11.1 INTRODUCTION

The recent trend is to construct all primary load bearing components of aerospace structures, with advanced composite materials. Usually Boron-Epoxy or Graphite -Epoxy composites are used due to their high strength to weight and high modulus to weight ratios. Many of the most important configurations are shells of revolutions made of many layers. There are many theories to analyse laminated composite shell structures, but their analytical solutions are limited to simple geometry, loading and boundary conditions.

The formulation described in this chapter can be used to analyse any axisymmetric shell of revolution, made of arbitrary number of perfectly bonded layers, each of which possesses different thickness, orientation of fibres and orthotropic elastic properties.

11.2 EXISTING THEORIES

In the composite literature classical lamination theory (CLT) is in vogue for quite a long time. This theory is based on the Kirchhoff hypothesis for plates and Kirchhoff-Love hypothesis for shells (43) namely

1. The normal to the undeformed mid surface remains straight and normal to the middle surface even after deformation. This is equal to ignoring the shearing
strain in places perpendicular to the mid-surface.

2. The normals are assumed to be inextensible. This is equal to ignoring the normal strain in the thickness direction.

The classical lamination theory results in under prediction of the energy due to neglect of transverse shear strains. Multilayered laminates in general, develop coupling between bending and extension. There is always inherent thickness-shear effect which cannot be neglected. Ambartsumayan (38) considered apriori a single continuous parabolic shear stress distribution through the entire laminate thickness. Hsu and Wang (38) proved that Ambartsumayan assumption was invalid for a laminated shell since it does not permit satisfaction of inter-laminar strain compatibility unless the insurface Poisson's ratios are identical in all the laminae.

By introducing the degeneration concept introduced by Ahmed etal (3,4), the normal is allowed to rotate and thereby by certain amount of shear deformation can be taken into account.

Multilayered shell theory has been developed by Hattelmaier et al (40). The only large deformation analysis of laminated composite shell that can be found from literature is due to Noor and Hartley (56) in which nonlinear triangular and quadrilateral elements were derived based on a shallow shell theory and the effect of shell deformation is included.
11.3 ASSUMPTIONS

Following assumptions are made so as to be compatible with thin shell theory.

1. The elastic modulus in the normal direction is taken as zero to avoid normal stresses.

2. The original normals to the middle surface of the shell are taken as inextensible and straight.

3. Besides the above the definition of independent rotation and displacement degrees of freedom in the degeneration concept permits the transverse shear deformation to be taken into account. Thus the displacement of a typical midsurface node has three components defined by linear displacement $U$ and $V$ along the two orthogonal global coordinates $R$ and $Z$ and one rotation of the nodal normal in the $RZ$ plane. This results in an approximation that all layers rotate through the same angle as against taking separate normal rotation for such layer.

4. The strain compatibility is enforced at the layer interface, by formulating the strain displacement relation with respect to mid surface of the shell as the reference surface.

11.4 CONSTITUTIVE LAW FOR LAYERED MODEL

The same displacement model is used to define the displacement at any point in the shell and the shell is discretized into so many line elements obtained by degeneration concept as in Fig,11.4 (i.e by lumping in the
thickness direction). The element contains all the layers with different orientation in the thickness direction of the shell.

Consider local orthogonal coordinate, \( z' \) parallel to surface \( \zeta \) constant within the shell and \( r' \) created truly normal to that with the corresponding displacement \( U' \) and \( V' \) (shown in Fig.11.2)

The strain in the local coordinates are given by

\[
\begin{bmatrix}
\varepsilon_x' \\
\varepsilon_\theta \\
\varepsilon_{z'}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial U'}{\partial z'} \\
\frac{U'}{r'} \\
\frac{\partial V'}{\partial r'} + \frac{\partial U'}{\partial z'}
\end{bmatrix}
\]

(11.1)

To evaluate these local strain components we need two sets of transformation, noting that the second term is directly available.

Firstly the \( r, z \) derivatives of \( U \) and \( V \) are derived from their \( \xi, \eta \) derivatives and secondly by another transformation to local displacement direction (i.e. \( r' \) and \( z' \)) complete evaluation of the local strains \( \varepsilon' \) are obtained. (The same transformation given in previous chapter for isotropic shells is used.)

11.5 STRESS STRAIN RELATIONS

Each layer in the laminate is assumed to have two normal stress and one shear stress as shown. For a laminate in \( \Theta, Z \) plane, the stresses are shown in Fig.11.3.
The constitutive law in $\tilde{\Omega}Z$ system is given below.

$$
\begin{bmatrix}
\tilde{\sigma}_{x} \\
\tilde{\sigma}_{y} \\
\tilde{\sigma}_{z} \\
\tilde{\tau}_{xy} \\
\tilde{\tau}_{xz} \\
\tilde{\tau}_{yz}
\end{bmatrix} =
\begin{bmatrix}
\tilde{D}_{11} & \tilde{D}_{12} & 0 & 0 & 0 \\
\tilde{D}_{21} & \tilde{D}_{22} & 0 & 0 & 0 \\
0 & 0 & \tilde{D}_{33} & 0 & 0 \\
0 & 0 & 0 & \tilde{D}_{44} & 0 \\
0 & 0 & 0 & 0 & \tilde{D}_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{x} \\
\varepsilon_{y} \\
\varepsilon_{z} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
$$

(11.2)

where

$$
\tilde{D}_{11} = \frac{E_L}{1 - \gamma_{LT} \gamma_{TL}} \\
\tilde{D}_{22} = \frac{E_T}{1 - \gamma_{LT} \gamma_{TL}} \\
\tilde{D}_{12} = \frac{E_T \gamma_{LT}}{1 - \gamma_{LT} \gamma_{TL}} \\
\tilde{D}_{33} = G_{LT} \\
\tilde{D}_{44} = k_1 G_{LT} \\
\tilde{D}_{55} = k_2 G_{LT}
$$

$E_L$ - Young's modulus in longitudinal direction

$E_T$ - Young's modulus in transverse direction

$\gamma_{LT}$ - Poisson's ratio (\(\frac{E_T}{E_L}\))

$\gamma_{TL}$ - Poisson's ratio (\(\frac{G_{LT}}{E_T}\))

where $k_1$ and $k_2$ are shear correction factors (5/6) to account for the shear being parabolic as already discussed in the previous chapter.

If principal axes of the layer do not coincide with the local axes but are rotated through an angle $\phi$ (being positive if measured in the anticlockwise direction), the
elasticity matrix in Eq. 11.2 has to be transformed to the $\theta_1''z_1'$ axes as shown below.

From Eqn 11.2

$$\bar{\epsilon}_{\theta, \bar{z}, \bar{r}} = \bar{D} \epsilon_{\theta, \bar{z}, \bar{r}}'$$

$$\delta_{\theta, \bar{z}, \bar{r}} = T \delta_{\theta, \bar{z}, \bar{r}}'$$

$$\bar{\epsilon}_{\theta, \bar{z}, \bar{r}} = T' \bar{\epsilon}_{\theta, \bar{z}, \bar{r}}'$$

(11.3)

where $T$ and $T'$ represent the stress and strain transformation matrices.

where $T = \begin{bmatrix} c_s^2 & s_s^2 & 2c_s s_s & 0 & 0 \\ s_s^2 & c_s^2 & -2c_s s_s & 0 & 0 \\ -c_s & c_s & c_s^2 - s_s^2 & 0 & 0 \\ 0 & 0 & 0 & c_s & s_s \\ 0 & 0 & 0 & -s_s & c_s \end{bmatrix}$

(11.4)

and $T' = \begin{bmatrix} c_s^2 & s_s^2 & c_s & 0 & 0 \\ s_s^2 & c_s^2 & -c_s & 0 & 0 \\ -2c_s & 2c_s & c_s^2 - s_s^2 & 0 & 0 \\ 0 & 0 & 0 & c_s & s_s \\ 0 & 0 & 0 & -s_s & c_s \end{bmatrix}$

(11.5)

$$\bar{\epsilon}_{\theta, \bar{z}, \bar{r}}' = \bar{D} \epsilon_{\theta, \bar{z}, \bar{r}}$$

$$\delta_{\theta, \bar{z}, \bar{r}}' = T^{-1} \delta_{\theta, \bar{z}, \bar{r}}$$

$$\bar{\epsilon}_{\theta, \bar{z}, \bar{r}}' = T^{-1} \bar{D} \epsilon_{\theta, \bar{z}, \bar{r}}$$

$$\bar{D} \epsilon_{\theta, \bar{z}, \bar{r}}' = T^{-1} \bar{D} T' \epsilon_{\theta, \bar{z}, \bar{r}}'$$

(11.6)
\[
D = \Gamma^{-1} \bar{D} \Gamma^T \\
\Gamma^{-1} = [\Gamma^T]^T \\
D = [\Gamma^T]^T [\bar{D}] [\Gamma^T] \\
\]

\[
[D] = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & 0 & 0 \\
\bar{A}_{12} & \bar{A}_{22} & \bar{A}_{23} & 0 & 0 \\
\bar{A}_{13} & \bar{A}_{23} & \bar{A}_{33} & 0 & 0 \\
0 & 0 & 0 & \bar{A}_{44} & \bar{A}_{45} \\
0 & 0 & 0 & \bar{A}_{45} & \bar{A}_{55}
\end{bmatrix} \\
\]

(11.7)

where

\[
\bar{A}_{11} = \bar{D}_{11} \, c^4 + 2 \bar{D}_{12} \, c^2s^2 + \bar{D}_{22} \, s^4 + 4 \, \bar{D}_{33} \, c^2s^2 \\
\bar{A}_{12} = \bar{D}_{11} \, c^2s^2 + \bar{D}_{12} \, (c^4+s^4) + \bar{D}_{22} \, c^2s^2 - 4 \, \bar{D}_{33} \, c^2s^2 \\
\bar{A}_{13} = c^2s^2 (\bar{D}_{11} - \bar{D}_{12}) + \bar{D}_{22} \, c^2s^2 + \bar{D}_{33} \, c^2s^2 \\
\bar{A}_{22} = \bar{D}_{11} \, s^4 + 2 \bar{D}_{12} \, c^2s^2 + \bar{D}_{22} \, c^4 + 4 \, \bar{D}_{33} \, c^2s^2 \\
\bar{A}_{23} = c^2s^2 (\bar{D}_{11} - \bar{D}_{12}) + \bar{D}_{22} \, c^2s^2 + 2 \bar{D}_{33} \, c^2s^2 \\
\bar{A}_{33} = c^2s^2 (\bar{D}_{11} - 2\bar{D}_{12} + \bar{D}_{22}) + \bar{D}_{33} \, (c^2s^2)^2 \\
\bar{A}_{44} = \bar{D}_{44} \, c^2 + \bar{D}_{55} \, s^2 \\
\bar{A}_{55} = \bar{D}_{44} \, s^2 + \bar{D}_{55} \, c^2 \\
\bar{A}_{45} = \bar{D}_{44} \, c^2 (\bar{D}_{44} - \bar{D}_{55}) \\
\]

(11.8)
From Eq. 11.9 it is clear that transformation for the out of plane shear stresses are independent of the transformation for the inplane normal and shear stresses. Because of axisymmetry, the inplane shear stress and the out of plane shear stress associated with \( \Theta \) direction are zero. If \( D_{\theta\theta} \) and \( D_{\phi\phi} \) are taken equal to \( G_{LT} \), the shear stress can be directly written as \( k G_{LT} \). Therefore the resulting transformed constitutive law is

\[
\begin{bmatrix}
\sigma_{\theta}' \\
\sigma_z' \\
\tau_{r z}'
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & 0 \\
\tilde{A}_{12} & \tilde{A}_{22} & 0 \\
0 & 0 & kG_{LT}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{\theta}' \\
\varepsilon_2' \\
\gamma_{yz}'
\end{bmatrix}
\] (11.10)

\( k \) is usually taken as \( 5/6 \)

where \( \tilde{A}_{11}, \tilde{A}_{22}, \tilde{A}_{12} \) are defined above.

11.6 FORMULATION OF STIFFNESS MATRIX

From the principle of virtual work,

\[
\int_V \mathbf{S} \mathbf{e}^T \mathbf{e} \, dv = \sum \mathbf{u}_i \mathbf{F}_i \] (11.11a)

\[
\langle \mathbf{S} \mathbf{u} \rangle \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dv \langle \mathbf{u} \rangle = \langle \mathbf{S} \mathbf{u} \rangle \int \mathbf{F} \] (11.11b)

\[
(\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dv) \langle \mathbf{u} \rangle \{ \mathbf{y} \} = \{ \mathbf{F} \} \] (11.11c)

\[
[\mathbf{kI}] \{ \mathbf{u} \} = \{ \mathbf{F} \} \]

(11.11d)

The integral is to be evaluated using Gauss Quadrature formula. But in the layered shell, the elasticity matrix
varies from layer to layer. In order to apply the Gauss quadrature integration, the limits should vary from -1 to 1. This is achieved by modifying the variable $\xi_j$ to $\xi_k$ in any $k$th layer such that $\xi_k$ varies -1 and 1 in that layer.

Thus $[K]$ is obtained as

$$
[K] = \sum_{k=1}^{n} \int_{-1}^{1} \int_{-1}^{1} B_K^T D_k B_K \, d\xi_k \, d\eta_k
$$

$$
[K] = \sum_{k=1}^{n} \left\{ \left[ \sum_{\xi=1}^{2} \sum_{\eta=1}^{2} f(\xi_k, \eta_k) \, w_{\xi_k} \, w_{\eta_k} \right] \right\}
$$

Two Point Gauss quadrature is applied for numerical integration. Properties of all the layers are assumed to be symmetric about the middle. Layer has to be numbered sequentially, starting from the inner surface of the shell. Each layer contains stress points at Gauss points. The stress components are computed at these points and if there is one Gauss point the stresses are assumed to be constant over the thickness of each layer, so that actual stress distribution is modelled by piecewise constant approximation as in Fig.11.4.

Layer of different thickness can be employed as well as different number of layers and orientation per element. The above process of integration is computationally more expensive but is more appropriate for thick and problems of varying thickness.

11.7 CONSISTENT LOAD VECTOR

Considering the uniform lateral pressure, it can be transformed as work equivalent load at the nodes by the
principle of virtual work as detailed in chapter 10.

11.8 NUMERICAL EXAMPLES

The following layered axisymmetric shell problems are solved using the computer program developed in the chapter G.

11.8.1 Cylinder With Uniform Pressure (p = 1.0 psi)

The problem shown in Fig.11.5 has been solved with isotropic properties as given below and with three layers.

Radius (r) = 5.125"
Height (z) = 10.0"
Thickness (t) = 0.081"
Youngs modulus (E) = 2.8 x 10^7 psi
Poisson's ratio (\(\nu\)) = 0.3

The problem has been solved with 2, 4 and 8 elements and the results are coinciding with that obtained for cylinder with single layer. The variation of radial displacement along depth is given in Fig.11.6

11.8.2 Cylinder With Constant Edge Couple (Moment =1)

The cylinder shown in Fig.11.7 with three layers has been solved using the following properties for Glass/epoxy layer.

radius (r) = 50 cm
e-height (z) = 50 cm
thickness (t) = 1.5 cm
Youngs' modulus (\(E_L\)) = \(.386 \times 10^6\) kg / cm^2
Youngs' modulus (\(E_T\)) = \(0.827 \times 10^5\) kg / cm^2
Poisson's ratio (\(\nu_{LT}\)) = 0.26
shear modulus \( G = 0.414 \times 10^5 \) kg / cm\(^2\)

The problem has been solved using 2, 4 and 8 elements and the graphs for \( u \) vs \( z \) is given in Fig.11.8

11.8.3 Spherical Cap Under Uniform Pressure (\( p = 1.0 \))
This problem is shown in Fig.11.9. The following geometric and material properties are used for Kelvar/Epoxy layer with three layers.

radius \( r \) = 90.0 cm
thickness \( t \) = 3.0 cm
Young's modulus \( E_\perp \) = 0.76 \times 10^6 \) kg / cm\(^2\)
Young's modulus \( E_T \) = 0.55 \times 10^6 \) kg / cm\(^2\)
Poisson's ratio \( \gamma_{LT} \) = 0.34
Shear modulus \( G \) = 0.23 \times 10^5 \) kg / cm\(^2\)

The problem has been solved with 2, 4 and 8 elements and the graph hoop force vs \( \theta \) is given in Fig.11.10

11.8.4 Cylinder With Edge Load (load = 1)
The problem is shown in Fig.11.11. The following geometric and material properties are used for 3 layers.

radius \( r \) = 5.0 "
height \( z \) = 5.0 "
thickness \( t \) = 0.01 "
Young's modulus \( E_\perp \) = 1 \times 10^6 \) psi
Young's modulus \( E_T \) = 1 \times 10^6 \) psi
Poisson's ratio \( \gamma_{LT} \) = 0.1667
Shear modulus \( G \) = 3.8461 \times 10^5 \) psi

The problem has been solved and the graph \( u \) vs \( z \) is given in
11.9 SUMMARY

The advent of advanced fibre-reinforced composite materials has been called the biggest technical revolution in aerospace vehicles. Such advanced composites have two major advantages among others; improved strength and stiffness. The finite element (static) analysis of layered composite axi-symmetric shells of revolution has been discussed in this chapter. The finite element formulation for classical stability of isotropic as well as layered composite shells of revolution will be discussed in next chapter.
Fig: 11.1 GLOBAL DISPLACEMENTS IN AN AXI-SYMMETRIC LAYERED SHELL

Fig: 11.2 LOCAL ORTHOGONAL CO-ORDINATES
Fig: 11.3 A TYPICAL AXI-SYMMETRIC SHELL ELEMENT

Fig: 11.4 LAYER MODEL AND CORRESPONDING STRESS REFERENCES
FIG. 11.5 CYLINDER WITH UNIFORM PRESSURE

FIG. 11.7 CYLINDER WITH HARMONIC MOMENTS
Fig 11.6 VARIATION OF RADIAL DISPLACEMENT ALONG DEPTH
Fig. 11.8 VARIATION OF $U/t$ ALONG DEPTH $Z$
Fig. 11-10 VARIATION OF HOOP FORCE WITH ANGLE $\phi$
FIG. 11.11 CYLINDER WITH EDGE LOAD

FIG. 11.9 A SPHERICAL CAP WITH UNIFORM INTERNAL PRESSURE
Fig. 11.12 VARIATION OF RADIAL DISPLACEMENTS ALONG DEPTH