CHAPTER 4

QUASI-STATIONARY DISTRIBUTIONS
4.1 Introduction

In certain reliability problems the behaviour of the system after its first breakdown may not be of much interest either because the failure of a unit is irrevocable or because the system recovery is expensive. It would then be of great interest to investigate the limit distribution of the residual lifetime of the system conditioned on its failure free performance. Let $T$ denote the system failure and $X(t)$, the residual lifetime of the system at any time $t$. The period $T$ may be sufficiently long to allow the process $X(t)$ to settle down to a state of equilibrium within this period. Therefore, we study the limit of the following conditional probability, as $t \to \infty$,

$$H(x,t) = \text{Pr}(X(t) \leq x / T > t), \ 0 < x < \infty$$

The existence of the limit distributions of above nature known as quasi-stationary distributions has been the subject of interest of many probabilists. Darroch and Seneta [1965, 67] have discussed quasi-stationary distributions in absorbing discrete-time and continuous-time finite Markov chains. For the discrete-time parameter, the results of Darroch and Seneta have been applied to genetic problems by Seneta [1966]. The general discussion for a denumerably infinite discrete-time and continuous-time Markov chains have been given by Seneta and Vere-Jones [1967] and Vere-Jones [1969] respectively. Daley [1969] illustrated the work of Seneta and Vere-Jones for a left-continuous random walk. The quasi-stationary distribution of the virtual waiting time process of the $M/G/1$ queue when certain conditions are imposed on the service
time distribution has been discussed in Kyprianou [1971], while that of
the queue size process of the M/G/1 queue under the same conditions
has been considered in Kyprianou [1970]. Kyprianou [1972] studies the
existence of quasi-stationary distributions of the virtual waiting time
process of the GI/M/1 queue. Cavender [1978] extended the concept of
quasi-stationary distributions to birth-and-death processes. Kalpakam and
Hameed [1983] introduced this concept in reliability analysis by finding
the quasi-stationary distributions for a two-unit standby redundant system.
In this chapter we find the quasi-stationary distributions for two complex
repairable systems. Section 4.2 deals with quasi-stationary distributions
for a two-unit warm standby redundant system with imperfect switch
supported by a single repair facility. In Section 4.3 we establish the
quasi-stationary distribution for a two-unit cold standby redundant system
with imperfect switch supported by two repair facilities.

4.2 A Two-unit Warm Standby System Supported by a Single
Repair Facility

Consider a reliability system with two identical repairable units,
one functioning online and the other kept in warm standby. The system
is supported by a single repair facility. At the instant of the failure of
the online unit, the standby unit, if operable, is switched online, while
the failed unit enters repair. A switching device is required to transfer
a unit from standby state to online. The switch is imperfect in the
sense that the switching device is subject to breakdown. That is, it has
its own pattern of lifetime distribution. The switching device and the
failed units are repaired at the same repair facility. Initially, we assume
that at $t=0$, one unit goes online, the repair of the other unit just
begins and the switching device is operable. The failure times of online
units are independently and identically distributed random variables with
p.d.f. $f(\cdot)$. The standby unit and the switching device have constant
failure rates $\lambda$ and $\mu$, respectively. Repair time of both online and
standby failures are generally distributed with common p.d.f. $g_1(\cdot)$. The
repair time of the switching device is denoted by $g_2(\cdot)$. Furthermore,
we assume that the Laplace-Stieltjes Transforms of $f(\cdot)$, $g_1(\cdot)$ and $g_2(\cdot)$
are all rational functions of their arguments.

We define

$E_0$ event that one unit has just come online and the other in
standby.

$E_1$ event that one unit has just come online, the repair for the
other unit just begins and the switching device is operable.

$N(\theta, t)$ number of $\mathcal{O}$ events in $(0, t]$

$T_i$ time to system failure starting with an $E_i$ - event, $i=0,1$

$P_i(t)$ p.d.f of $T_i$, $i=0,1$

Then

$$421 \quad H_i(x, t) = \frac{P_i(t) - P_i(x+t)}{P_i(t)} = 1 - \frac{P_i(x+t)}{P_i(t)}$$
and

\[ H_i(x) = \lim_{t \to \infty} H_i(x,t) = 1 - \lim_{t \to \infty} \frac{P_i(x+t)}{P_i(t)} \]

We observe that \( \overline{P}_i(t) \) is the reliability of the system starting with an \( E_i \)-event initially. The above limit is obtained by writing the integral equations for \( \overline{P}_i(t) \) using regeneration point method and then applying the Laplace Transform technique to get an asymptotic expansion for \( \overline{P}_i(t) \), \( i=0,1 \). To this end we derive the reliability function for the system.

When one unit is continuously operating online, the behaviour of the other unit and the switching device can be identified with that of a two-unit parallel system with a single repair facility. To describe the behaviour of this subsystem we define the following events:

- \( \epsilon_0 \) both the switching device and the unit are operable
- \( \epsilon_1 \) repair of the failed unit just begins while the switching device is operable
- \( \epsilon_2 \) repair of the switching device just begins while the other unit is operable

We observe that \( \epsilon_i \)-events, \( i=0,1,2 \) are regenerative and that between two successive events of the same type at least one event of the other two types occur. Next we define the following auxiliary functions which are helpful in characterizing the subsystem.
$$q_{j}(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr\{N(\epsilon_j, t+\delta t) = 1, \ N(\epsilon_m, t) = 0 \mid \epsilon_k \text{ at } t=0\}, \quad i=1,2 \quad k, j, m = 0,1,2$$

We observe that

$$q_{j}(t) = 0, \quad j=0,1,2$$

and by simple probabilistic arguments, we get

$$q_{01}(t) = \lambda e^{-(\lambda + \mu)t}, \quad q_{02}(t) = \mu e^{-(\lambda + \mu)t},$$

$$q_{10}(t) = g_1(t) e^{-\mu t}, \quad q_{20}(t) = g_2(t) e^{-\lambda t},$$

$$q_{12}(t) = g_1(t) [1-e^{-\mu t}], \quad q_{21}(t) = g_2(t) [1-e^{-\lambda t}],$$

Next we observe that the interval between two successive events of the same type are i.i.d random variables. The following functions are well defined

$$\beta_{mj}(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr\{N(\epsilon_j, t+\delta t) = 1, \ N(\epsilon_m, t) = 0 \mid \epsilon_m \text{ at } t=0\}, \quad m=0,1$$

$$k \rho_{jl}(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr\{\epsilon_j \text{ in } (t,t+\delta), \ N(\epsilon_k, t) = 0 \mid \epsilon_j \text{ at } t=0\}, \quad k, j = 0,1,2 \quad k \neq j$$

$$\rho_{mj}(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr\{\epsilon_j \text{ in } (t,t+\delta t) \mid \epsilon_m \text{ at } t=0\}, \quad m, j = 0,1$$

66
We observe that $\rho_{jl}(\cdot)$ gives the renewal density of $\epsilon_j$ events.

Using simple renewal theoretic arguments (Cox [1962]), we obtain

$$4.2.10 \quad \rho_{jl}(t) = \sum_{n=1} \{\beta_{jl}(t)\}^{(n)}$$

and

$$4.2.11 \quad \rho_{mj}(t) = \beta_{mj}(t) + \beta_{mj}(t) \otimes \rho_{jl}(t), \ m=0,1 \quad m \neq j$$

Furthermore, since $\rho_{jl}(t)$ is the renewal density of $\epsilon_k$ avoiding $\epsilon_j$ events, we obtain

$$4.2.12 \quad 0\rho_{11}(t) = \sum_{n=1} \{q_{12}(t) \otimes q_{21}(t)\}^{(n)} = 0\rho_{22}(t)$$

$$4.2.13 \quad 1\rho_{00}(t) = \sum_{n=1} \{q_{02}(t) \otimes q_{20}(t)\}^{(n)} = 1\rho_{22}(t)$$

$$4.2.14 \quad 2\rho_{00}(t) = \sum_{n=1} \{q_{01}(t) \otimes q_{10}(t)\}^{(n)} = 2\rho_{11}(t)$$

Observing that $\beta_{mj}(t)\delta t$ is the conditional probability of the occurrence of the first $\epsilon_j$-event in $(t, t+\delta t)$, given an $\epsilon_m$-event at $t=0$, and considering the following mutually exclusive and exhaustive possibilities, we derive the expression for $\beta_{mj}(t); \ m \neq j$

(i) the interval $(0, t)$ is not intercepted by an $\epsilon_m$ event

(ii) the interval $(0, t)$ is intercepted by at least one $\epsilon_m$ event
\[ \beta_{10}(t) = q_{10}(t) + q_{12}(t) \odot q_{20}(t) + \rho_{11}(t) \odot \{q_{10}(t) + q_{12}(t) \odot q_{20}(t)\} \]

\[ \beta_{20}(t) = q_{20}(t) + q_{21}(t) \odot q_{10}(t) + \rho_{22}(t) \odot \{q_{20}(t) + q_{21}(t) \odot q_{10}(t)\} \]

\[ \beta_{01}(t) = q_{01}(t) + q_{02}(t) \odot q_{21}(t) + \rho_{00}(t) \odot \{q_{01}(t) + q_{02}(t) \odot q_{21}(t)\} \]

\[ \beta_{02}(t) = q_{02}(t) + q_{01}(t) \odot q_{12}(t) + 2\rho_{00}(t) \odot \{q_{02}(t) + q_{01}(t) \odot q_{12}(t)\} \]

\[ \beta_{12}(t) = q_{12}(t) + q_{10}(t) \odot q_{02}(t) + 2\rho_{11}(t) \odot \{q_{12}(t) + q_{10}(t) \odot q_{02}(t)\} \]

\[ \beta_{21}(t) = q_{21}(t) + q_{20}(t) \odot q_{01}(t) + \rho_{22}(t) \odot \{q_{21}(t) + q_{20}(t) \odot q_{01}(t)\} \]

When \( m=j \), the following expressions for \( \beta_{mj}(t) \) are obtained by using the fact that between two successive \( \epsilon_k \) events at least one \( \epsilon_m \) event (\( k \neq m \)) should occur. Therefore

\[ \beta_{00}(t) = q_{01}(t) \odot \beta_{10}(t) + q_{02}(t) \odot \beta_{20}(t) \]

\[ \beta_{11}(t) = q_{12}(t) \odot \beta_{21}(t) + q_{10}(t) \odot \beta_{01}(t) \]

\[ \beta_{22}(t) = q_{20}(t) \odot \beta_{02}(t) + q_{21}(t) \odot \beta_{12}(t) \]
We observe that when one unit is continuously operating online the behaviour of the other unit and the switching device is completely described by the functions \( \rho_m(t) \) and \( \beta_m(t) \).

To discuss the behaviour of the system consisting of two-units and a switching device we require the following events:

- **\( E_0 \)** event that one unit just begins to operate online, the other goes to standby and the switching device is operable.

- **\( E_1 \)** event that one unit begins to operate online, the repair of the other unit just begins. At this instant the switching device is operable.

To render the analysis simple, we define the following function:

\[
4.2.24 \quad \alpha(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr\{N(E_1, t + \delta t) = 1, \ N(E_1, t) = 0, \ \text{system is up in} \ (0, t) \mid E_1 \ \text{at} \ t = 0\}
\]

We note that the function \( \alpha(t)\delta t \) is the probability density function of the random variable representing the interval between two successive \( E_1 \) events, the system being in the upstate during this interval. For the system to be in the upstate in \((0, t)\), it is necessary that at the failure instants of the online unit both the standby unit and the switching device should be in operable condition.

Hence:

\[
4.2.25 \quad \alpha(t) = f(t) \{ \rho_{10}(t) \odot e^{-(\lambda + \mu)t} \}
\]
Now, we are in a position to write an expression for the reliability function of the system

The reliability function \( R_1(t) \) of the system is defined by

\[
R_1(t) = P_r \{ \text{the system is up in } (0,t] / E_1 \text{ at } t = 0 \}
\]

and is given by

\[
4.2.26 \quad R_1(t) = \bar{F}(t) + \{ \sum_{n=1} \alpha^{(n)}(t) \} \odot R_1(t)
\]

The above expression is derived by considering the fact that the interval \((0,t)\) is not intercepted by an \( E_1 \) event or intercepted by at least one \( E_1 \) event.

Taking Laplace-Transform on both sides of 4.2.26, we obtain

\[
4.2.27 \quad R_1^*(s) = P_1^*(s) = \frac{\bar{F}^*(s)}{1 - \alpha^*(s)}
\]

Furthermore,

\[
4.2.28 \quad \alpha^*(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(z) \, dz \int_0^\infty [\rho_{10}(t) \odot e^{-(\lambda + \mu)}] e^{-(s-z)t} \, dt
\]

\[
= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(z) \rho_{10}(s-z)}{(\lambda + \mu + s-z)} \, dz
\]

where

\[
\rho_{10}(s) = \begin{bmatrix}
q_{10}^* + q_{12}q_{20}^*
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - q_{12}q_{21}^* - q_{01}q_{10}^* - q_{01}q_{12}q_{20}^* - q_{02}q_{20}^* - q_{02}q_{21}q_{10}^*
\end{bmatrix}
\]

70
\[ q_{01}(s) = \frac{\lambda}{(s+\lambda+\mu)} \quad q_{02}(s) = \frac{\mu}{(s+\lambda+\mu)} \]

\[ q_{10}(s) = g_1^*(s+\mu) \quad q_{12}(s) = g_1^*(s) - g_2^*(s+\mu) \]

\[ q_{20}(s) = g_2^*(s+\lambda) \quad q_{21}(s) = g_2^*(s) - g_2^*(s+\lambda) \]

We observe that \( \rho_{10}(s) \), being a Laplace transform, the degree of the numerator is strictly less than the degree of the denominator. Furthermore, the fact that \( g_1(\cdot) \) and \( g_2(\cdot) \) are rational functions guarantees that \( \rho_{10}(\cdot) \) is a rational function.

Furthermore, let \( f^*(z) = \frac{A(z)}{B(z)} \) and \( \rho_{10}^*(z) = \frac{C(z)}{D(z)} \);

where \( B(z) \) and \( D(z) \) are polynomials of degree \( n \) and \( m \) respectively and \( A(z) \) and \( C(z) \) are polynomials of degree at most \( n-1 \) and \( m-1 \) respectively.

Then from 4.2.28 we observe that \( \alpha^*(s) \) is also a rational function.

Let \( \alpha^*(z) = \frac{P(z)}{Q(z)} \) where \( P(z) \) and \( Q(z) \) are polynomials with complex coefficients. Therefore, all the singularities of \( \alpha^*(s) \) are poles.

Thus, from 4.2.27, we have

\[ 4.2.29 \quad \tilde{p}_1^*(s) = \frac{Q(s)[B(s) - A(s)]}{s \cdot B(s)[Q(s) - P(s)]} \]
\[ p_1^* (s) = 1 - sP_1^* (s) \]
\[ = \frac{Q(s)A(s) - B(s)P(s)}{B(s)[Q(s) - P(s)]} \]

We notice that \( p_1^* (s) \) and \( P_1^* (s) \) are rational functions of their arguments and their singularities are essentially same. Let \( s = \eta \) be the pole of \( p_1^* (s) \) situated furthest to the right of \( s \)-plane. Then \( \eta \) is real and negative and is the abscissa of convergence of \( p_1^* (s) \) (Widder [1971]). Furthermore, for the same reason the pole \(-\gamma\) \((\gamma > 0)\) of \( f^* (s) \) situated closest to the right of \( s \)-plane is real and negative.

**Lemma 4.2.1:** The pole \( \eta \) of \( p_1^* (s) \) is such that \(-\gamma < \eta < 0\) and is a simple pole.

**Proof** Now consider \((\lambda + \mu + s)Q^* (s)\). This is a rational function and has a pole at \( s = 0 \). Furthermore,

\[ (\lambda + \mu + s)Q^* (s) \bigg|_{s=0} = 0 \]

and

\[ \frac{d}{ds}[(\lambda + \mu + s)Q^* (s)] \bigg|_{s=0} = \omega \]

where \( \omega = \lambda + \mu \left[ g_2^* (0) (g_1^* (\mu) - 1) + g_1^* (0) (g_2(\lambda - 1)) \right] \\
+ g_1(\mu) \left[ 1 - \alpha g_2^* (0) \right] + g_2(\lambda) \left[ 1 - g^* (\alpha) - \lambda g^* (0) \right] \]
We observe that all the terms on the right hand side of 4.2.32 are positive. Therefore

\[ 4.2.33 \quad \frac{d}{ds}[\lambda \mu \ + \ s \ Q^*(s)] \bigg|_{s=0} > 0 \]

Hence, \( s=0 \) is a zero of \((\lambda + \mu + s) \ Q^*(s)\) with multiplicity one. Thus

\[ 4.2.34 \quad \alpha^*(s) = \frac{1}{2\pi i} \int_{\gamma_{-\infty}}^{\gamma_{+\infty}} \frac{f^*(z) P^*(\lambda+z)}{(\lambda + \mu + \lambda - z) Q^*(\lambda - z)} \ dz, \quad 0 < \sigma < \text{Re}(s) \]

\[ = \frac{1}{2\pi i} \int_{\gamma_{-\infty}}^{\gamma_{+\infty}} \frac{f^*(z) P^*(\lambda+z)}{(\lambda + \mu + \lambda - z) Q^*(\lambda - z)} \ dz \]

\[ - f^*(s) \lim_{\lambda \to s} \frac{(\lambda-s) P^*(\lambda-s)}{(\lambda + \mu + \lambda - s) Q^*(\lambda - s)} \]

\[ = \frac{f^*(s)}{\omega} + \frac{1}{2\pi i} \int_{\gamma_{-\infty}}^{\gamma_{+\infty}} \frac{f^*(z) P^*(\lambda+z)}{(\lambda + \mu + \lambda - z) Q^*(\lambda - z)} \ dz \]

If \( \Phi(s) = 1 - \alpha^*(s) \), then

\[ 4.2.35 \quad \Phi(s) = 1 - \frac{f^*(s)}{\omega} - \frac{1}{2\pi i} \int_{\gamma_{-\infty}}^{\gamma_{+\infty}} \frac{f^*(z) P^*(\lambda+z)}{(\lambda + \mu + \lambda - z) Q^*(\lambda - z)} \ dz \]

We observed that

\[ 4.2.36 \quad \Phi(0) = 1 - \alpha^*(0) \]

\[ = 1 - \int_0^\infty f(t) \left[ \varphi_{10}(t) \ast e^{-(\lambda + \mu)t} \right] \ dt \]
Observing that \( \rho_{10}(t) \triangleq e^{-(\lambda + \mu)t} \) is the probability that an operable standby unit is switched over to online, conditioned on the commencement of repair of online failed unit at \( t = 0 \), we have

\[
\int_{0}^{\infty} f(t) \left[ \rho_{10}(t) \triangleq e^{-(\lambda + \mu)t} \right] dt < 1
\]

We also observe that

\[
\Phi(-\gamma) = 1 - \alpha^*(\gamma) = \infty
\]

Equation 4.2.38 follows from 4.2.35 and from the fact that \(-\gamma\) is a singularity of \( f(s) \). Hence from 4.2.36 and 4.2.38 it can be seen that there exists at least one zero, say \( \xi \), of \( \Phi(s) \) such that \(-\gamma < \Re \xi < 0\). As \( \eta \) is assumed to be the pole of \( p_1^*(s) \) closest to the right of s-plane we get \(-\gamma < \eta < 0\). Since \( \Phi(\eta) = 1 - \alpha'(\eta) = 0 \) [which implies \( \alpha^*(\eta) = 1 \)], we get \( \alpha^*(s) \) to be convergent for \( s = \eta \) and

\[
\Phi'(\eta) = \int_{0}^{\infty} t \alpha(t) e^{-\eta t} dt, \text{ exists and is greater than 0}
\]

Therefore, \( \eta \) is a root of \( 1 - \alpha^*(s) \) with multiplicity one. Thus, \( \eta \) is a simple pole of \( p^*(s) \) and also of \( \tilde{P}'_1(s) \). Hence the lemma.

As a consequence of 4.2.29 we see that \( \eta \) is a zero of \([Q(s) - P(s)]\) and not that of \( B(s) \).

**Theorem 4.2.1** The quasi-stationary distribution \( H_1(x) \) is exponential with parameter \(-\eta, \eta < 0\)
Proof Since $\eta$ is the simple pole closest to the origin, on inverting 4.2.27, we get, as $t \to \infty$

4.2.40 \[ \overline{P}_1(t) = \overline{F}^*(\eta) \Lambda e^{\eta t} + o(e^{\eta t}) \]

and

4.2.41 \[ \overline{P}_1(x+t) = \overline{F}(\eta) \Lambda e^{\eta(x+t)} + o(e^{\eta t}) \]

where \[ \Lambda = \lim_{s \to \eta} \frac{s - \eta}{1 - \alpha^*(s)} \]

Hence

4.2.42 \[ \frac{\overline{P}_1(x+t)}{\overline{P}_1(t)} \to \lim_{t \to \infty} \frac{\overline{F}(\eta) \Lambda e^{\eta(x+t)} + o(e^{\eta t})}{\overline{F}(\eta) \Lambda e^{\eta t} + o(e^{\eta t})} = e^{\eta x} \]

Therefore,

4.2.43 \[ H_1(x) = 1 - e^{\eta x}, \quad \eta < 0 \]

That is, the quasi-stationary distribution of a two-unit warm standby system when the switch is subject to failure is exponential with parameter $-\eta$, $\eta < 0$.

The theorem shows that irrespective of nature failure time and repair times of switching device as well as that of units, the quasi-stationary distribution is exponential.

Illustration 1 For the purpose of illustration we consider a special case in which
\[ f(t) = \alpha e^{-\alpha t}, \quad \alpha > 0 \]

\[ g_i(t) = \beta_i e^{-\beta_i t}, \quad \beta_i > 0, \quad i = 1, 2 \]

Then

\[ q_01(s) = \frac{\lambda}{\lambda + \mu + s} \quad q_{10}(s) = \frac{\beta_1}{\beta_1 + \mu + s} \]

\[ q_{02}(s) = \frac{\mu}{\mu + \lambda + s} \quad q_{20}(s) = \frac{\beta_2}{\beta_2 + \lambda + s} \]

\[ q_{12}(s) = \frac{\beta_1 \mu}{(\beta_1 + s)(\beta_2 + \mu + s)} \quad q_{21}(s) = \frac{\beta_2 \lambda}{(\beta_2 + s)(\beta_2 + \lambda + s)} \]

and

\[ \Phi(s) = 0 \Rightarrow 1 - \frac{\alpha \rho_{10}(s + \alpha)}{\alpha + \lambda + \mu + \xi} = 0 \]

Equivalently, on simplification

\[ \Phi(s) = 0 \Rightarrow \sum_{i=1}^{7} w_i s^{8-i} = 0 \]

where

\[ w_i = q_i - u_i, \quad i = 1 \text{ to } 6 \]

\[ q_1 = b_4 \rho_1 \]

\[ w_7 = q_7 \]

\[ q_8 = p_7 \]

\[ u_1 = t_1 r_1 \]

\[ u_2 = t_1 r_2 + t_2 r_1 \]

\[ u_3 = t_1 r_3 + t_2 r_2 + t_3 r_1 \]

\[ u_4 = t_2 r_1 + t_3 r_2 + r_1 \]

\[ u_5 = t_3 r_3 + r_2 \]

\[ u_6 = r_3 \]
\[ t_1 = b_4^2 b_5 \quad t_2 = 2b_4^2 b_5 \quad t_3 = b_5 + 2b_4 \]

\[ r_1 = a_1(b_1 + b_3) \quad r_2 = a_1(b_1 + b_3) \quad r_3 = a_1 \]

\[ p_4 = l_4 - (d_4 + c_4) \quad p_5 = l_5 - (d_5 + c_5) \]

\[ p_6 = l_6 \quad p_7 = l_7 \]

\[ k_1 = b_4^2 k_3 \quad k_2 = 2b_4 k_3 \quad k_3 = a_1 a_2 \lambda \mu \]

\[ m_1 = b_1 b_2 m_3 \quad m_2 = (b_1 + b_2)m_1 \quad m_3 = a_2 \lambda^2 \mu \]

\[ n_1 = b_4 b_5 n_3 \quad n_2 = (b_4 + b_5)n_1 \quad n_3 = a_1 a_2 \lambda^2 \mu \]

\[ d_1 = a_1 a_2 b_4 b_5 b_1 b_3 \quad d_2 = a_1 \lambda \quad [b_4 b_5 (b_1 + b_3) + (b_4 + b_5)b_1 b_3] \quad d_3 = a_1 \lambda \quad [b_4 b_5 + (b_4 + b_5)(b_1 + b_3) + b_1 b_3] \]

\[ c_1 = b_1 b_2 b_4 b_5 \mu \quad c_2 = \mu \quad [b_1 b_2(b_4 + b_5) + (b_1 + b_2)b_4 b_5] \quad c_3 = \mu \quad [(b_1 + b_2) (b_4 + b_5) + b_4 b_5] \]

\[ b_i = a_i + \alpha \quad i = 1 \text{ to } 5 \]

and

\[ a_1 = \beta_1, \quad a_2 = \beta_1 + \mu, \quad a_3 = \beta_3 + \lambda, \quad a_4 = \lambda + \mu, \quad a_5 = \beta_2 \]

For known values of \((\alpha, \mu, \lambda, \beta_1, \beta_2)\) resulting simple negative roots of \(\Phi(s) = 0\) can be obtained using Newton-Raphson method and thus the root closest to the origin can be identified. Table 4.2.1 provides simple
negative root closest to the origin. For instance when \((\alpha, \mu, \lambda, \beta_1, \beta_2) = (1.5, 20, 25, 30, 50)\) simple negative roots in \((-5000, 0)\) are \(-243720, -4645729\) and \(-7154138\). The root \(\eta\) closest to the origin is given by \(\max (-2.43720, -4645729, -7154138) = -2.43720\). Therefore, the quasi-stationary distribution is exponential with parameter 2.43720.

**Illustration 2** For the purpose of illustration, we consider

\[ f(t) = \beta \ e^{-\beta t}, \ \beta > 0, \ g_1(t) = \gamma \ e^{-\gamma t}, \ \gamma > 0 \text{ and } u = 0 \]

In this case, \(\alpha^\bullet(s) = \frac{\beta \rho_0^\bullet(s+\beta)}{(s+\beta+\lambda)}\)

\[ \rho_0^\bullet(s) = \frac{\gamma}{\gamma + s} \left[ 1 + h^\bullet(s) \right] \]

\[ h^\bullet(s) = \lambda \gamma \ [(s+\lambda)(s+\gamma) - \lambda \gamma]^1 \]

By Lemma 4.2.1, \(\eta\) is a simple root of \([1 - \alpha^\bullet(s)]\) with multiplicity one. Furthermore, \(1 - \alpha^\bullet(s) = 0 \Rightarrow (s+\lambda+\beta)(s+\lambda+\gamma) - \lambda \gamma = 0\)

The real, negative root \(\eta\) closest to the origin is given by

\[ \eta = \frac{1}{2} \left\{ - (2\lambda + \beta + \gamma) + [(\beta + \gamma)^2 + 4\lambda \gamma]^{1/2} \right\} \]

Hence, the quasi-stationary distribution of a two-unit warm standby system is exponential with parameter

\[ \frac{1}{2} \left\{ (2\lambda + \beta + \gamma) - [(\beta + \gamma)^2 + 4\lambda \gamma]^{1/2} \right\} \]
4.3 A Two-unit Cold Standby System Supported by Two Repair Facilities

Consider a reliability system with two identical repairable units. One working online and the other kept in cold standby. When the online unit fails, an operable standby is switched over to online by a switching device. The switching device has its own pattern of failure time and repair time distributions. There are two repair facilities, one exclusively meant for the switching device and the other for the units. Repair completely restores the properties of the units. It is assumed that the switchover times are negligible.

Let $\lambda e^{-\lambda t}$ and $g(t)$ be p.d.f of the failure time and repair time of the units respectively and $\alpha e^{-\alpha t}$ and $h(t)$ be p.d.f of the failure time and repair time of the switching device respectively. It is assumed that the Laplace-Stieljes transform of $g(t)$ and $h(t)$ are rational functions of their arguments.

We define

$e_0$ commencement of operation of an unit while the other unit as well as the switching device are in operable condition

$e_1$ commencement of repair of an online failed unit (at this instant the other unit begins to operate online and the switching device is found in operable condition)

$T_i$ time to system failure starting with an $e_i$-event initially, $i=0,1$
\( p_i(t) \) p.d.f of \( T_i \), \( i=0,1 \)

\( X(t) \) residual life time of the system at any instant \( t \)

Let

\[
4.3.1 \quad H_i(x,t) = \Pr \{ X(t) \leq x \mid T_i > t \}, \quad i=0,1
\]

Equivalently,

\[
4.3.2 \quad H_i(x,t) = 1 - \frac{\overline{P}_i(x+1)}{\overline{P}_i(t)}
\]

where \( \overline{P}_i(t) \) is the reliability of the system starting with an \( e_i \)-event.

Therefore

\[
4.3.3 \quad H_i(x) = \lim_{t \to \infty} H_i(x,t) = 1 - \lim_{t \to \infty} \frac{P_i(x+1)}{\overline{P}_i(t)}, \quad i=0,1
\]

The above limit is obtained by first writing the integral equations for \( \overline{P}_i(t) \) using the regeneration point method and then by applying the Laplace-Transform technique to get an asymptotic expansion for \( \overline{P}_i(t) \).

The behaviour of the switching device during a failure free operation of online unit is discussed in Section 2.2.3. Using the equations \( 2.2.3.2 \) through \( 2.2.3.5 \) and observing that \( e_i \)-events are regenerative in nature, equation for \( \overline{P}_1(t) \) can be written by classifying the realisations into two mutually exclusive and exhaustive cases, namely the interval \((0,t)\) is intercepted by an \( e_1 \)-event or not intercepted by an \( e_1 \)-event.
Thus, the reliability of the system is given by

\[ P_1(t) = e^{-\lambda t} + \alpha(t) \odot \bar{P}_1(t) \]

\[ \alpha(t) = \lambda e^{-\lambda t} G(t) Q_{oo}(t) \]

Taking Laplace-Transform on both sides of 4.3.4 and 4.3.5, we obtain

\[ P_1^*(s) = \frac{1}{s + \lambda} \left[ 1 - \alpha^*(s) \right] \]

\[ \alpha^*(s) = \int_0^\infty \lambda e^{-(\lambda + s)t} G(t) Q_{oo}(t) dt \]

Clearly, \( \alpha^*(s) \) is a rational function

Let

\[ \alpha^*(s) = \frac{A(s)}{B(s)} \]

where \( A(s) \) and \( B(s) \) are polynomials of degree \( n \) and \( m \) \((n > m)\) respectively.

Then

\[ P_1^*(s) = \frac{1}{s + \lambda} \frac{B(s)}{[B(s) - A(s)]} \]

and

\[ p_0^*(s) = 1 - sP_1^*(s) = \frac{\lambda[B(s) - A(s)] - sH(s)}{(s + \lambda)[B(s) - A(s)]} \]
We observe that 4.3.7 and 4.3.8 are rational functions of their arguments and their singularities are essentially same. Moreover, all the singularities of $P_i^*(s)$ are poles. Let $s = \eta$ be the pole of $p_i^*(s)$ situated closest to the right of s-plane. Then $\eta$ is real and negative and is the abscissa of convergence of $p_i^*(s)$.

**Lemma 4.3.1** The pole $\eta$ of $p_i^*(s)$ is such that $-\lambda < \eta < 0$ and is a simple pole.

**Proof** If $\Phi(s) = 1 - a^*(s)$, then

\[ 4.3.9 \quad \Phi(s) = 1 - \int_0^\infty \lambda G(t)Q_{oo}(t) e^{-\lambda t + \lambda t} dt \]

and

\[ 4.3.10 \quad \Phi(0) = 1 - \int_0^\infty \lambda e^{-\lambda t} G(t)Q_{oo}(t) dt \]

Observing that

\[ \int_0^\infty \lambda e^{-\lambda t} G(t)Q_{oo}(t) dt < \int_0^\infty \lambda e^{-\lambda t} dt = 1 \]

[since $G(t)Q_{oo}(t)$ is the probability that both the standby unit and the switching device are good at time $t$ given that a repair has commenced initially]

We conclude that $\Phi(0)$ is greater than zero. Moreover,

\[ 4.3.11 \quad \Phi(-\lambda) = 1 - a^*(-\lambda) < 0 \]
The expression 4.3.11 follows from 4.3.9 and from the fact that \( \lambda \) is a singularity of \( \frac{1}{s+\lambda} \). Hence from 4.3.10 and 4.3.11 it can be seen that there exists at least one \( \xi \) of \( \Phi(\xi) \) such that \( -\lambda < \text{Re} \xi < 0 \). As \( \eta \) is assumed to be the pole of \( p_1^*(s) \) which is closest to the right of s-plane, we have \( -\lambda < \eta < 0 \). Since \( \Phi(\eta) = 1 - \alpha^*(\eta) = 0 \), we have \( \alpha^*(\eta) = 1 \). This implies that \( \alpha^*(s) \) is convergent for \( s = \eta \) and

\[
4.3.12 \quad \Phi'(\eta) = \int_0^1 \alpha(t) e^{-\eta t} dt > 0 \text{ exists}
\]

Thus, \( \eta \) is a root of \( 1 - \alpha^*(s) = 0 \) with multiplicity one. This implies that \( \eta \) is a simple pole of \( p_1^*(s) \) and also of \( P_1^*(s) \). Consequently \( \eta \) is a zero of \([B(s) - A(s)]\).

**Theorem 4.3.1** The quasi-stationary distribution \( H_1(x) \) is exponential with parameter \(-\eta\), \((\eta < 0)\).

**Proof:** Using lemma 4.3.1 and inverting 4.3.6, we get,

\[
4.3.13 \quad \bar{P}_1(t) = \frac{1}{\eta + \lambda} \Lambda e^{\eta t} + o(e^{\eta t}), \quad \text{as } t \to \infty
\]

and

\[
4.3.14 \quad \bar{P}_1(x+t) = \frac{1}{\eta + \lambda} \Lambda e^{\eta x + \eta t} + o(e^{\eta t}), \quad \text{as } t \to \infty
\]

where

\[
\Lambda = \lim_{s \to \eta} \frac{s - \eta}{1 - \alpha^*(s)}
\]
Proceeding on lines as given in theorem 4.2.1, we get,

\[ \lim_{t \to \infty} \frac{P_1(x+t)}{P_1(t)} = e^{\eta x}, \quad \eta < 0 \]

which in turn gives

\[ H_1(x) = 1 - e^{\eta x}, \quad \eta < 0 \]

Hence the theorem

Now consider \( \overline{P}(t) \), starting with an \( e_0 \)-event initially. The equation governing \( \overline{P}(t) \) conditioned on \( e_0 \)-event at \( t=0 \) is given by

\[ \overline{P}_0(t) = e^{-\lambda t} + \alpha_0(t) \circ \overline{P}_1(t) \]

where

\[ \alpha_0(t) = \lambda e^{-\lambda t} \quad Q_{oo}(t) \]

Taking Laplace transform on both the sides of 4.3.17 and 4.3.18 and using 4.3.6, we obtain, after simplification,

\[ \overline{P}_0^*(s) = \frac{1}{s + \lambda} \left( 1 - \alpha^*(s) + \alpha_0^*(s) \right) \]

**Theorem 4.3.2:** The quasi-stationary distribution \( H_0(x) \) is exponential with parameter \( -\eta \) \((\eta < 0)\)

Proof. Conditioned on an \( e_0 \)-event at \( t=0 \), for a cold standby system, \( Q_{oo}(t) \) yields the probability that both the standby unit and the switching device are in operable condition and its value is always less than unity.
Hence, \( \alpha_0(t) \leq \lambda e^{-\lambda t} \) and the abscissa of convergence \( \alpha_0^*(s) \) is less than the abscissa of convergence of \( \lambda e^{-\lambda t} \). Moreover from Lemma 4.3.1 \( \alpha_0^*(\eta) \) exists. Observing that

\[
\alpha^*(\eta) = 1 \quad \text{and} \quad [1 - \alpha^*(\eta) + \alpha_0^*(\eta)] = \alpha_0^*(\eta)
\]

we conclude from 4.3.19 that the singularity closest to the origin of \( \overline{P}_0^*(s) \) is \( \eta \) and it is a simple pole. Consequently, \( \eta \) is the zero of \( 1 - \alpha^*(s) \).

Inverting 4.3.6 yields,

\[
4.3.20 \quad \overline{P}_0(t) = \frac{1}{\eta + \lambda} \Lambda_0 e^{\eta t} + o(e^{\eta t}), \quad \text{as} \; t \to \infty
\]

where

\[
\Lambda_0 = \lim_{s \to \eta} \frac{s - \eta}{1 - \alpha_0^*(s)}
\]

and

\[
4.3.21 \quad \overline{P}_0(x+t) = \frac{1}{\eta + \lambda} \Lambda_0 e^{\eta(x+t)} + o(e^{\eta t}), \quad \text{as} \; t \to \infty
\]

Therefore,

\[
4.3.22 \quad \lim_{t \to \infty} \frac{\overline{P}_0(x+t)}{\overline{P}_0(t)} = e^{\eta x}, \quad \eta < 0
\]

which implies

\[
4.3.23 \quad H_0(x) = 1 - e^{\eta x}, \quad \eta < 0
\]

85
Theorem 4.3.1 and theorem 4.3.2 show that irrespective of the nature the life and repair time distributions, the quasi-stationary distribution is exponential and is independent of the initial event.

Illustration 1 For the purpose of illustration we assume that the repair time density of the unit is exponentially distributed with parameter $\mu$.

Thus

$$\alpha^* (s) = \lambda \left[ Q_{00}^* (s + \lambda) - Q_{00}^* (s + \lambda + \mu) \right]$$

and

$$Q_{00}^* (s) = \frac{s + \beta}{s(s + \alpha + \beta)}$$

Furthermore,

$$\Phi(s) = 0 \Rightarrow 1 - \lambda \left[ Q_{00}^* (s + \lambda) - Q_{00}^* (s + \lambda + \mu) \right] = 0$$

Equivalently, on simplification,

$$\Phi(s) = 0 \Rightarrow \sum_{i=1}^{4} m_{i(4-i)} s^i = 0$$

where

$$m_1 = d_1, \quad m_i = d_i - \lambda(c_{i-1} - b_{i-1}), \quad i = 2, 3, 4$$

$$b_1 = a_1 + a_4 + \lambda, \quad c_1 = a_2 + a_3 + a_5$$

$$b_2 = a_4(a_1 + \lambda + \lambda a_1), \quad c_2 = a_5(a_2 + a_3) + a_2 a_3$$

$$b_3 = \lambda a_1 a_4, \quad c_3 = a_2 a_3 a_5$$
\[ d_1 = a_1 + a_2 + a_3 + \lambda \quad d_3 = (a_1 + \lambda)a_1a_3 + a_1(a_2 + a_3) \]
\[ d_2 = a_1a_3 + (a_1 + \lambda)(a_2 + a_3) + a_3 \lambda \quad d_4 = a_1a_2a_3 \]

and
\[ a_1 = \lambda + \beta + \alpha \quad a_4 = \lambda + \mu + \beta \]
\[ a_2 = \lambda + \mu \quad a_5 = \lambda + \beta \]
\[ a_3 = \lambda + \mu + \alpha + \beta \]

For given values of \((\lambda, \mu, \alpha, \beta)\) resulting negative roots of \(\Phi(s) = 0\) can be obtained using Newton-Raphson method and thus the root closest to the origin can be identified. Table 4.3.1 provides simple negative root closest to the origin. For instance when \((\lambda, \mu, \alpha, \beta) = (3, 15, 50, 100)\) simple negative roots in \((-5000, 0)\) are \(-129308\) and \(-976844\). The root \(\eta\) closest to the origin is given by \(\text{Max} (-129308, -976844) = -129308\). Therefore, the quasi-stationary distribution is exponential with parameter 1.29308.

Illustration 2: If we assume that the switch is failure free then
\[ \alpha(t) = \int_0^\infty \lambda e^{-\lambda t} \left[ 1 - e^{-\mu t} \right] \, dt \]
\[ \Phi(s) = 1 - \alpha^*(s) = 0 \Rightarrow (\lambda + s)(\lambda + \mu + s) - \lambda \mu = 0 \]

The real negative root \(\eta\) closest to the origin is given by
\[ \eta = -\frac{1}{2} \left[ -(2\lambda + \mu) + (\mu^2 + 4\lambda \mu)^{\frac{1}{2}} \right] \]
Hence the quasi-stationary distribution of a 2-unit cold standby system is exponential with parameter

\[ \eta = \frac{1}{2} \left[ (2\lambda + \mu) - (\mu^2 + 4\lambda\mu)^{1/2} \right] \]

This is in agreement with Kalpakam and Hameed [1983]

It would be interesting to investigate the quasi-stationary distribution for the system considered in 4.3 with the standby subject to failure.