4.13) \]

From L.H.S. of (4.11), we have
\[ \langle v^*_{n_0}, y^*_{n_0}, x \rangle = \langle y^*_{n_0}, v_{n_0} x \rangle \]
\[ = y^*_{n_0} \left( v_{n_0}(x) \right) \]
Thus, we get
\[
\langle v_{n_0}^*, y_{n_0}^*, x \rangle \neq 0.
\] (4.14)

Again, from R.H.S. of (4.11), we have
\[
\begin{align*}
&\left\langle -\sum_{n \neq n_0} v_n^*, y_n^*, x \right\rangle = -\sum_{n \neq n_0} \langle v_n^*, y_n^*, x \rangle \\
&\quad = -\sum_{n \neq n_0} \langle y_n^*, v_n x \rangle \\
&\quad = -\sum_{n \neq n_0} y_n^* v_n x \\
&\quad = -\sum_{n \neq n_0} y_n^* \delta_{n n_0} y_n \\
&\quad = 0.
\end{align*}
\]

Thus, we get
\[
\left\langle -\sum_{n \neq n_0} v_n^*, y_n^*, x \right\rangle = 0.
\] (4.15)

Therefore, from (4.14) and (4.15). We get a contradiction.
Hence, \( \{ v_n \} \) is an independent \( X_\phi \)-frame.

**Theorem 4.7.4.** An independent \( X_\phi \)-frame \( \{ v_n \} \) for \( E \) with respect to \( X_\phi \) must have a dual frame.

**Proof.** Let \( T = \{ v_n \} \) be an independent \( X_\phi \)-frame for \( E \) with respect to \( X_\phi \). Then from Theorem 4.7.3, its analysis operator \( R_T^* \) is invertible. Thus by Proposition 1.3.9, \( R_T^* \) is invertible.

For any \( n \in \mathbb{N} \), put \( u_n = P_n R_T^{-1} \), where \( P_n \) is the coordinate operator on \( X_\phi^* \). Then, \( u_n^* \in L\left( E^*, G_n^* \right) \), \( \forall n \in \mathbb{N} \) and for any \( x^* \in E^* \), there exists a sequence \( \{ y_n^* \} \in X_\phi^* \).
such that
\[ x^* = \sum_{n=1}^{\infty} v_n^* y_n^*. \]

So
\[ \{ u_n(x^*) \} = \{ P_n R_T^{-1} x^* \} = \{ P_n \{ y_n^* \} \} = \{ y_n^* \} \in X_\phi^* \]

and for all \( x^* \in E^* \), we have
\[ \| x^* \| = \left\| R_T^{-1} R_T^* (x^*) \right\| \leq \left\| R_T^{-1} \right\| \left\| R_T^* (x^*) \right\| \leq \left\| R_T^{-1} \right\| \| x^* \|.
\]

Therefore,
\[ \left\| \frac{x^*}{\| R_T^* \|} \right\| \leq \left\| \left\{ u_n(x^*) \right\} \right\|_{X_\phi} = \left\| \left\{ P_n R_T^{-1} (x^*) \right\} \right\|_{X_\phi} = \left\| \left\{ R_T^{-1} (x^*) \right\} \right\|_{X_\phi} \leq \left\| R_T^{-1} \right\| \left\| x^* \right\|.
\]

Hence, \( Q = \{ u_n \} \) is a \( X_\phi \)-frame for \( E^* \) with respect to \( X_\phi^* \).

For any \( x \in E \) and \( x^* \in E^* \), we have
\[
\left\langle \sum_{n=1}^{\infty} v_n^* u_n, x \right\rangle = \sum_{n=1}^{\infty} \left\langle v_n^* u_n, x^* \right\rangle = \sum_{n=1}^{\infty} \left\langle u_n, x^* \right\rangle = \sum_{n=1}^{\infty} \left\langle P_n R_T^{-1} u_n, x^* \right\rangle = \sum_{n=1}^{\infty} \left\langle P_n \{ y_n^* \}, x^* \right\rangle = \sum_{n=1}^{\infty} \left\langle y_n^*, x^* \right\rangle = \sum_{n=1}^{\infty} \left\langle y_n^*, x \right\rangle = \left\langle x^*, x \right\rangle.
\]

and
\[
\left\langle x^*, \sum_{n=1}^{\infty} u_n^* v_n \right\rangle = \sum_{n=1}^{\infty} \left\langle x^*, u_n^* v_n \right\rangle = \sum_{n=1}^{\infty} \left\langle u_n x^*, v_n x \right\rangle.
\]
\[ = \sum_{n=1}^{\infty} \langle P_n R_n^{-1} u_n x^*, v_n x \rangle \]
\[ = \sum_{n=1}^{\infty} \langle P_n \{ y_n^* \}, v_n x \rangle \]
\[ = \sum_{n=1}^{\infty} \langle y_n^*, v_n x \rangle \]
\[ = \left( \sum_{n=1}^{\infty} v_n^* y_n^* x \right) \]
\[ = \langle x^*, x \rangle. \]

From above, we must have
\[ x = \sum_{n=1}^{\infty} u_n^* v_n x, \quad \forall x \in E \]
and
\[ x^* = \sum_{n=1}^{\infty} v_n^* u_n x^*, \quad \forall x^* \in E^*. \]

This show that \( Q = \{u_n\} \) is a dual frame of \( T = \{v_n\} \), i.e., \( (T, Q) \) is called a pair of dual frames for \( E \).

### 4.8 \( X_{\Phi^*} \)-Riesz Bases for Banach Spaces

In this section, we will study \( X_{\Phi^*} \)-Riesz bases for Banach spaces and related concepts with an independent \( X_{\Phi^*} \)-frames. We begin this section by giving the following definition.

**Definition 4.8.1.** A family of operators \( T = \{v_n\} \) is called a \( X_{\Phi^*} \)-Riesz bases for Banach space \( E \) with respect to \( X_{\Phi} \) if it satisfies:

(a) \( \{v_n\} \) is total and

(b) there exist constants \( C, D \) with \( 0 < C \leq D < \infty \) such that

\[ C \left\| \{y_n^*\} \right\|_{X_{\Phi^*}} \leq \left\| \sum_{n=1}^{\infty} v_n^* y_n^* \right\|_{X_{\Phi^*}} \leq \left\| \{y_n^*\} \right\|_{X_{\Phi^*}}, \quad \forall \{y_n^*\} \in X_{\Phi^*}. \]
The positive constants $C$ and $D$, respectively, are called lower and upper bounds for the $X_\phi$-Riesz bases for $E$.

Immediately, here we know that every $X_\phi$-Riesz bases is always a $X_\phi$-frame. It is clear from the following conclusion.

**Theorem 4.8.2.** Assume that $E$ and $\{G_n\}$ are Banach spaces, $v_n \in L(E,G_n)$ $\forall n \in \mathbb{N}$ and $T = \{v_n\}$ be a $X_\phi$-Bessel sequence for $E$ with respect to $X_\phi$. Then the following statements are equivalent:

1. $\{v_n\}$ is an independent $X_\phi$-frame for $E$ with respect to $X_\phi$.
2. $\{v_n\}$ is a $X_\phi$-Riesz bases for $E$ with respect to $X_\phi$.

**Proof.** Assume that (1) holds. Let $R_T$ is an analysis operator of $T = \{v_n\}$. By Theorem 4.7.3., $R_T$ is an invertible. So $R_T^*$ is an invertible and for any $\{y^*_n\} \in X_\phi^*$, we have

$$R_T^*(\{y^*_n\}) = \sum_{n=1}^{\infty} y^*_n v_n.$$ 

Now, again for any $\{y^*_n\} \in X_\phi^*$, we have

$$\|\{y^*_n\}_{X_\phi^*}\|_{X_\phi} = \|R_T^{-1} R_T^* \{y^*_n\}\|_{X_\phi} \leq \|R_T^{-1}\| \|R_T^* \{y^*_n\}\|_{E^*}.$$ 

Thus, we get

$$\|\{y^*_n\}_{X_\phi^*}\|_{X_\phi} \leq \|R_T^{-1}\| \|R_T^* \{y^*_n\}\|_{E^*}.$$

Put $C = \|R_T^{-1}\|^{-1}$, $D = \|R_T^*\|$. Then, we have

$$C \|\{y^*_n\}_{X_\phi^*}\|_{X_\phi} \leq \|R_T^* \{y^*_n\}\|_{E^*}.$$
Therefore, we have
\[
C \left\| \left\{ \gamma_n^* \right\}_{n=1}^{\infty} \right\|_{X^*_{\phi}} \leq \left\| \sum_{n=1}^{\infty} \gamma_n^* \gamma_n^* \right\|_{E^*} \leq D \left\| \left\{ \gamma_n^* \right\}_{n=1}^{\infty} \right\|_{X^*_{\phi}}.
\]
Again, \( R_T^* \) is invertible, the \( \text{ran} \left( R_T^* \right) \) is dense in \( E^* \), i.e.
\[
\left( \text{ran} \left( R_T^* \right) \right) \perp = \text{ker} \left( R_T^* \right) = \{0\}.
\]
Thus, \( \{v_n \} \) is total. Hence, (2) is hold.

Conversely, assume that (2) holds. Define an operator \( Q \) as follows

\[
Q : X^*_\phi \to E^*,
\]

such that

\[
Q \left( \left\{ \gamma_n^* \right\} \right) = \sum_{n=1}^{\infty} \gamma_n^* \gamma_n^*.
\]

From the definition of \( X^*_\phi \)-Riesz bases, we know that \( Q \) is well defined, bounded linear and invertible.

By computing, we can get
\[
Q^* (x) = \left\{ v_n (x) \right\} \in X^*_\phi, \quad \forall x \in E.
\]

and
\[
\left\| \left\{ v_n (x) \right\} \right\|_{X^*_\phi} = \left\| Q^* (x) \right\|_{X^*_\phi} \leq \left\| Q^* \right\| \| x \|, \quad \forall x \in E.
\]

Let \( B = \| Q^* \| \), then we get
\[
\left\| \left\{ v_n (x) \right\} \right\|_{X^*_\phi} \leq B \| x \|, \quad \forall x \in E.
\]

This show that \( \{v_n \} \) is a \( X^*_\phi \)-Bessel sequence for \( E \) with respect to \( X^*_\phi \) and with bound \( B = \| Q^* \| \).

Suppose \( R_T \) is the analysis operator of \( \{v_n \} \). It is easy to check that

\[
R_T^* = Q.
\]
Since $Q$ is invertible this implies $R_f$ is invertible. Thus, by Theorem 4.7.3, we get (1).

Now, we give a necessary and sufficient condition for a Banach space have to a $X_\phi$-Riesz bases.

**Theorem 4.8.3.** A Banach space $E$ has a $X_\phi$-Riesz bases with respect to $X_\phi$ if and only if $E$ is an isomorphic to $X_\phi$.

**Proof.** Assume that $T = \{v_n\}$ is a $X_\phi$-Riesz bases for $E$ with respect to $X_\phi$. Then by Theorem 4.8.2, $\{v_n\}$ is an independent $X_\phi$-frame for $E$ with respect to $X_\phi$ and Theorem 4.7.3, implies that its analysis operator $R_f$ is an invertible. Hence, $E$ is an isomorphic to $X_\phi$.

Conversely, let $U$ is an isomorphic from $E$ to $X_\phi$. From the proof of the converse part of the Theorem 4.4.2.

Let $\{P_n\}$ be the sequence of coordinate operators on $X_\phi$, then

$$P_n(\{y_m\}) = y_n, \text{ for all } n \in \mathbb{N}.$$  

Define, $v_n = P_nU$, then $v_n : E \rightarrow G_n$ is linear and bounded for every $n \in \mathbb{N}$, such that

$$\{v_n(x)\} = \{P_nU(x)\} \in X_\phi, \text{ for all } x \in E.$$  

Then, for all $x \in E$, we have

$$\|x\| = \|U^{-1}(x)\| \leq \|U^{-1}\|\|U(x)\|.$$  

Therefore,

$$\frac{\|x\|}{\|U^{-1}\|} \leq \|v_n(x)\|_{X_\phi} = \|P_nU(x)\|_{X_\phi} = \|U(x)\|_{X_\phi} \leq \|U\|\|x\|.$$  

Thus, $\{v_n\}$ is a $X_\phi$-frame for $E$ with respect to $X_\phi$.

Suppose $R_f$ is the analysis operator of $\{v_n\}$. For any $x \in E$, we have
\[ R_r(x) = \{ v_n(x) \} = \{ P_n U(x) \} = U(x). \]

Thus, \( R_r = U \) and so \( R_r \) is invertible. By Theorem 4.7.3, \( \{ v_n \} \) is an independent and Theorem 4.8.2, implies that \( \{ v_n \} \) is a \( X_\phi \)-Riesz bases for \( E \) with respect to \( X_\phi \).