Chapter-3

Construction of Generalized Atomic Decompositions in Banach spaces
CHAPTER-3
CONSTRUCTION OF GENERALIZED ATOMIC DECOMPOSITIONS IN BANACH SPACES

3.1 Introduction

In this chapter, we generalize the classical construction of Pelczynski [75]. In Section 3.2, $G$-atomic decompositions for Banach spaces with respect to model space of the sequences have been introduced and studied as a generalization of atomic decompositions. In Section 3.3, examples and counter-examples have been proved to show existence of $G$-atomic decomposition. In Section 3.4, we shall prove that an associated Banach space for $G$-atomic decomposition always has a complemented subspace. In Section 3.5, the notion of the representation system of Banach space is defined and exhibits its relationship with $G$-atomic decomposition. In Section 3.6, we classify $G$-atomic decomposition in terms of bases of subspaces for Banach space. Also it has been observed that $G$-atomic decompositions are exactly compressions of Schauder decompositions for a larger Banach space. Finally, in Section 3.7, we shall give a characterization for a finite dimensional $G$-atomic decomposition in terms of a finite-dimensional expansion of identity.

3.2 Generalized Atomic Decomposition in Banach Space

The theory of spaces of sequences of scalars admits a natural generalization to a vector sequence spaces. If $\Phi = \{G_n\}$ is a sequence of Banach spaces, a sequence space $X_\Phi$ associated with $\{G_n\}$ is a linear subspace of $\prod_{n=1}^{\infty} G_n$ (the collection of all sequences $\{y_n\}$ with $y_n \in G_n$, $n = 1, 2, \ldots$, endowed with product topology).

The coordinate transformations $P_n : X_\Phi \to G_n$ are defined by
\[ P_n\left( \{ y_i \} \right) = y_n, \quad n = 1, 2, \ldots \]

Then \( X_\phi \) is called a generalized BK-space induced by \( \{ G_n \} \) if \( X_\phi \) is a Banach space and \( P_n \) is a continuous operator, for every \( n \in \mathbb{N} \). The scalar BK-spaces containing all unit vectors \( e_n \) are generalized by the spaces \( X_\phi \) containing all canonical subspaces

\[ F_n = \{0\} \times \{0\} \times \ldots \{0\} \times G_n \times \{0\} \times \ldots (G_n \neq \{0\}, n = 1, 2, \ldots). \]

These \( F_n \)'s closed linear subspaces of \( X_\phi \). We refer to the space \( X_\phi \) as a model space.

The following is the example of such type of a model space.

Let \( \Phi = \{ G_n \} \) be a sequence of closed linear subspaces of a Banach space \( E \).

Consider the linear space \( X_\phi \) of the system \( \Phi \), that is, the space of all elements sequences \( y = \{ y_n \}_{n=1}^\infty \) for which the series \( \sum_{n=1}^\infty y_n \) is convergent equipped with the norm

\[ \left\| y \right\|_{X_\phi} = \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^n y_i \right\|_{E}, \quad y_n \in G_n (n = 1, 2, \ldots). \quad (3.1) \]

Thus the space \( X_\phi \) is complete with respect to (3.1). Indeed, clearly (3.1) define a norm on \( X_\phi \). Now, let \( \{ y_n^{(k)} \} (k = 1, 2, \ldots) \) be a Cauchy sequence in \( X_\phi \). Then for every \( \varepsilon > 0 \) there exists a positive integer \( N(\varepsilon) \) such that

\[ \left\| \{ y_n^{(k)} \} - \{ y_n^{(m)} \} \right\|_{X_\phi} = \sup_{1 \leq n < \infty} \left| \sum_{i=1}^n (y_i^{(k)} - y_i^{(m)}) \right| < \varepsilon \quad (k, m > N(\varepsilon)) \quad (3.2) \]

Then

\[ \left\| y_n^{(k)} - y_n^{(m)} \right\| \leq \left\| \sum_{i=1}^n (y_i^{(k)} - y_i^{(m)}) \right\| + \left\| \sum_{i=1}^{n-1} (y_i^{(k)} - y_i^{(m)}) \right\| < 2\varepsilon \quad (k, m > N(\varepsilon); n = 1, 2, \ldots). \]

whence, since by our assumption each \( G_n \) is complete, \( \lim_{k \to \infty} y_n^{(k)} = y_n \in G_n \).
\[(n = 1, 2, \ldots)\]. Hence, from the inequalities (3.2)

\[
\left\| \sum_{i=1}^{n} (y_{i}^{(k)} - y_{i}^{(m)}) \right\| < \varepsilon \quad (k, m > \mathbb{N} (\varepsilon); \; n = 1, 2, \ldots).
\]

We obtain, for \(m \to \infty\), we obtain

\[
\left\| \sum_{i=1}^{n} (y_{i}^{(k)} - y_{i}) \right\| \leq \varepsilon \quad (k > \mathbb{N} (\varepsilon), n = 1, 2, \ldots).
\]

Then

\[
\left\| \sum_{i=n+1}^{n+l} y_{i} \right\| \leq 2\varepsilon + \left\| \sum_{i=n+1}^{n+l} y_{i}^{(k)} \right\| \quad (k > \mathbb{N} (\varepsilon); \; n, l = 1, 2, \ldots).
\]

Consequently, since each series \(\sum_{i=1}^{\infty} y_{i}^{(k)}\) converges and since \(E\) is complete, it follows that \(\sum_{i=1}^{\infty} y_{i}\) converges, i.e., \(\{y_{n}\} \in X_{\Phi}\). Moreover, by the above we have

\[
\left\| \{y_{n}^{(k)}\} - \{y_{n}\} \right\|_{X_{\Phi}} = \sup_{1 \leq m < \infty} \left\| \sum_{i=1}^{n} (y_{i}^{(k)} - y_{i}) \right\| \leq \varepsilon \quad (k > \mathbb{N} (\varepsilon)),
\]

which shows that the space \(X_{\Phi}\) is complete with respect to this norm.

The system \(\{F_{n}\}\) defined by above is a Schauder decomposition of \(X_{\Phi}\).

Clearly, any model space \(X_{\Phi}\) can be obtained by the method described above, indeed, if \(E\) is a model space of sequence of subspaces \(\Phi = \{F_{n}\}\) then \(X_{\Phi} = E\).

We begin with the following generalization of Atomic decomposition

**Definition 3.2.1.** Let \(\Phi = \{G_{n}\}\) be a sequence of non-trivial subspaces of a Banach space \(E\) and \(\{v_{n}; \; v_{n} \in L(E, G_{n})\}\) be a sequence of linear operators (not necessarily projections). Let \(X_{\Phi}\) be a model space associated with \(E\). Then, we say \((\{G_{n}\}, \{v_{n}\})\) is \(G\) - atomic decomposition for \(E\) with respect to \(X_{\Phi}\) if

1. \(\{v_{n}(x)\} \in X_{\Phi}\), for all \(x \in E\)
(b) there exist constants $A, B$ with $0 < A \leq B \leq \infty$ such that
\[
A \|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{V}_E} \leq B \|x\|_E, \text{ for all } x \in E
\]

(c) \[x = \sum_{n=1}^{\infty} v_n(x), \text{ for all } x \in E.\]

The positive constants $A$ and $B$, respectively, are called lower and upper atomic bounds for the $G$-atomic decomposition $\{G_n\}, \{v_n\}$.

Next, we recall the Lemma which is the modified form of Lemma 1.4.11 as below.

**Lemma 3.2.2.** Let $\{G_n\}$ be a sequence of subspaces of $E$ and $\{v_n\} \subset L(E, G_n)$ be a sequence of operators $\forall n \in \mathbb{N}$. If $\{v_n\}$ is total over $E$, then $X = \{\{v_n(x)\} : x \in E\}$ is a Banach space with the norm given by $\|\{v_n(x)\}\|_X = \|x\|_E, x \in E$.

### 3.3 Examples and Counter-examples.

In this section, we give examples and counter-examples to exhibits relation among various types of $G$-atomic decomposition. The modified sequence $\{G_n\}$ used below was constructed in [66].

**Example 3.3.1.** Consider the Banach space
\[
E = l^\infty(\mathcal{X}) = \left\{ \{x_n\} : x_n \in \mathcal{X}; \sup_{1 \leq n < \infty} \|x_n\|_\mathcal{X} < \infty \right\}
\]
equipped with the norm
\[
\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_\mathcal{X}, \{x_n\} \in E,
\]
where $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Define a sequence $\{G_n\}$ of subspaces of $E$ by
\[
G_{2n-1} = \left\{ \delta^x_{2n-1} + 2^n \delta^x_{2n} : x \in \mathcal{X} \right\}
\]
\[
G_{2n} = \left\{ \delta^x_{2n} : x \in \mathcal{X} \right\}
\]
where \( \delta^n_i = (0, 0, \ldots, x, \ldots, 0, 0, \ldots) \) for all \( n \in \mathbb{N} \) and \( x \in \mathcal{X} \).

Define operators \( v_n : l^n(\mathcal{X}) \rightarrow l^n(\mathcal{X}) \) by

\[
v_{2n-1}(x) = \delta^{2n-1}_{2n-1} + 2^n \delta^{2n}_{2n},
\]

\[
v_{2n}(x) = 2^n \left( \frac{1}{2^{n+1}} \delta^{2n}_{2n} \right), \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}.
\]

Then, by Lemma 3.2.2, there exists an associated model space

\[
X = \left\{ \{v_n(x)\} : x \in E \right\}
\]

with the norm given by

\[
\left\| \{v_n(x)\} \right\|_X = \|x\|_E, \quad x \in E.
\]

Also,

\[
\sum_{n=1}^{\infty} v_n(x) = x, \quad x \in E.
\]

Therefore, \( (\{G_n\}, \{v_n\}) \) is G-atomic decomposition of \( E \) with respect to model space \( X \).

**Example 3.3.2.** Let \( E = c_0 \) and \( \{e_n\} \) be the unit vector basis in \( c_0 \).

Write

\[
G_n = [x_n]
\]

and

\[
v_n(x) = f_n(x) x_n, \quad n \in \mathbb{N},
\]

where \( \{x_n\} \subset E \) and \( \{f_n\} \subset E^* \) are given by

\[
x_{2n-1} = 2^{n-1} e_{2n-1} - e_{2n}, \quad x_{2n} = e_{2n} \quad (n \in \mathbb{N})
\]

\[
f_{2n-1} = 2^n h_{2n-1}, \quad f_{2n} = 2^{n-1} h_{2n-1} + h_{2n} \quad (n \in \mathbb{N}),
\]

Then, \( \{h_n\} \) being the sequence of coordinate functionals to \( \{e_n\} \). Then, it can be easily prove that there exist an associated model space \( X = \left\{ \{v_n(x)\} : x \in E \right\} \), such that \( (\{G_n\}, \{v_n\}) \) is G-atomic decomposition for \( E \) with respect to \( X \).
Example 3.3.3. Let $E$ be a Banach space defined as
\[ E = l^2(\mathcal{X}) = \left\{ \{x_n\} : x_n \in \mathcal{X} : \sum_{n=1}^{\infty} \|x_n\|_\mathcal{X}^2 < \infty \right\}, \]
where $(\mathcal{X}, \|\cdot\|)$ is a Banach space, equipped with the norm given by
\[ \|\{x_n\}\|_E = \left( \sum_{n=1}^{\infty} \|x_n\|_\mathcal{X}^2 \right)^{\frac{1}{2}}. \]
Define for $n \in \mathbb{N}$,
\[ G_n = \{ \delta_n^{x_n} + \delta_{n+1}^{x_n} : x \in \mathcal{X} \} \]
and
\[ v_n(x) = \delta_n^{x_n} + \delta_{n+1}^{x_n}, \quad x = \{x_n\} \in E, \]
where $\delta_n^{x_n} = (0,0,\ldots,\frac{x_n}{n},0,0,\ldots)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{X}$.

But, since for any $0 \neq x \in \mathcal{X}$, $\delta_2^x = (0,x,0,\ldots) \in E$ is such that
\[ v_n(\delta_2^x) = 0, \quad \text{for all } n \in \mathbb{N}, \]
there exist no associated model space $X$ such that \( \{G_n\}, \{v_n\} \) is a $G$-atomic decomposition for $E$ with respect to $X$.

Remark 3.3.4. Any Banach space $E$ admits the trivial $G$-atomic decomposition $\{G_n\}$, where
\[ G_1 = E \quad \text{and} \quad G_n \neq \{0\} \quad (n = 2,3,\ldots) \]
are arbitrary with operators
\[ v_1 = I_E, \]
\[ v_n = 0 \quad (n = 2,3,\ldots). \]

3.4 Existence of G-atomic decomposition for Banach Spaces
In this section, we shall give existence of G-atomic decomposition in terms of complemented coefficient subspace of model space and an isomorphism from Banach space into model space.

**Theorem 3.4.1.** If \( \{ G_n \}, \{ v_n \} \) is a G-atomic decomposition for \( E \) with respect to \( X_\phi \), then there exist a complemented coefficient subspace \( G \) of \( X_\phi \) and an isomorphism \( T \) from \( E \) into \( X_\phi \) such that \( X_\phi = T(E) \oplus G \).

**Proof.** Let \( \{ G_n \}, \{ v_n \} \) be a G-atomic decomposition of \( E \) with respect to \( X_\phi \), where

\[
X_\phi = \left\{ \{ y_n \} \subset E : \sum_{n=1}^{\infty} y_n \text{ converges}; y_n \in G_n \ (n = 1, 2, \ldots) \right\}
\]  
(3.3)
equipped with norm

\[
\left\| \{ y_n \} \right\|_{X_\phi} = \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} y_i \right\|.
\]

Then the mapping \( T : E \to X_\phi \) defined by

\[
T(x) = \{ v_n(x) \}, \quad x \in E
\]
is an isomorphism from \( E \) into \( X_\phi \).

Since, \( \sum_{n=1}^{\infty} v_n(x) \) converges to \( x \) by (3.3) and

\[
\| x \|_E = \left\| \sum_{i=1}^{\infty} v_i(x) \right\|
\leq \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i(x) \right\|
= \left\| \{ v_n(x) \} \right\|_{X_\phi}
\leq B \| x \|_E , \ x \in E,
\]
where

58 | P a g e
\[ B = \sup_{n < \infty} \|S_n\| < \infty \]

and
\[ S_n(x) = \sum_{i=1}^{\infty} v_i(x). \]

Now, define \( S : X_\varphi \to E \) by
\[ S(\{x_n\}) = \sum_{i=1}^{\infty} x_i, \quad \{x_n\} \in X_\varphi, \ n \in \mathbb{N}. \]

Then \( S \) is the bounded linear operators from \( X_\varphi \) to \( E \).

Put \( G = \ker S \).

Then
\[ G = \left\{ \{x_n\} \subset E : x_n \in G_n (n = 1, 2, \ldots), \sum_{i=1}^{\infty} x_i = 0 \right\}, \]

is called subspace of \( X_\varphi \).

Furthermore, if \( \{v_n(x)\} \in G \) for some \( x \in E \), then
\[ 0 = S(\{v_n(x)\}) = \sum_{n=1}^{\infty} v_n(x) = x. \]

So
\[ T(E) \cap G = \{0\}. \]

Now, let \( \{x_n\} \in X_\varphi \) be arbitrary such that
\[ x = \sum_{i=1}^{\infty} x_i. \]

Then \( \{v_n(x)\} \in T(E) \) such that
\[ \sum_{i=1}^{\infty} (x_i - v_i(x)) = \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{\infty} v_i(x) = x - x = 0. \]

Therefore, \( \{x_n - v_n(x)\} \in G \) such that
\[ \{x_n\} = \{v_n(x)\} + \{x_n^0\}, \]

where
\[ \{v_n(x)\} \in T(E) \text{ and } \{x_n^{(0)}\} = \{x_n - v_n(x)\} \in G. \]

Hence, we obtain
\[ X_\Phi = T(E) \oplus G. \]

### 3.5 Representation System of Banach Spaces

This section, begin by giving the following definition of representation system of Banach space regarding model space as.

**Definition 3.5.1.** A system \( \Phi = \{G_n\} \) of closed linear subspaces of a Banach space \( E \), with \( G_n \neq \{0\} (n = 1, 2, \ldots) \) is called a representation system of \( E \) with respect to model space \( X_\Phi \) if for every \( x \in E \), there exists a sequence \( \{x_n\} \subseteq E \) with \( x_n \in G_n (n = 1, 2, \ldots) \) such that
\[
x = \sum_{n=1}^{\infty} x_n
\]
and
\[
G = \left\{ \{x_n\} \subseteq E : \sum_{n=1}^{\infty} x_n = 0 \right\}
\]
is a complemented coefficient subspace of \( X_\Phi \).

We give necessary and sufficient condition for a representation system to be a \( G \)-atomic decomposition in Banach space.

**Theorem 3.5.2.** Let \( E \) be a Banach space and \( X_\Phi \) be an associated Banach space indexed by \( \mathbb{N} \). Then \( \Phi = \{G_n\} \) is a representation system if and only if \( (\{G_n\}, \{v_n\}) \) is a \( G \)-atomic decomposition with respect to \( X_\Phi \).

**Proof. Necessity.** Let \( \Phi = \{G_n\} \) be a representation system of \( E \). Then for \( x \in E \), there exists a sequence for \( \{x_n\} \subseteq E \) with \( x_n \in G_n \), such that
\[ x = \sum_{n=1}^{\infty} x_n, \quad \forall n \in \mathbb{N}. \]

Let \( G = \left\{ \{x_n\} \subset E : \sum_{n=1}^{\infty} x_n = 0 \right\} \) be a complemented coefficient subspace of \( X_\phi \).

Then, we have

\[ X_\phi = G \oplus F \]

and \( F \) is complemented to \( G \).

Define \( S : X_\phi \to E \) by

\[ S(\{x_n\}) = \sum_{n=1}^{\infty} x_n, \quad \{x_n\} \in X_\phi, \quad n \in \mathbb{N}. \]

As in Theorem 3.4.1, \( T \) is an isomorphism from \( E \) into \( X_\phi \).

Then, \( S|_F \) is an isomorphism from \( F \) onto \( E \).

Indeed, if \( S(\{x_n\}) = 0 \) for some \( \{x_n\} \subset F \), then

\[ \sum_{n=1}^{\infty} x_n = 0. \]

Hence, \( \{x_n\} \in G \cap F = \{0\} \) which proves that \( S|_F \) is one to one.

Also, if \( y \in E \) then, since \( \Phi \) is a representation system, there exists a sequence \( \{y_n\} \subset X_\phi \) such that

\[ y = \sum_{n=1}^{\infty} y_n \quad \text{with} \quad y_n \in G, \quad \forall n \in \mathbb{N}. \]

\[ = S(\{y_n\}), \]

write

\[ \{y_n\} = \{x_n^{(0)}\} + \{x_n\}, \]

where

\[ \{x_n^{(0)}\} \in G \quad \text{and} \quad \{x_n\} \in F. \]

Then,

\[ y = S(\{x_n^{(0)}\}) + S(\{x_n\}) = S(\{x_n\}), \]
which proves that $S|_{F}$ is onto.

Now, let $x \in E$ be an arbitrary element and let $\{v_{n}(x)\} = \left( S|_{F} \right)^{-1}(x) \in F$, then

$$\{x_{n}\} = \{v_{n}(x)\} + \{x_{n}^{(0)}\},$$

where

$$\{x_{n}^{(0)}\} \in G.$$

So, we have

$$S(\{x_{n}\}) = \sum v_{n}(x) + \sum x_{n}^{(0)} = S(\{v_{n}(x)\}).$$

Therefore,

$$x = S(\{v_{n}(x)\}) = \sum_{n=1}^{\infty} v_{n}(x).$$

Since, $F \subset X_{\phi}$, we have

$$v_{n}(x) \in G, \ x \in E, \ n \in \mathbb{N}.$$ 

Thus,

$$x = S(\{v_{n}(x)\}) = \sum_{n=1}^{\infty} v_{n}(x), \ x \in E$$

and each $v_{n}$ is linear on $E$ and satisfies

$$\left\| v_{n}(x) \right\| \leq 2 \sup_{1 \leq k < \infty} \left\| \sum_{i=1}^{k} v_{i}(x) \right\|$$

$$= 2 \left\| \{v_{n}(x)\} \right\|$$

$$\leq 2 \left\| (S|_{F})^{-1} \right\| \left\| x \right\|. \ (x \in E, \ n = 1, 2...).$$

Also, by the the principle of uniform boundedness,
\[ \|x\|_E = \left\| \sum_{n=1}^{\infty} v_n(x) \right\| \]
\[ \leq \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i(x) \right\| \]
\[ = \left\| \{v_n(x)\} \right\|_{X_\phi} \]
\[ \leq B \|x\|, \]

where
\[ B = \sup_{1 \leq n < \infty} \|S_n\| < \infty \]

and
\[ S_n(x) = \sum_{i=1}^{n} v_i(x). \]

Therefore, \( \{G_n, \{v_n\}\} \) is a G-atomic decomposition of \( E \) with respect to \( X_\phi \).

**Sufficiency**, follows with the argument of the proof of Theorem 3.4.1.

In the following result, we shall show that a G-atomic decomposition for a Banach space produces another G-atomic decomposition for the Banach space.

**Theorem 3.5.3.** If \( \{G_n, \{v_n\}\} \) is a G-atomic decomposition for \( E \) with respect to \( X_\phi \), then there exists a projection \( P \) of \( X_\phi \) onto \( T(E) \) along \( G \) such that \( \{T^{-1}P(F_n), \{v_n\}\} \) is a G-atomic decomposition for \( E \) with respect to \( X_\phi \), where \( \{F_n\} \) is the Schauder decomposition of \( X_\phi \).

**Proof.** Let \( P \) be a projection of \( X_\phi \) onto \( T(E) \) along \( G \). Then
\[ P(\{v_n\}) = \left\{ v_n \left( \sum_{i=1}^{\infty} x_i \right) \right\}, \quad \{x_n\} \in X_\phi. \]

Since for every \( \{v_n(x)\} \in T(E) \), we have
\[ P\left( \{ v_n(x) \} \right) = \{ v_n(x) \} = \left\{ v_n \left( \sum_{i=1}^{\infty} v_i(x) \right) \right\} \]

and since for every \( \{ x_n \} \in G \), we have

\[ P\left( \{ x_n \} \right) = 0 = \{ v_n(0) \} = \left\{ v_n \left( \sum_{i=1}^{\infty} x_i \right) \right\} \]

We have in particular, for any \( \{ \delta_{nk} x_n \} \in F_k, k = 1,2,... \)

\[ P\left( \{ \delta_{nk} x_n \} \right) = \left\{ v_n \left( \sum_{i=1}^{\infty} \delta_{ik} x_i \right) \right\} = \left\{ v_n(x_k) \right\} = T(x_k) \]

where

\[ \delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k. \end{cases} \]

Since \( T \) is invertible then we write

\[ x_k = T^{-1}\left( P\left( \{ \delta_{nk} x_n \} \right) \right), \quad x_k \in G_k, k = 1,2,... \]

Therefore,

\[ G_k = T^{-1}P(F_k), \quad k = 1,2,... \]

Hence, \( \left( \{ T^{-1}P(F_n) \}, \{ v_n \} \right) \) is a \( G \)-atomic decomposition for \( E \) with respect to model space \( X_\phi \).
3.6 Classification of $G$-atomic Decomposition in Terms of Bases of Subspaces for Banach Space

In this section, we classify $G$-atomic decomposition in terms of bases of subspaces (Schauder decomposition) for Banach space and observed that $G$-atomic decompositions are exactly compression of Schauder decompositions for a larger Banach space.

**Theorem 3.6.1.** If $D$ is any Banach space with Schauder decomposition $\{F_n\}$ and an isomorphism $T$ from $E$ into $D$ and a projection $P$ from $D$ into $T(E)$ such that $G_n = T^{-1}P(F_n)$. Then, there exists an associated sequence of vectors $\{v_n\}$ such that $(\{G_n\},\{v_n\})$ is a $G$-atomic decomposition for $E$.

**Proof.** Since $\{F_n\}$ is Schauder decomposition for $D$.

Assume that $\{u_n\} \subset L(D,D)$ is an associated sequence of coordinate projection to $\{F_n\}$.

Then, for $y \in P(D)$ we have

$$y = P(y) = P\left(\sum_{n=1}^{\infty} u_n(y)\right) = \sum_{n=1}^{\infty} P(u_n(y)).$$

Thus we have

$$y = \sum_{n=1}^{\infty} P(u_n(y)). \tag{3.4}$$

Since,

$$Pu_n|_{P(D)} \in L(P(D),P(F_n)), \quad n = 1,2,\ldots,$$

Therefore, $(\{P(F_n)\},\{Pu_n\})$ is $G$-atomic decomposition of $P(D) = T(E)$ by (3.4).
Now, $T$ is an isomorphism from $E$ onto $T(E)$ and

$$G_n = T^{-1}P(F_n).$$

Put $Pu_n = v_n$, $n = 1, 2, \ldots$.

It follows that $(\{G_n\}, \{v_n\})$ is $G$-atomic decomposition for $E$.

In the following theorem we generalize Theorem 2.6 [19], which is a classical construction of Pelczynski [75].

**Theorem 3.6.2.** Let $E$ be a Banach space. Then the following are equivalent:

(i). There is a Banach space of scalar valued sequences $X_\Phi$, so that

$$(\{G_n\}, \{u_n\})$$

satisfies Definition 3.1 (i.e., is an $G$-atomic decomposition for $E$).

(ii). There is a Banach space $D$ with a Schauder decomposition $\psi = \{A_n\}$ so that $E \subset D$ and there is a bounded linear projection $P: D \to E$ with

$$P\{A_n\} = G_n,$$

for all $n \in \mathbb{N}$.

**Proof.** Let $(i) \Rightarrow (ii)$.

This implication is obvious with $D = E$ and $\{A_n\} = \{G_n\}$ and $P = I_E$ in Theorem 3.5.3.

Conversely, let $(ii) \Rightarrow (i)$.

This follows with the argument of the above proof of Theorem 3.6.1.
3.7 Characterization of a G-atomic Decomposition

In this section, we examine the general relationship between a finite-dimensional G-atomic decompositions and approximation property in Banach space theory.

We begin this section by giving the following definition:

**Definition 3.7.1 ([74]).** A sequence of non-zero finite rank operators \( \{v_i\} \) from a Banach space \( E \) into itself is called a finite dimensional expansion of the identity of \( E \), if
\[
x = \sum_{i=1}^{\infty} v_i(x), \quad x \in E.
\]

In view of above definition, we give a necessary and sufficient condition for a finite dimensional G-atomic decomposition in Banach Space \( E \) having a finite dimensional expansion of the identity of \( E \).

**Theorem 3.7.1** A Banach space \( E \) has a finite-dimensional \( G \)-atomic decomposition \( \{G_n, \{v_n\}\} \) (i.e, such that \( \dim G_n < \infty \) for all \( n = 1,2,... \)) if and only if \( E \) admits a finite-dimensional expansion \( \{v_n\} \) of the identity of \( E \).

**Proof.** Let \( \{G_n, \{v_n\}\} \) be finite dimensional \( G \)-atomic decomposition for \( E \). Then
\[
\dim G_n < \infty \text{ for all } n = 1,2,\ldots
\]
and
\[
x = \sum_{n=1}^{\infty} v_n(x), \quad x \in E.
\]

Therefore, an associated sequence of operators \( \{v_n\} \) for \( \{G_n\} \) is a finite-dimension expansion of \( E \).

Conversely, if \( \{v_n\} \) is a finite-dimensional expansion of the identity of \( E \). Then
\[ \{G_n\} = \{v_n(E)\} \]
i.e., \( \{G_n\}, \{v_n\} \) is a finite-dimensional \( G \)-atomic decomposition of \( E \).

**Remark 3.7.2.** With the help of above result, we can classified finite \( G \)-atomic decompositions in terms of several forms of the approximation property [19, 78, and 79] for Banach spaces.