CHAPTER 3

PRIORITY REDUNDANT SYSTEMS
3.1 Introduction

Studies relating to priority redundant repairable systems are not many even though there are several real life situations, such as anti-aircraft systems where such a study is applicable. A priority redundant system consists of \( n \) (\( \geq 2 \)) units in which one of the units is called the priority unit (p-unit) and the others termed as non-priority units (o-units). The p-unit is always in operation except when it is failed. The o-units are put into operation only for the duration of the repair of the p-unit. Osaki [1970] studied a two-unit priority standby redundant system and obtained the Laplace-Stieltjes transform of the distribution of the time-to-system-failure. Buzacott [1971] extended Osaki's model and derived the steady state availability. Later Nakagawa and Osaki [1975] also obtained expected number of system-downs during a finite interval, distribution of the busy period of the repair time, distribution of time to system recovery in addition to availability and MTSF for a two-unit priority system with arbitrary failure time and repair time distributions for the p-unit and the o-unit having an exponential distribution in service. Unlike Nakagawa and Osaki [1975], Venkatakrishnan [1975] considered a two-unit warm standby priority system and obtained the availability and reliability functions assuming pre-emptive and non-pre-emptive repair policies for the o-unit. Arora [1976a] analysed a two-unit priority system with restricted repair while Kodama et al [1976] investigated seven models of two-unit warm standby systems in which the units have constant failure rates and the repair time distribution is arbitrary. Subramanian and Ravichandran [1980] considered a two-unit
warm standby priority system with a discretionary policy for the repair of the o-unit and obtained the expression for the reliability, availability and interval reliability of the system. In all the models mentioned above, the o-unit while in standby cannot fail or has a constant failure rate. Natarajan [1980] has obtained the reliability characteristics of a two-unit warm standby redundant system assuming Erlang failure time distribution for the o-unit in standby.

In the analysis of priority redundant repairable systems either of the following policies of repair are adopted.

(i) Either pre-emptive or non-preemptive priority for the repair of the o-unit.

(ii) In case of pre-emptive priority, if the p-unit fails in service during the repair of the o-unit, the repair of the o-unit is interrupted and the p-unit is taken up for repair.

After repair completion of the p-unit, the o-unit is taken up for repair with either of the following policies:

1. Pre-emptive resume policy

2. Pre-emptive repeat policy

We observe that even though policy 1 simplifies the analysis, policy 2 is preferable.
In the section to come we analyse a two-unit priority standby redundant system in which the failure time of the o-unit in standby has phase type distribution, the repair time of the o-unit is Erlang distributed and while in service the o-unit is subject to preventive maintenance. The interpretation that for an Erlang distribution the sojourn times in each stage is distributed exponentially enables us to assume pre-emptive repeat policy for the repair of the o-unit. In section 3.3 we study a priority standby redundant system in which the failure time distribution of the p-unit is different after each repair.

3.2 A Complex Priority Standby Redundant System

In this section we study a two-unit warm standby priority redundant system in which the failure time distribution of non-priority unit in standby state has a phase type distribution. The phase type distribution introduced by Neuts [1975] is more general in the sense that the Erlang distribution is a particular case of the former. Neuts and Meier [1980], Ravichandran [1984] and Gururajan et al [1986] have discussed the applicability of phase type distribution in reliability problems. The section to follow describes the system.

3.2.1 System description and notation

1. The system consists of two repairable units. One unit, the p-unit, is always in service except when it is failed. The other unit is allowed to operate online only when the p-unit is under repair.

† A modified version of this section has appeared in Microelectron Reliab., 30, 453-455, 1990
2. The o-unit while in standby has phase type failure time distribution with $m_1$ phases and is represented by $(\alpha, T)$, where $\alpha$ is an $m_1$-probability vector and $T$ is a matrix of order $m_1$.

3. When the p-unit fails it is replaced by the o-unit, if operable. If, at the epoch of failure of the p-unit, the o-unit is undergoing repair, the repair of the o-unit is interrupted and the p-unit is taken up for repair immediately. The repair of the o-unit is continued from where it was left once the repair of the p-unit is completed.

4. When the repair of the p-unit is completed it is switched into service immediately and the o-unit is taken up for repair or preventive maintenance accordingly as it is in failed state or in operating state at the epoch of the p-unit repair completion.

5. The repair time and preventive maintenance time of the o-unit are Erlang distributed random variables with $m_2$ stages (mean $\frac{m_2}{\alpha}$)

and $m_3$ stages (mean $\frac{m_3}{b}$) respectively.

6. There is only one repair facility.

7. All switchover times are negligible and the switch is perfect.

8. Repair completely restores the property of a unit.

$$f_p()$$ pdf of failure time of the p-unit.
\[ g_p(.) \text{ p.d.f. of repair time of the } p\text{-unit} \]

\[ f_j(.) \text{ p.d.f of failure time of the } o\text{-unit when switched into service from } j\text{-th phase in the standby state, } j = 1, 2, \ldots, m_1 \]

\[ B(m_2, 1, a, t) = \frac{(at)^{m_2-1} e^{-at}}{(m_2 - 1)!}, \text{ repair time density of the } o\text{-unit} \]

\[ B(m_3, 1, b, t) = \frac{(bt)^{m_3-1} e^{-bt}}{(m_3 - 1)!}, \text{ preventive maintenance time density of the } o\text{-unit} \]

\[ Q_i(t) = \Pr \{ \text{the Markov chain corresponding to phase type distribution with representation } (\alpha, T) \text{ is in phase } j \text{ at } t \text{ if it was found in phase } i \text{ at } t=0 \}, \quad i = 1, 2, \ldots, m_1; \quad j = 1, 2, \ldots, m_1 \]

\[ Q_i(t)\delta t = \Pr \{ \text{the absorption in the Markov chain corresponding to phase type distribution with representation } (\alpha, T) \text{ occurs between } t \text{ and } t+\delta t \text{ if it was found in phase } i \text{ at time } t=0 \}, \quad i = 1, 2, \ldots, m_1 \]

\[ B^k(m_2, 1, a, t) = \alpha_k B(m_2, 1, a, t), \quad k = 1, 2, \ldots, m_1 \]

\[ B^k(m_3, 1, b, t) = \alpha_k B(m_3, 1, b, t), \quad k = 1, 2, \ldots, m_1 \]
3.2.2 The stochastic behaviour of the o-unit in standby

The behaviour of the o-unit in standby during a period of continuous operation of the p-unit in service can be described by a stochastic process \( \{Z(t), t \geq 0\} \). The random variable \( Z(t) \) takes values on the state space \( \{o,r\} \) where 'o' denotes that the unit is operable and 'r' denotes that the unit is under repair at any instant \( t \). The process \( \{Z(t), t \geq 0\} \) can be identified with an alternating renewal process Cox [1962] and can be characterized using the functions \( P_{\theta_1, \eta_j}(t) \) defined below:

\[
P_{\theta_1, \eta_j}(t) = \Pr\{Z(t) = (\eta, j) / Z(0) = (\theta, i)\}
\]

\( \theta, \eta = o \) (operable), \( r \) (repair), \( l \) (preventive maintenance)

\( i = 1, 2, ..., m_1 \), or \( 1, 2, m_2 \) or \( 1, 2, m_3 \) according as \( \theta = o \) or \( r \) or \( l \)

\( j = 1, 2, ..., m_1 \), or \( 1, 2, m_2 \) or \( 1, 2, m_3 \) according as \( \eta = o \) or \( r \) or \( l \)

The ordered pair \((\theta, m)\) represents the state of the o-unit in standby \( \theta (= o, r, l) \) represents the state of the o-unit and \( m \) represents its stage (phase) in that state.

The functions \( P_{o_i, o_j}(t) \) are given by

\[
3.2.2.2 \quad P_{o_i, o_j}(t) = Q_0(t) + \sum_{k=1}^{m_1} \{Q_1(t) \cap B^k(m_2, 1, a, t) \} \cap P_{o_i, o_j}(t),
\]

\( i, j = 1, 2, ..., m_1 \)
The expression 3.2.2.2 is obtained by considering the following mutually exclusive and exhaustive cases

(i) the standby unit does not fail up to \( t \)

(ii) the standby unit fails before \( t \)

The first case implies that the absorption in the Markov chain corresponding to phase type distribution with representation \((\alpha; T)\) has not occurred till \( t \). This probability is given by \( Q_{ij}(t) \). The second case corresponds to the absorption in the Markov chain between \( u \) and \( u + \delta u, \ 0 < u < t \). Consequently, the standby unit is taken up for repair, the repair is completed between \( v \) and \( v + \delta v, \ 0 < u < v < t \) and the repair completed unit is found in phase \( j \) in the standby state. This probability is given by

\[
\sum_{k=1}^{m_1} \left\{ Q_i(t) \otimes B^k (m_2,1,a,t) \right\} \otimes P_{ok,ij}(t),
\]

\( i,j = 1,2, \ldots, m_1 \)

Using similar renewal theoretic arguments, we get

3.2.2.3. \( P_{o1,j}(t) = Q_i(t) \otimes P_{r1,ij}(t), \)

\( i = 1,2, \ldots, m_1, \ j = 1,2, \ldots, m_2 \)

3.2.2.4. \( P_{r1,j}(t) = e^{-at} \frac{(at)^{j-1}}{(j-1)!} + \sum_{i=1}^{m_1} \left\{ B^i (m_2,1,a,t) \otimes P_{o1,ij}(t) \right\}, \)

\( j = 1,2, \ldots, m_2 \)
3.2.2.5 \( P_{\text{n},\text{oj}}(t) = \sum_{k=1}^{m_1} \{B^k (m_{2,i,a},t) \odot P_{\text{o},\text{oj}}(t)\}, \)
\[ i = 1,2, ,m_2 , \quad j = 1,2, ,m_1 \]

3.2.2.6 \( P_{\text{n},\text{ij}}(t) = \delta_{ij} e^{-at} + \beta(j-j) e^{-at} \frac{(at)^{-1}}{(j-1)!} \)
\[ + \sum_{k=1}^{m_1} \{B^k (m_{2,i,a},t) \odot P_{\text{o},\text{ij}}(t)\}, \]
\[ i = 1,2, ,m_2 , \quad j = 1,2, ,m_2 \]

3.2.2.7 \( P_{\text{l},\text{oj}}(t) = \sum_{k=1}^{m_1} \{B^k (m_{3,i,b},t) \odot P_{\text{o},\text{oj}}(t)\}, \)
\[ i = 1,2, ,m_3 , \quad j = 1,2, ,m_1 \]

3.2.2.8 \( P_{\text{l},\text{ij}}(t) = \sum_{k=1}^{m_1} \{B^k (m_{3,i,b},t) \odot P_{\text{o},\text{ij}}(t)\}, \)
\[ i = 1,2, ,m_3 , \quad j = 1,2, ,m_2 \]

3.2.2.9 \( P_{\text{l},\text{ij}}(t) = \delta_{ij} e^{-bt} + \beta(j-j) e^{-bt} \frac{(bt)^{-1}}{(j-1)!} \)
\[ i = 1,2, ,m_3 , \quad j = 1,2, ,m_3 \]

3.2.3 Reliability analysis

For the system to be continuously operable in \((0, t)\) it is necessary that at the failure instants of the \(p\)-unit, the \(o\)-unit should be in operable condition and for the duration of the \(p\)-unit repair the \(o\)-unit should not fail in service. To this end we define the following regenerative events.
\( E_0 \) event that the p-unit is just put into service and the o-unit is just in standby

\( E_j \) event that the p-unit is just switched into service after repair and the o-unit is just under preventive maintenance (and found in j-th stage, \( j=1,2,...,m_3 \))

\( E_{2+j} \) event that the p-unit is just switched into service after repair while the o-unit is under repair (and found in j-th stage, \( j=1,2,...,m_2 \)).

\( E_{4+j} \) event that the p-unit is just under repair while the o-unit is just put into service from j-th phase in standby, \( j=1,2,...,m_1 \)

\( E_{5+j} \) event that the p-unit is just under repair while the o-unit is queueing up for preventive maintenance (and found in j-th stage just before switching, \( j=1,2,...,m_3 \))

\( E_{6+j} \) event that the p-unit is just under repair while the o-unit is queueing up for repair (and found in j-th stage just before switching, \( j=1,2,...,m_2 \))

The reliability function \( R(t) \) of the system, conditioned upon \( E_0 \) at \( t=0 \), is given by

\[
3.2.3.1 \quad R(t) = \overline{F}_p(t) + \sum_{j=1}^{m_1} \{f_p(t) \odot P_{o \rightarrow q}(t)\} \odot R_{E_{4+j}}(t),
\]

\( t=1,2,...,m_1 \)
The expression 3.2.3.1 is derived by considering the following mutually exclusive and exhaustive cases:

(i) the p-unit does not fail up to t

(ii) the p-unit fails before t, at this instant the o-unit is in operable condition (and found in j-th phase in the standby state)

For \( j = 1, 2, \ldots, m_j \), we have

\[
3.2.3.2 \quad R_{E_{4+j}}(t) = F_j(t) G_p(t) + \bar{F}_j(t) g_p(t) \odot R_j(t)
\]

The expression 3.2.3.2 is derived by considering following mutually exclusive and exhaustive possibilities:

(i) the repair of the p-unit is not over up to t and during this period the o-unit which was put into service from the j-th phase in standby, is found in operating condition

(ii) the repair of the p-unit is completed before t while the o-unit is found in operating condition throughout the repair of the p-unit.

For \( i = 1, 2, \ldots, m_3 \)

\[
3.2.3.3 \quad R_i(t) = \bar{F}_p(t) + \sum_{j=1}^{m_j} \{ f_p(t) P_{i,4+j}(t) \} \odot R_{E_{4+j}}(t)
\]

The expression 3.2.3.3 is derived by considering the following mutually exclusive and exhaustive possibilities
Conditioned on \( E_j \) event at \( t = 0 \)

(i) the \( p \)-unit does not fail up to \( t \)

(ii) the \( p \)-unit fails before \( t \), at this instant of the \( p \)-unit failure, the \( o \)-unit is in operable condition (and found in \( j \)-th phase in the standby state)

3.2.4 Availability analysis

To find the point-wise availability of the system, we consider the following mutually exclusive and exhaustive possibilities

(i) the \( p \)-unit does not fail up to \( t \)

(ii) the \( p \)-unit fails before \( t \), at this instant the \( o \)-unit is either operable (found in phase \( j \), \( j = 1,2, \ldots, m_1 \)) or undergoing repair (found in stage \( j \), \( j = 1,2, \ldots, m_2 \))

Hence

\[ A(t) = \overline{F}_p(t) + \sum_{j=1}^{m_1} \{ f_p(t) \ P_{o\alpha j}(t) \} \ \oplus \ A_{E_{4+j}}(t) \]

\[ + \sum_{j=1}^{m_2} \{ f_p(t) \ P_{o\alpha r j}(t) \} \ \oplus \ A_{E_{6+j}}(t) \]
For $i = 1, 2, \ldots, m_3$,

$$A_{E_i}(t) = \bar{F}_p(t) + \sum_{j=1}^{m_1} \{f_p(t) P_{i,0j}(t)\} \ominus A_{F_{i,j}}(t)$$

$$+ \sum_{j=1}^{m_3} \{f_p(t) P_{i,j}(t)\} \ominus A_{F_{i,j}}(t)$$

$$+ \sum_{j=1}^{m_2} \{f_p(t) P_{i,j}(t)\} \ominus A_{F_{i,j}}(t)$$

Furthermore, conditioned on $E_{2+j}$ at $t=0$, we have

$$A_{E_{2+j}}(t) = \bar{F}_p(t) + \sum_{j=1}^{m_1} \{f_p(t) P_{n,0k}(t)\} \ominus A_{E_{4+k}}(t)$$

$$+ \sum_{j=1}^{m_2} \{f_p(t) P_{n,j}(t)\} \ominus A_{E_{4+k}}(t)$$

Conditioned on $E_{4+j}$ at $t=0$, we obtain $A_{E_{4+j}}(t)$ considering the fact that the repair of the p-unit is either completed or not completed in the interval $(0,t)$

$$A_{E_{4+j}}(t) = \bar{F}_f(t) \bar{G}_p(t) + \bar{F}_f(t) g_p(t) \ominus A_{F_1}(t)$$

$$+ F_f(t) g_p(t) \ominus A_{E_{2+j}}(t)$$

We observe that when $E_{5+j}$, $j = 1, 2, \ldots, m_2$ or $E_{6+j}$, $j = 1, 2, \ldots, m_3$ occurs at $t=0$ the system enters non-operative state In order to have system in operative state at $t$ it is necessary that the repair of the p-unit should be completed before $t$. Thus
3.2.4.5 \( A_{E_{s+j}}(t) = g_p(t) \odot A_{E_j}(t), \ j = 1, 2, \ldots, m_3 \)

3.2.4.6 \( A_{E_{s+j}}(t) = g_p(t) \odot A_{E_{2+j}}(t), \ j = 1, 2, \ldots, m_2 \)

Once the functional nature of \( f_p(\cdot), g_p(\cdot), f_j(\cdot), \) and (\( \alpha, \gamma \)) are known equations 3.2.3.1 and 3.2.4.1 can be solved by Laplace Transform technique. The mean-time-to-system failure is given by \( R^*(0) \) and the steady state availability of the system can be computed from the following relation

\[
A_{\infty} = \lim_{s \to 0} sA^*(s)
\]

3.3 Reliability Analysis of a Two-unit Priority Redundant System with Imperfect Repair

In the previous model we have assumed that repair completely restores the property of a unit. That is the p-unit behaves like a new one after each repair. In this section we assume that the failure time distribution of the p-unit is different after each repair such that the average failure time of the p-unit decreases after each repair. The p-unit can be repaired at most k times. In other words, when the p-unit fails for the (k+1)th time it cannot be repaired further. At this instant the system breaks down permanently. The following section describes the system.

3.3.1 System description and notation

1. The system consists of two repairable units. We name them as priority unit (p-unit) and the other as non-priority unit (o-unit)
2. The o-unit in standby has Erlang failure time distribution with $m_1$ stages (and mean $\frac{m_1}{c}$).

3. When the p-unit fails it is replaced by the o-unit, if operable. If at the epoch of the failure of the p-unit, the o-unit is under repair, the repair of the o-unit is interrupted. The repair of the o-unit is continued from where it was left once the repair of the p-unit is completed.

4. When the repair of the p-unit is completed it is switched into service immediately and the o-unit is taken up for repair if it is in failed condition, otherwise it is put in into stage 1 in the standby state.

5. The repair time of the o-unit is Erlang distributed random variable with $m_2$ stages (mean $\frac{m_2}{d}$).

6. There is only one repair facility.

7. The repair of the p-unit is imperfect in the sense that the expected life of the p-unit decreases after each repair.

8. The p-unit can be repaired at most $k$ times. When the p-unit fails for $(k+1)$th time it cannot be repaired further.

9. The repair of the o-unit is perfect.

10. All switchover times are negligible. The switch is perfect.
The stochastic behavior of the n-unit in standby

The behavior of the n-unit in standby during a life cycle of the p-unit in service can be described by a stochastic process \( \{Z(t), t \geq 0\} \). The process \( \{Z(t); t \geq 0\} \) is a two-valued stochastic process taking the value on the state space \( \{0, 1\} \), '0' if the unit is operable and '1' if the units is undergoing repair. The process \( \{Z(t); t \geq 0\} \) during a continuous failure free interval of the p-unit in service can be identified by an alternating renewal process with renewal density.
\[ h(t) = \sum_{n=1} h^n(t) \]

Where \( h^n(t) \) is the \( n \)-fold convolution of \( B(m_1,1,c,t) \) and \( B(m_2,1,a,t) \).

However, the behaviour of the o-unit can also be described through the following functions:

\[ P_{\delta,\eta_1}(t) = P_{r}(Z(t) = (\eta_1) / Z(0) = (\eta_1)) \]

The functions \( P_{\delta,\eta_1}(t) \) characterizing the process \( \{Z(t), t \geq 0\} \) are given by:

\[ P_{o_1,o_j}(t) = \delta_{ij} e^{-\alpha t} + \beta(j-1) \frac{e^{-\alpha t} (ct)^{j-1}}{(j-1)!} \]

\[ + B(m_1,1,c,t) \ominus P_{r_1,o_j}(t), \]

\( i=1,2, \quad m_1 \quad j=1,2, \quad m_1 \)

\[ P_{r_1,o_j}(t) = B(m_2,1,a,t) \ominus P_{o_1,o_j}(t), \quad j=1,2, \quad m_1 \]

\[ P_{o_1,o_j}(t) = \frac{e^{-\alpha t} (ct)^{j-1}}{(j-1)!} + h(t) \ominus \frac{e^{-\alpha t} (ct)^{j-1}}{(j-1)!}, \]

\( j=1,2, \quad m_1 \)

\[ P_{r_1,o_j}(t) = B(m_2,1,a,t) \ominus P_{o_1,o_j}(t), \quad j=1,2, \quad m_1 \]

\[ P_{r_1,o_j}(t) = \frac{e^{-bt} (bt)^{j-1}}{(j-1)!} + h(t) \ominus \frac{e^{-bt} (bt)^{j-1}}{(j-1)!}, \]

\( j=1,2, \quad m_2 \)

58
\[ P_{\eta_1,\eta_2}(t) = \delta_{\eta_1} e^{bt} + \beta(j-1) \frac{e^{-bt} (bt)^{j-1}}{(j-1)!} + B(m_2,1,1,1) \odot P_{r_1,\eta_1}(t), \quad i=1,2, \quad m_2, \quad j=1,2, \quad m_2 \]

\[ P_{\sigma_1,\eta_2}(t) = B(m_1,1,c,1) \odot P_{r_1,\eta_1}(t), \quad i=1,2, \quad m_2 \]

Next we proceed to obtain the reliability function of the system.

### 3.3.3 Reliability analysis

For the system to be continuously operable in \((0,1]\), it is necessary that at the failure instants of the \(p\)-unit, the \(o\)-unit should be in operable condition and for the duration of \(p\)-unit repair, the \(o\)-unit should not fail in service. To this end we define the following events required for the analysis:

- **\(e_{p,o}\)** event that the \(p\)-unit is new and is just put into service. At this instant the \(o\)-unit is just in standby and found in stage 1 in the standby state.

- **\(e_{p,n}\)** event that the \(p\)-unit is just put into service after its \(n\)-th repair, \(n=1,2,\ldots,k\).

Furthermore, let

\[ Q_k(n,t)dt = P_r(e_{p,n}) \text{ occurs between } t \text{ and } t+dt \text{ and the system is operable in } (0,1] \text{ at } t=0, \quad n=1,2,\ldots,k \]
3.3.3.2 \( \Pi_f(n,t)\)\(\text{d}t = P_r\{e_{p_0} \} \) occurs between \( t \) and \( t + \text{d}t \) and the system is operable in \( (0,t] / e_{p_0} \text{ at } t=0 \), \( n = 1,2, \ldots k \).

We note that the functions \( Q_f(n,t) \) and \( \Pi_f(n,t) \) satisfy the following recurrence relation.

3.3.3 \( \Pi_f(n,t) = \Pi_f(n-1,t) \odot Q_f(n,t), \quad n = 2,3, \ldots k \)

3.3.4 \( \Pi_f(1,t) = Q_f(1,t) \)

Fixing our attention on the failure and repair instants of the p-unit in the interval \( (0,t) \), we obtain

3.3.5 \( Q_f(n,t) = \sum_{j=1}^{m_j} \{f_{p,n}(t) P_{o1,o_j}(t) \odot \tilde{F}_j(t) G_p(t) \}, \quad n = 1,2, \ldots k \)

conditioned on \( e_{p_0} \text{ at } t=0 \), the reliability function \( R(t) \) of the system is given by

3.3.6 \( R(t) = \bar{F}_{p1}(t) + \sum_{j=1}^{m_j} \{f_{p,n}(t) P_{o1,o_j}(t) \odot \bar{F}_j(t) \bar{G}_p(t) \}

+ \sum_{n=1}^{k-1} \Pi_f(n,t) \odot \{\bar{F}_{p,n+1}(t) \}

+ \sum_{j=1}^{m_j} \{f_{p,n+1}(t) P_{o1,o_j}(t) \odot \bar{F}_j(t) \bar{G}_p(t) \}

+ Q_f(k,t) \odot \bar{F}_{p,k+1}(t) \)
The expression 3.3.3.5 is obtained by considering the following mutually exclusive and exhaustive possibilities:

(i) the p-unit which is new at $t=0$, does not fail up to $t$

(ii) the o-unit which is switched over from $j$-th stage in the standby state, upon the failure of the p-unit, does not fail before $t$

(iii) the p-unit while operating $n$-th time ($n \geq 2$) does not fail up to $t$