CHAPTER 4
EXTENSION OF TWO POINT EXPANSION SCHEME
FOR MODEL FORMULATION OF LTICS

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4.1 INTRODUCTION

Stabilisation is a process by which the output of a given Linear Time Invariant System is controlled for a specified signal under certain performance indices which may be either in time domain or in frequency domain [76]. In classical control system, the stabilisation of LTICS is achieved by state variable feedback technique or selection of PID controller or phase compensator. The design of controllers and compensators for higher order LTICS involves computationally difficult and cumbersome tasks. Hence there is a need for the design of higher order LTICS through suitable reduced order models.

Further in the design of sub-optimal control systems by deriving sub-optimal strategies based on reduced order models, the computational effort of obtaining optimal and sub-optimal controllers is minimised. Reduced order models are also employed for analysis and stability of higher order nonlinear systems.

In this chapter, the two point expansion scheme (as detailed in Chapter 3) is extended to get suitable lower order models which minimizes the computational complexity and the procedure is found to be simpler in application.

4.2 DESIGN OF PID CONTROLLER

In general, series controllers are preferred over feedback controllers because for higher order systems, the large number of state variables would require a large number of transducers to sense during feedback. This makes the use of series controllers very common. The
controller is attached with the reduced order model and closed loop response is observed. The parameters of the controller are tuned to get a response, meeting the desired specifications. The tuned parameters are introduced into the higher order system for stabilisation processes.

The transfer function of PID controller is written for a continuous system as [76,77]:

\[ G(s) = K_p + \frac{K_i}{s} + K_d \cdot s \]  \hspace{1cm} (4.1)

The design involves the determination of the values of the constants \(K_p, K_i,\) and \(K_d,\) meeting the required performance specifications. The general block diagram of the PID controller is shown in Fig. 4.1.

![FIGURE 4.1 A FEEDBACK CONTROL SYSTEM WITH PID CONTROLLER](image)

The proportional control action multiplies the error signal with a constant to improve the overall gain of the system.
4.2.1 Performance Specifications

Performance specifications are considered with respect to the closed loop response of the compensated system to unit step input. The specifications are chosen as:

(i) Overshoot \( \leq 3\% \)
(ii) Settling time \( \leq 3 \) seconds
(iii) Steady state error \( \leq 2\% \)

Note: The above values are left to the choice of the designer.

4.2.2 System Description

The closed loop transfer function of a unity feedback system with \( G(s) \) as the open loop transfer function is given in equation (4.2) as:

\[
T(s) = \frac{G(s)}{1 + G(s)} \tag{4.2}
\]

If the system output response does not satisfy the specifications, the PID controller is added to the forward path and the closed loop transfer function of the system is given in equation (4.3) as:

\[
T_c(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)} \tag{4.3}
\]

where \( G_c(s) \) is the transfer function of the controller.
4.2.3 General Algorithm for the Design of Controller Using Reduced Order Model

Step 1: Read the open loop transfer function of the given higher order system.

Step 2: Form the closed loop transfer function.

Step 3: Obtain the step response of closed loop systems.

Step 4: Check the response for the required specifications.

Step 5: If the specifications are not met, get a reduced order model (through Steps 6 to 9) and design a controller for the reduced order model.

Step 6: The transfer function is represented by

\[
G_c(s) = \frac{A_{21} + A_{22}s + \cdots + A_{2,n}s^{n-1}}{A_{11} + A_{12}s + \cdots + A_{1,n+1}s^n}
\]  

(4.4)

Step 7: General expansion point 'a' is given by a rule

\[
a = \frac{(A_{1,n}/A_{1,n+1}) \pm (A_{2,n-1}/A_{2,n})}{n \pm (n-1)}
\]  

(4.5)

Step 8: Continued-fraction expansion technique is applied about two points (s = 0 and s = a) to \(G(s)\); Routh like and Reverse Routh Like tables are formed (as detailed in sections 3.2.1 and 3.2.2).

Step 9: Reduced order model is obtained from first two rows of reverse Routh like table from section 3.2.2 and represented as:

\[
R_k(s) = \frac{R_{21} + R_{22}s + \cdots + R_{2,k}s^{k-1}}{R_{11} + R_{12}s + \cdots + R_{1,k}s^{k-1} + R_{1,k-1}s^k}
\]  

(4.6)
Step 10 : If any of \( R_n \) is negative then compute \( a' = 1/a \). Goto Step 8.

Step 11 : Steps 8 and 9 are repeated for all values of \( \alpha' \) calculated from equation (4.5).

Step 12 : For the purpose of comparison of model responses, error index \( J \) is chosen and is defined as:

\[
J = \sum_{i=0}^{N} (Y_i - Y_{ri})^2
\]  

(4.7)

Step 13 : The model with minimum value of \( J \) is chosen for further analysis and design.

Step 14 : Get the initial values of the parameters \( K_p, K_i \) and \( K_d \) by pole-zero cancellation.

Step 15 : Attach the controller with the reduced order model and get the closed loop response with initial values of the controller parameters.

Step 16 : Find the optimum values for the controller parameters which satisfy the required specifications.

Step 17 : With the optimum values, attach this controller with the original system.

Step 18 : Get the closed loop step response of the system with the controller.

Step 19 : If the specifications are met, exit; else tune the parameters of the controller till it meets the required specifications.

Step 20 : Stop.
4.3 ILLUSTRATIONS

Example 4.1

Consider the transfer function from [34]:

\[ G(s) = \frac{194480 + 482964s + 511812s^2 + 278376s^3 + 82402s^4 + 13285s^5 + 1086s^6 + 35s^7}{9600 + 28880s + 37492s^2 + 27470s^3 + 11870s^4 + 3017s^5 + 437s^6 + 33s^7 + s^8} \]

By applying the proposed two point expansion scheme (as detailed in sections 3.2.1 and 3.2.2), the reduced order model for \( a = 1.971 \) is:

\[ R(s) = \frac{33.709476s + 48.708}{s^2 + 2.85061s + 2.404344} \]

and the value of \( J = 1.5785 \).

Applying pole-zero cancellation method to the reduced model, the initial values of \( K_p, K_i \) and \( K_d \) are obtained as [78]:

\[ K_p = 3, \quad K_i = 2.5, \quad K_d = 1 \]

Using the simulation procedure, initial parameters are tuned to get unit step response of the compensated system to meet the required specifications.

The tuned values obtained are as follows:

\[ K_p = 3, \quad K_i = 2.5, \quad K_d = 0.0005 \]

The designed PID controller \( G_c(s) \) is attached with the original higher order system. The closed loop response is found to meet the required specifications as given in section 4.2.1. Figure 4.2 shows the step response of the original system. Figures 4.3 and 4.4 shows the step response of the reduced and higher order system with PID controller respectively.
FIGURE 4.2 STEP RESPONSE OF ORIGINAL SYSTEM FOR EXAMPLE 4.1
FIGURE 4.3 STEP RESPONSE OF REDUCED ORDER MODEL WITH PID CONTROLLER FOR EXAMPLE 4.1
FIGURE 4.4 STEP RESPONSE OF ORIGINAL SYSTEM WITH PID CONTROLLER FOR EXAMPLE 4.1
Example 4.2

Consider the transfer function

\[
G(s) = \frac{194480 + 482964s + 511812s^2 + 278376s^3 + 82402s^4 + 13285s^5 - 1086s^6 + 35s^7}{17760 + 45952s + 46350s^2 + 24469s^3 + 7669s^4 + 1558s^5 + 220s^6 + 21s^7 + s^8}
\]

By applying the proposed two point expansion scheme (as detailed in sections 3.2.1 and 3.2.2), the reduced order model for \( a = 52.0285 \) is:

\[
R(s) = \frac{35.055664s + 404.873474}{s^2 + 1.554757s + 36.973228}
\]

and the value of \( J = 18.4858 \).

Applying pole-zero cancellation method to the reduced order model, the initial values of \( K_p, K_i \), and \( K_d \) are obtained as:

\[
K_p = 1.6, \quad K_i = 37, \quad K_d = 1
\]

Using simulation procedure, the initial parameters are tuned to get unit step response of the compensated system to meet the required specifications.

The tuned values obtained are as follows:

\[
K_p = 9, \quad K_i = 1, \quad K_d = 0.0001
\]

The designed PID controller \( G_c(s) \) is attached with the original higher order system. The closed loop response is found to meet the required specifications. Fig. 4.5, 4.6 and 4.7 show the step response of the original system, reduced order system with PID controller and higher order system with PID controller respectively.

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FIGURE 4.5 STEP RESPONSE OF ORIGINAL SYSTEM FOR EXAMPLE 4.2
FIGURE 4.6 STEP RESPONSE OF REDUCED ORDER MODEL WITH PID CONTROLLER FOR EXAMPLE 4.2
FIGURE 4.7 STEP RESPONSE OF ORIGINAL SYSTEM WITH PID CONTROLLER FOR EXAMPLE 4.2

[Diagram showing a step response graph with axes labeled: Time in sec. vs. Output Response.]

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Example 4.3

Consider the transfer function from [60]:

\[ G(s) = \frac{2 + 6s + 8s^2}{2 + 5s + 4s^2 + s^3} \]

By applying the proposed two point expansion scheme (as detailed in sections 3.2.1 and 3.2.2), the reduced order model for \( a = 3.25 \) is:

\[ R(s) = \frac{7.891353s + 4.741064}{s^2 + 3.655894s + 4.741064} \]

and the value of \( J = 0.3537 \).

Applying pole-zero cancellation method to the reduced model, the initial values of \( K_p, K_i, \) and \( K_d \) are obtained as:

\[ K_p = 4, \quad K_i = 5, \quad K_d = 1 \]

Using simulation procedure, the initial parameters are tuned to get unit step response of the compensated system to meet the required specifications.

The tuned values obtained are as follows:

\[ K_p = 1.5, \quad K_i = 6.5, \quad K_d = 0.0005 \]

The designed PID controller \( G_c(s) \) is attached with the original higher order system. The closed loop response is found to meet the required specifications. Fig 4.8, 4.9 and 4.10 show the step response of the original system, reduced order system with PID controller and higher order system with PID controller respectively.
FIGURE 4.8 STEP RESPONSE OF ORIGINAL SYSTEM FOR EXAMPLE 4.3
FIGURE 4.9 STEP RESPONSE OF REDUCED ORDER MODEL WITH PID CONTROLLER FOR EXAMPLE 4.3
FIGURE 4.10 STEP RESPONSE OF ORIGINAL SYSTEM WITH PID CONTROLLER FOR EXAMPLE 4.3
4.4 DESIGN OF COMPENSATOR

In the selection of a phase compensator, the transfer function is assumed to be one of the following [76,77]:

For Lead Phase Compensator,

\[ G_c(s) = \frac{K(s + A)}{(s + B)} \]  

(4.8)

For Lag Phase Compensator,

\[ G_c(s) = \frac{(1 + As)}{(1 + Bs)} \]  

(4.9)

where the parameters \( K, A \) and \( B \) are to be suitably chosen.

In the process of selection of controller or compensator, the different sets of \((K_p, K_i, K_d)\) or \((K, A, B)\) or \((A, B)\) are chosen and the corresponding output responses are compared for optimum response on the basis of specified overshoot, quick settling time and steady state error. An appropriate controller or compensator can thus be obtained giving an optimum response of the system for unit step input. The performance specifications are chosen as:

(i) Overshoot \( \leq 3\% \)
(ii) Settling time \( \leq 3 \) seconds
(iii) Steady state error \( \leq 2\% \)

Note: These values are left to the choice of the designer.
4.4.1 Design of Lead Phase Compensator

The transfer function for a Lead Compensator is assumed to be:

\[ G_c(s) = \frac{K(s + A)}{(s + B)} \]

with \( B > A, \ K > 0, \ A > 0 \) and \( B > 0 \).

Now, the proposed two point scheme (as detailed in Chapter 3) is applied to this transfer function and different sets of \( K, A, B \) values are chosen. The corresponding output responses are compared for optimum response and suitable values of \( K, A, B \) are obtained in order to meet the desired specifications.

4.4.2 Design of Lag Phase Compensator

The transfer function for a Lag Compensator is assumed to be:

\[ G_c(s) = \frac{(1 + As)}{(1 + Bs)} \]

with \( B > A, \ A > 0 \) and \( B > 0 \).

Proposed scheme for model reduction about two points is applied to this transfer function and different sets of \( A, B \) values are chosen. The corresponding output responses are compared for optimum response and suitable values of \( A, B \) are obtained in order to meet the desired specifications.
4.5 ILLUSTRATION

Example 4.4

Consider the transfer function from [69]:

\[
G(s) = \frac{19.82s^7 + 429.26156s^6 + 4843.8098s^5 + 45575.892s^4 + 241544.75s^3 + 905812.05s^2 + 1890443.1s + 842597.05}{s^8 + 30.41s^7 + 358.4295s^6 + 2913.8638s^5 + 18110.567s^4 + 67556.983s^3 + 173383.58s^2 + 149172.19s + 37752.826}
\]

By applying the two point expansion scheme (as detailed in sections 3.2.1 and 3.2.2), the reduced order model for minimum J is found as:

\[
R(s) = \frac{15.9292s + 16.23224}{s^2 + 1.96823s + 0.7272}
\]

Applying pole-zero cancellation method to the reduced order model, the values of A and B are obtained as:

\[
A = 0.4929 \quad \text{and} \quad B = 1.019
\]

Using the simulation procedure, the parameter K is tuned to get the unit step response of the compensated system to meet the required specifications. The tuned value of K is 20.

The transfer function of the Lead Phase Compensator is obtained as:

\[
G_c(s) = \frac{20(s + 0.4929)}{(s + 1.019)}
\]

Fig. 4.11, 4.12 and 4.13 show the step response of the original system, reduced order system with Lead Phase Compensator and original system with Lead Phase Compensator.
FIGURE 4.11 STEP RESPONSE OF ORIGINAL SYSTEM FOR EXAMPLE 4.4
FIGURE 4.12 STEP RESPONSE OF REDUCED ORDER MODEL WITH LEAD PHASE COMPENSATOR FOR EXAMPLE 4.4
FIGURE 4.13 STEP RESPONSE OF ORIGINAL SYSTEM WITH LEAD PHASE COMPENSATOR FOR EXAMPLE 4.4
4.4 DESIGN OF FEEDBACK CONTROLLER AND OBSERVER

The process of input stabilisation of Linear Time Invariant Continuous System is achieved by using state variable feedback technique in modern control systems. The state variables are fed back through constant gains to obtain the control of outputs. In cases where all the states are not available for measurement and control, a state observer is designed for use in reconstruction of state variables along with certain specifications of pole locations. The design of controller and observer for higher order system is carried out by an equivalent lower order model using the proposed two point expansion scheme as detailed in Chapter 3. Schemes have been devised to choose the given settings of a state variable controller and observer without specifying the location of poles. The controller and observer are designed by separation principle [78-81] and gains so chosen are used for overall structure of the state feedback control system.

4.4.1 State Feedback Controller

The state vector differential equations of a single input and single output continuous system are written as [78-80] :

\[ x = Ax + Bu \] (4.10)
\[ y = Cx + Du \] (4.11)

where

\( x \) is the \( n \times 1 \) state vector,
\( u \) is the scalar input,
\( y \) is the scalar output,
\( A, B \) and \( C \) are respectively \( n \times n \), \( n \times 1 \) and \( 1 \times n \) real constant matrices.

The state variable feedback for the system is a scalar function which takes the form:

\[
Kx = \begin{bmatrix} k_1, k_2, \ldots, k_n \end{bmatrix}
\]

where

\( K \) is the state gain vector.

The input vector and the state equations are reconstructed as:

\[
u = r - Kx
\]

\[
x = (A - BK)x + Br
\]

where

\( r \) is the system input.

with all the states available it should be noted that the pair \((A, B)\) should be state controllable for the system design. The design problem for state variable feedback control involves the estimation of the \( K \) matrix such that the closed loop system will have a unit step response satisfying the design specifications. The reduced order matrix \( K_r \) is estimated using the proposed two point expansion scheme.
4.4.2 State Observer

For implementing the control law as given by the equation (4.12), all the state variables should be available to provide feedback. But in practical situations all the state variables may not be available for measurement and control. In such cases, an observer is used to reconstruct the states of the system from the information contained in the input as well as the output variables of the system and the output of the observer is used for control purpose.

The state equation of the observer is written from [78-80] as:

\[ x_1 = (A - GC) x_1 + GC x + Bu \]  

(4.15)

where

- \( x_1 \) is the observed state vector and
- \( G \) is the observer gain column vector.

The error between the actual state and the observer state is obtained from the equations (4.10) and (4.15) as:

\[ x_2 = \dot{x} - \dot{x}_1 = (A - GC)(x - x_1) \]  

(4.16)

The design of state observer involves estimation of observer matrix \( G \) with the state error, \( \dot{x}_2 \) reduced to zero and the pair \((A, C)\) should be observable.
4.5 ALGORITHMS FOR THE DESIGN OF CONTROLLER AND OBSERVER GAINS

4.5.1 Algorithm I: Design of State Feedback Controller

Step 1: Transform the state space form into transfer function form.

Let it be

\[ G(s) = C(sI - A)^{-1}B \]  \hspace{1cm} (4.17)

Step 2: Obtain a second order reduced model using Steps 7 to 12 (as detailed in section 4.2.3).

Step 3: Let the real and imaginary parts of the roots of \( D_r(s) = 0 \) be \(-\alpha\) and \(\beta\).

Step 4: By adjusting the values of \(|\alpha|\) and \(|\beta|\) and through simulation obtain the unit step response such that the design specifications viz.,

(i) Overshoot \(\leq 3\%\)
(ii) Settling time \(\leq 3\) seconds
(iii) Steady state error \(\leq 2\%\)

are satisfied.

Step 5: Transform \( R(s) \) into phase variable form as:

\[ \dot{x}_r = A_r x + B_r u \]  \hspace{1cm} (4.18)
\[ y = C_r x \]  \hspace{1cm} (4.19)

Step 6: Form the following equation with state variable feedback.

\[ \dot{x}_r = (A_r - B_r k_r) x + B_r u \]  \hspace{1cm} (4.20)
\[ y = C_r x \]  \hspace{1cm} (4.21)
Step 7: Reconstruct the reduced order model with state feedback control gains $K_1$ and $K_2$ as $R_c'(s)$.

Step 8: By reversing the process of model reduction in the proposed scheme reconstruct a higher order controller system, $G_c'(s)$ for $R_c'(s)$.

Step 9: Compare the original higher order system $G(s)$ with $G_c'(s)$ and derive the value of the feedback gain vector $K = [k_1, k_2, \ldots]$.

Step 10: The output response obtained by using the feedback gain vector is again sharpened to get the optimum response if needed.

4.7.2 Algorithm II: Design of State Observer

The design Steps 1 to 4 of Algorithm I are employed to get the reduced order model.

Step 5: Transform $R(s)$ into phase variable form as:

$$x_r = A_r x + B_r u \quad (4.22)$$

$$y = C_r x \quad (4.23)$$

Step 6: Form the following equation with state observer.

$$\dot{x}_2 = (A_r - G_r C_r)(x - x_1) \quad (4.24)$$

Step 7: Reconstruct the final $D_r(s)$ and read values $g_1$ and $g_2$ for the reduced order model.

Step 8: By tuning procedure, the state observer gain matrix for higher order gain matrix is obtained from the reduced order state observer.
The above algorithms are applied to the following illustrations.

4.6 ILLUSTRATIONS

Example 4.5 [60]

Consider a third order system

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

where

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -5 & -4
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
2 & 6 & 8
\end{bmatrix}
\]

\[
D = [0]
\]

Design of a State Feedback Controller Gain

The transfer function model is obtained using

\[ G(s) = C (sI - A)^{-1} B \]

as:

\[ G(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \]

Using the two point expansion scheme the reduced order model for

\[ a = 3.25 \]

is formulated as:
The phase variable form of the above reduced order model is given as:

\[ \dot{x}_r = A_r x + B_r u \]
\[ y = C_r x \]

where

\[
A_r = \begin{bmatrix} 0 & 1 \\ -4.741064 & -3.655894 \end{bmatrix}
\]

\[
B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
C_r = \begin{bmatrix} 4.741064 & 7.891353 \end{bmatrix}
\]

The state equation of the closed loop state feedback controller is given as:

\[ \dot{x}_r = (A_r - B_r k_r) x + B_r u \] and
\[ y = C_r x \]

The transfer function of the above state equation is written as:

\[ R_1(s) = C_r \left[ sI - (A_r - B_r k_r) \right]^{-1} B_r \]

\[ R_1(s) = \frac{7.891353 s + 4.741064}{s^2 + (3.655894 + k_1)s + (4.741064 + k_2)} \]

From the elements of matrix \( C_r \), the value of \( k_1 \) is obtained as:
\[ k_1 = 7.891353 - 4.741064 = 4.7809 \]
\[ k_2 = 0 \text{ (for steady state error to be zero)} \]
Fig. 4.14 shows the step response of the original system. Fig. 4.15 shows the step response of the reduced order model with state feedback controller.

The \( R_c' \) is obtained as:

\[
R_c' (s) = \frac{(7.891353 s + 4.741064)s + 4.741}{s^2 + 8.436794 s + 4.741}
\]

By reverse procedure \( G_c' \) is found as:

\[
G_c' (s) = \frac{12.78 s^2 + 7.139 s + 2}{s^3 + 8.776s^2 + 6.134 s + 2}
\]

The gain of the state feedback controller is:

\[
K = \begin{bmatrix} 0 & 1.1342 & 4.7759 \end{bmatrix}
\]

The step response of the original system with feedback controller is shown in Fig. 4.16 for the gain factor \( K \) and is found to give satisfactory results and meets the required design specifications.

**Design of Observer Gain**

The design of observer gain matrix \( G_r \) is obtained in the same fashion as that of the feedback gain matrix \( K_r \) and the values of \( g_1 \) and \( g_2 \) are evaluated as:

\[
g_1 = 0.127 \quad \text{and} \quad g_2 = 0.5295
\]
FIGURE 4.14 STEP RESPONSE OF ORIGINAL SYSTEM FOR EXAMPLE 4.5
FIGURE 4.15 STEP RESPONSE OF REDUCED ORDER MODEL WITH STATE FEEDBACK CONTROLLER FOR EXAMPLE 4.5
FIGURE 4.16 STEP RESPONSE OF ORIGINAL SYSTEM WITH STATE FEEDBACK CONTROLLER FOR EXAMPLE 4.5
The step response for the reduced order system is shown in Fig. 4.17.

The design of higher order state observer is obtained by tuning procedure by taking initial values as:

\[ G_{\text{initial}} = \begin{bmatrix} 0 & 0.127 & 0.5295 \end{bmatrix}^T \]

By tuning procedure, the final values are obtained as:

\[ G = \begin{bmatrix} 0 & 0.127 & 0.5095 \end{bmatrix}^T \]

The higher order system with state observer is obtained as:

\[ G_{\text{ob}}(s) = \frac{8s^2 + 6s + 2}{s^3 + 8.838s^2 + 6.279s + 2} \]

The step response of the original system with state observer, shown in Fig. 4.18 for the observer gain \( G \), is found to give satisfactory results and meets the required design specifications.
FIGURE 4.17 STEP RESPONSE OF REDUCED ORDER MODEL WITH STATE OBSERVER FOR EXAMPLE 4.5
FIGURE 4.18 STEP RESPONSE OF ORIGINAL SYSTEM WITH STATE OBSERVER FOR EXAMPLE 4.5
Example 4.6

Consider a system transfer function from [65]:

\[
G(s) = \frac{24s^3 + 60s^2 + 72s + 12}{s^4 + 10s^3 + 35s^2 + 50s + 24}
\]

Using the two point expansion scheme, the reduced order model is formulated as:

\[
R(s) = \frac{32.61s + 7.414}{s^2 + 12.82s + 14.83}
\]

The reduced order system with state feedback controller is:

\[
R_c'(s) = \frac{32.61s + 7.414}{s^2 + (12.82s + k_2)s + (14.83 + k_1)}
\]

\[
K = \begin{bmatrix} -7.416 & 20.042 \end{bmatrix}
\]

From the values of matrix K and coefficients of \(R_c'(s)\) the reduced order model can be formed as:

\[
R_c'(s) = \frac{32.61s + 7.414}{s^2 + 32.862s + 7.414}
\]

Fig. 4.19 shows the step response of the original system and Fig. 4.20 shows the step response of the reduced order model with feedback controller and the reduced order system gives satisfactory results meeting the required design specifications.

Applying the reverse procedure to \(R_c'(s)\), obtain \(G_c'(s)\) as:

\[
G_c'(s) = \frac{29.76s^3 + 62.79s^2 + 66.84s + 12}{s^4 + 26.23s^3 + 61.75s^2 + 71.08s + 12}
\]

The gain of the state feedback controller is formed using the coefficients of \(G_c'(s)\) as:

\[
K = \begin{bmatrix} -12 & 21.08 & 26.75 & 16.23 \end{bmatrix}
\]
FIGURE 4.19 STEP RESPONSE ORIGINAL SYSTEM FOR EXAMPLE 4.6
FIGURE 4.20 STEP RESPONSE OF REDUCED ORDER MODEL WITH STATE FEEDBACK CONTROLLER FOR EXAMPLE 4.6
The step response of the original system with feedback controller is shown in Fig. 4.21. This response is in good agreement with the response shown in Fig. 4.20.

**Design of Observer Gain**

The observer gain matrix $G_r$ is obtained in the same fashion as that of the feedback gain matrix $K_r$ and the values of $g_1$ and $g_2$ are evaluated as:

$$g_1 = 0.0307 \text{ and } g_2 = 0.6075$$

The step response of the reduced order model with state observer is shown in Fig 4.22.

The design of higher order state observer is obtained through simulation by taking initial values as,

$$G_{\text{initial}} = \begin{bmatrix} 0 & 0 & 0.0307 & 0.6075 \end{bmatrix}^T$$

By simulation, the final values are obtained as:

$$G_{\text{final}} = \begin{bmatrix} 0 & 0 & 0.042 & 0.59 \end{bmatrix}^T$$

The higher order system with state observer is obtained as:

$$G_{\text{ob}}(s) = \frac{24s^3 + 60s^2 + 72s + 12}{s^4 + 26.68s^3 + 63.34s^2 + 72.82s + 12}$$

The step response of the original system with state observer, shown in Fig. 4.23 for the observer gain matrix $G_{\text{final}}$, is found to give satisfactory results and meets the required design specifications.
FIGURE 4.21 STEP RESPONSE OF ORIGINAL SYSTEM WITH STATE FEEDBACK CONTROLLER FOR EXAMPLE 4.6
FIGURE 4.22 STEP RESPONSE OF REDUCED ORDER MODEL WITH STATE OBSERVER FOR EXAMPLE 4.6
FIGURE 4.23 STEP RESPONSE OF ORIGINAL SYSTEM WITH STATE OBSERVER FOR EXAMPLE 4.6
4.9 SUB-OPTIMAL CONTROL USING REDUCED ORDER MODEL

The design of optimal control systems involves considerable computational implementation for higher order models. The proposed scheme for model reduction about two points is applied for extracting reduced order models from the given higher order systems. The cost functions of optimal control of a higher order system and of the sub-optimal control of reduced order models are compared to study the reliability of using reduced order models for such techniques.

4.9.1 Optimal and Sub-optimal Control [82]

An \( n^{th} \) order regulator system is considered as given in equation (4.25) as:

\[
\dot{x} = Ax + Bu
\]  

(4.25)

where

- \( x \) is an \( n \) dimensional vector,
- \( u \) is a scalar control vector,
- \( A \) is \( n \times n \) constant matrix and
- \( B \) is \( n \times 1 \) constant vector.

It is desired to obtain the closed loop control, which minimises the quadratic performance index:

\[
J = \frac{1}{2} \int_0^\infty (x'Qx + u'Ru)dt
\]  

(4.26)

where \( R \) is a scalar and \( Q \) is a \( n \times n \) positive semi definite matrix.

The performance index is obtained as follows:

Step 1: The \( P \) matrix is determined by solving the Lyapunov equation.

\[
A'P + PA = -Q
\]  

(4.27)

Step 2: The optimal performance index \( J_{\text{opt}} \) is found as:

\[
J_{\text{opt}} = \frac{1}{2} x'(0)Px(0)
\]  

(4.28)
Step 3: The optimal feedback control $U_{\text{opt}}(x,t)$ is found as:

$$U_{\text{opt}}(x,t) = -K'x \quad (4.29)$$

where

$$K = PBR^{-1}$$

is the feedback coefficient.

The sub-optimal control and performance index are evaluated in a similar manner using a model of order $k$.

The above scheme is applied to the following illustrations.

4.10 ILLUSTRATIONS

Example 4.7:

Consider a fourth order system [83]:

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-120 & -180 & -102 & -18
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}$$

The performance index $J$ with $x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is given as

$$J = \frac{1}{2} \int_0^\infty (x_1^2 + x_2^2 + u^2) \, dt$$

Let

$$Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad R = [1]$$
The $P$ matrix is determined by solving the equation (4.27)

$$P = \begin{bmatrix} 1.4550 & 0.5575 & 0.0824 & 0.0042 \\ 0.5575 & 0.7156 & 0.1126 & 0.0059 \\ 0.0824 & 0.1126 & 0.0203 & 0.0011 \\ 0.0042 & 0.0059 & 0.0011 & 0.0001 \end{bmatrix}$$

Using this $[P]$ the optimal cost function is calculated as:

$$J_{\text{opt}} = \frac{1}{2} x'(0)Px(0) = 1.6427$$

By the application of the proposed two point scheme, the fourth order system is replaced by a second order model and is given by:

$$A_r = \begin{bmatrix} 0 & 1 \\ -1.7066 & -4.2035 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_r = [1]$$

The solution of the Lyapunov equation yields

$$P_r = \begin{bmatrix} 1.5535 & 0.2930 \\ 0.2930 & 0.1886 \end{bmatrix}$$

$$K = \begin{bmatrix} 1.5535 & 0.2930 \\ 0.2930 & 0.1886 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2930 \\ 0.1886 \end{bmatrix}$$

$$K' = \begin{bmatrix} 0.2930 \\ 0.1886 \end{bmatrix}$$

$$U_{\text{sub}} = -K' x = -\begin{bmatrix} 0.2930 & 0.1886 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -(0.2930 x_1 + 0.1886 x_2)$$

Substitution of this sub-optimal control in the original fourth order system yields the following:
Solution of Lyapunov equation of this modified system gives

\[
A^* = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-120.293 & -180.1886 & -102 & -18
\end{bmatrix}
\]

\[
Q^* = \begin{bmatrix}
1.0858 & 0.0553 & 0 & 0 \\
0.0553 & 1.0356 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
R = [1]
\]

The sub-optimal cost function is:

\[
J_{\text{sub}} = \frac{1}{2} x'(0) P^* x(0) = 1.7298
\]

The ratio of sub-optimal to optimal cost is given by

\[
\frac{J_{\text{sub}}}{J_{\text{opt}}} = \frac{1.7298}{1.6427} = 1.0530
\]

Since the ratio of \(\frac{J_{\text{sub}}}{J_{\text{opt}}}\) is approximately equal to 1.05, the controller designed using the second order model is found to be satisfactory.
Example 4.8

Consider a transfer function from [60]:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -5 & -4
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

The performance index \( J \) with

\[
x = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}
\]

is given as:

\[
J = \frac{1}{2} \int_0^\infty (x_1^2 + x_2^2 + u^2) dt
\]

Let

\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad R = [1]
\]

The P matrix is determined by solving the equation (4.27),

\[
P = \begin{bmatrix}
1.9167 & 1.1667 & 0.2500 \\
1.1667 & 1.500 & 0.3333 \\
0.2500 & 0.3333 & 0.0833
\end{bmatrix}
\]

The optimal cost function is calculated as:

\[
J_{opt} = \frac{1}{2} x'(0)Px(0) = 0.9583
\]

By the application of the proposed two point scheme, the third order system is replaced by a second order model and is given by:

\[
A_r = \begin{bmatrix}
0 & 1 \\
-4.74108 & -3.65
\end{bmatrix}, \quad B_r = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
\[ Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R_r = [1] \]

The solution of the Lyapunov equation yields:

\[
P_r = \begin{bmatrix} 1.1714 & 0.1055 \\ 0.1055 & 0.1659 \end{bmatrix} \]

\[
K = \begin{bmatrix} 1.1714 & 0.1055 \\ 0.1055 & 0.1659 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1055 \\ 0.1659 \end{bmatrix} \]

\[
K' = \begin{bmatrix} 0.1055 \\ 0.1659 \end{bmatrix} \]

\[
U_{\text{sub}} = -K'x = -\begin{bmatrix} 0.1055 & 0.1659 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[
U_{\text{sub}} = -(0.1055x_1 + 0.1659x_2) \]

Substituting this sub-optimal control in the original system yields the following:

\[
A^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2.1055 & -5.1659 & -4 \end{bmatrix} \quad Q^* = \begin{bmatrix} 1.0111 & 0.0175 & 0 \\ 0.0175 & 1.0275 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \]

\[
R = [1] \]

Solution of Lyapunov equation of this modified system gives

\[
P^* = \begin{bmatrix} 1.8920 & 1.1277 & 0.2401 \\ 1.1277 & 1.4413 & 0.3178 \\ 0.2401 & 0.3178 & 0.0794 \end{bmatrix} \]

The sub-optimal cost function is

\[
J_{\text{sub}} = \frac{1}{2} x'(0)P^*x(0) = 0.9460 \]

The ratio of sub-optimal to optimal cost is given by

\[
\frac{J_{\text{sub}}}{J_{\text{opt}}} = \frac{0.9460}{0.9583} = 0.9871 = 1 \quad (\text{approx.}) \]
Since the ratio of $\frac{J_{\text{sub}}}{J_{\text{opt}}}$ is approximately equal to 1.0, the controller designed using the second order model is found to be satisfactory.

4.11 DESIGN OF CRITICAL GAIN FOR CONTINUOUS SYSTEMS

Absolute stability of nonlinear systems can be defined by employing digital computer methods. Several methods exist for analyzing the stability of nonlinear systems. The Popov criterion is one among the other methods used in the design of compensators for the LTICS having nonlinear elements.

Popov Criterion [84-88]

Fig. 4.24 shows a basic nonlinear system.

![FIGURE 4.24 SCHEMATIC DIAGRAM OF A BASIC NONLINEAR SYSTEM](image)

The transfer function of the linear plant is represented as:

$$G(s) = \frac{K(s + z_1)(s + z_2)\ldots(s + z_m)}{(s + p_1)(s + p_2)\ldots(s + p_n)}$$

Let $G(j\omega)$ represent $G(s)$ in frequency domain

$$G(j\omega) = X(\omega) + jY(\omega)$$

To apply the Popov criterion $G(s)$ is modified as:

$$G^*(j\omega) = \text{Re} G^*(j\omega) + j \text{Im} G^*(j\omega)$$

where $\text{Re}[G^*(j\omega)] = X(\omega)$ and $\text{Im}[G^*(j\omega)] = \omega Y(\omega)$
The necessary and sufficient conditions for the system to be absolutely stable, as suggested by Popov, are as follows:

**Necessary Condition**

For a given sector \([0, k]\), the Popov locus should never intersect or enclose the forbidden zone extending from the point \(-\frac{1}{k}\) to \(\infty\) along the real axis.

**Sufficient Condition**

For a free dynamic system to be absolutely stable in the sector \([0, k]\), it is sufficient that there exists a finite real number such that for all real values of \(\omega (\omega > 0)\), the inequality condition

\[
\text{Re}\left(\left(1 + j\omega \right)G(j\omega)\right) + \frac{1}{k} \geq 0
\]

is satisfied.

If the Popov locus lies to the right of the Popov line, then the Popov criterion is satisfied. Popov line is a straight line passing through a point and having a slope of \(1/k\).

These conditions reveal that absolute stability of nonlinear continuous systems could be achieved in two ways:

(a) By shifting the Popov line to the left of the Popov locus or

(b) By shifting the Popov locus to the right of the Popov line.

Shifting the Popov line is very difficult as the nonlinearity sector value \(K\) is fixed. Hence, it is better to shift the Popov locus to the right of the Popov line, in order to achieve absolute stability.
4.12 DETERMINATION OF CRITICAL GAIN USING PROPOSED SCHEME

The two point expansion scheme is applied to formulate the reduced order model of $G(s)$ (as detailed in Chapter 3). This reduced model is further used to estimate the critical gain $k_c$. The critical gains $k_c$ are evaluated with the help of Nyquist plot and results are compared with that of original higher order system. This scheme is illustrated with two examples.

4.13 ILLUSTRATIONS

Example 4.9

Consider an 8th order system whose transfer function is given by:

$$G(s) = \frac{194480 + 482964s + 511812s^2 + 278376s^3 + 82402s^4}{17760 + 45952s + 46350s^2 + 24469s^3 + 7669s^4 + 1558s^5 + 220s^6 + 21s^7 + s^8}$$

Applying the criterion for necessary and sufficient conditions, we obtain the value of $k_c$ from $1/k_c = 7.9935$ as:

Critical gain $k_c = 0.1251$
G(s) is reduced to a lower order model by applying the proposed two point scheme (as detailed in Chapter 3). The obtained reduced order model is given by \( G'(s) \) as:

\[
G'(s) = \frac{35.55664s + 404.873474}{s^2 + 1.5547s + 36.9732}
\]

Applying the criterion for Popov stability, the value of \( k_c \) is obtained from \( 1/k_c = 9.3628 \) as:

Critical gain \( k_c = 0.1268 \)

The ratio of the two values of critical gain gives a value approximately equal to unity. Hence it is observed that the critical gain of the reduced order model is comparable with that of the original system.

**Example 4.10**

Consider an all pole continuous system given by \( G(s) \) as:

\[
G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}
\]

Applying the necessary and sufficient criterion for Popov stability, the value of critical gain \( k_c \) is obtained from \( 1/k_c = 0.0355 \) as:

Critical gain \( k_c = 28.2080 \)

\( G(s) \) is reduced to a lower order model by applying the proposed two point scheme (as detailed in Chapter 3).
The obtained reduced order model is given by $R(s)$ as:

$$
R(s) = \frac{-0.028775s + 0.215827}{s^2 + 2.197842s + 1.294964}
$$

Applying the criterion for Popov stability, the value of $k_c$ is obtained from $1/k_c=0.0318$ as:

Critical gain $k_c=31.4334$

The ratio of the two values of critical gain gives a value approximately equal to unity. Hence it is observed that the critical gain of the reduced order model is comparable with that of the original system.

4.14 SUMMARY

In this chapter the newly proposed model reduction scheme, given in chapter 3, is suitably extended to design PID Controller, Lead Compensator, Lag Compensator, State Feedback Controller and Observer and Sub-optimal Controller. This approach minimises the complexity involved in the direct design of PID Controller, Lead Compensator, Lag Compensator, State Feedback Controller and Sub-optimal Control with a higher order continuous system. It is found that the newly deduced set of values $(K_p, K_i, K_d)$, $(K, A, B)$ and $(A, B)$ for PID Controller, Lead Phase Compensator and Lag Phase Compensator respectively, are good enough to get the optimal response as per design specifications.

Further, the proposed scheme is applied for the estimation of nonlinear critical gain of continuous systems. The proposed design procedure is found to be acceptable.