CHAPTER 2
MAGNETOHYDRODYNAMICS WAVES

The gas dynamics theory of nonlinear flows which are either time dependent or involve supersonic speeds is well developed. Such flows tend to contain planes across which the huge changes in gas condition occur just about discontinuity. These discontinuities which are called shock waves turn out to be thin to the point that the dispersal rates inside of them turn out to be vast. One of the prime reasons for interest in shock waves is that they provide a mechanism for changing over flow kinetic energy in front of the wave into thermal energy behind of the wave and thus provide a controlled method for delivering high temperature. Elliot [1] discussed self similar solutions for spherical blast wave in air using Rossland’s diffusion approximation under the assumption that there is no effect of heat flux at the centre of symmetry. Laumbach and Probstein [2] have studied the radiation effects on shocks in an exponential medium. Summer [3] considered the magnetic field. Singh [4] examined propagation of strong shocks in an optically thin atmosphere taking different models. Park and Hyun [5] and Park [6] concentrated on the one dimensional common convection of viscous stratified fluid. Shapiro and Fedorovich [7] studied the unsteady convectively determined flow of steadily stratified fluid along a plate, considering pressure work and ambient thermal stratification into account while Magyari et al. [8] restudied for a porous medium.

Mixed convective MHD mass exchange flow past an accelerated infinite vertical porous plate was examined by Ramana et al.[9].Chaudhary and Jain [10] researched MHD heat and mass diffusion flow by natural convection past a vertical plate in porous medium. The fluid properties may change by large amount. For this situation the gradients will be small. These regions may be treated by a strategy which is essentially a generalization to the nonlinear case of concepts developed in linear wave propagation in non dispersive media. In the propagation of electromagnetic signals in vacuum or in acoustic a disturbance of arbitrary shape at one instant of time have the same shape. If this disturbance propagated through a medium in which the propagation speed varied with position, due to gradual changes in the index of refraction or gas
temperature. Mathematically, the description of gas dynamic flow is a special case of the theory of characteristics which is applicable to certain types of hyperbolic partial differential equations.

Consider discontinuity is in x-direction. Nonlinearity can give rise to shocks, contact, rarefactions and compound waves in 1D MHD. Different wave speeds exist in MHD. The Eigen values of Jacobian $J = \frac{\partial F}{\partial \mathbf{U}}$ define the characteristic (wave) speeds in MHD. The basic properties of magnetohydrodynamic blast waves as determined by the conservation laws (the Rankine – Hugoniot relations) have been developed further by Friedrichs [11], Helfer [12], Lust [13, 14], Bazer & Ericksons[15], Kanwal[16] and many others, and they are now well understood; but the more complex question of their existence in nature has yet to be exhaustively treated inspite of efforts in this direction by several Russian authors. One can also encounter compound waves in ideal MHD. These compound waves can consist of a slow shock which travels with its maximal propagation speed and a rarefaction fan directly attached to it. Brio & Wu [17] detected those compound waves in numerical simulations which have become classical test problems for numerical codes.

2.1 LINEAR MHD WAVES

Our first system of interest is ideal Magnetohydrodynamics (MHD). MHD is a single fluid treatment of the long time and large length scale behavior of a plasma. The theory of characteristic describes the non-linear flow as composed of small amplitude step function waves. Therefore, first we obtain the small amplitude waves which result from the equations (1.1) to (1.5) for non dispersive medium. Its derivation requires for the system to be magnetized, $\rho i/L \ll 1$, and for strong collisionality, $\lambda_m/L \ll 1$, where $\lambda_m$ is the collisional mean free path and L is the system size. Using the equation it is impossible to form either a basic length or a basic time from the quantities in the equations. Therefore, if examined the linear waves by the more usual method of assuming sinusoidal shaped of waves, we would get that the phase velocity of these waves is independent of wavelength. The medium is non dispersive. The extension to the nonlinear case is more direct if use the linear wave properties in terms of step functions.
It is convenient to analyze the wave in a coordinate system in which the wave is stationary. In this coordinate system the fluid will be moving on both sides of the wave. The fluid velocity ahead of the wave must be equal and opposite to the propagation speed of the wave relative to the fluid in order for the wave to be stationary. We denote velocity by \( c \), which is the characteristic propagation speed of the wave. Consider the coordinate system such that the wave is normal in the \( x \) direction. The magnetic field is ahead of the wave in the \( x-y \) plane, i.e., \( B_z = 0 \), ahead of the wave. In the coordinate system the time derivatives are zero and the operator \( \nabla \) reduces to a derivative in the \( x \)-direction. Integrating the equations (1.1) to (1.6) across the wave. We obtain relations for the changes in the flow properties across the wave. Taking terms only to first order in the changes, we obtain for the continuity equation

\[
\delta(\rho u_x) = -c \delta p + \rho \delta u_x = 0 ,
\]

(2.1)

Three components of the momentum equation become

\[
- \rho c \delta u_x + \delta p + B_y \delta B_y = 0 ,
\]

(2.2)

\[
\rho c \delta u_y + B_x \delta B_y = 0 ,
\]

(2.3)

\[
\rho c \delta u_z + B_x \delta B_z = 0 ,
\]

(2.4)

The components of the equation (1.4) become

\[
B_y \delta u_x - c \delta B_y - B_x \delta u_y = 0 ,
\]

(2.5)

\[
c \delta B_z - B_x \delta u_z = 0 ,
\]

(2.6)

The equations (1.3) and (1.5) become

\[
c \left( \delta p - \frac{\gamma p}{\rho} \delta \rho \right) = 0 ,
\]

(2.7)

The equation (1.6) becomes

\[
\delta B_x = 0 ,
\]

(2.8)

The above equations are homogenous in the unknown corresponding to the changes across the wave. Therefore, for the linearized case, the differential equations do not determine the amplitude of the wave. However, using the equations and unknowns we determine the speed of propagation of the wave and the changes of flow properties across the wave in terms of the wave amplitude.
2.2 ENTROPY DISCONTINUITIES

There is a solution which is not really a wave, since it relates to zero propagation speed. We observed a solution by substituting \( c = 0 \) into equation (2.1) through (2.8) which allows a change in density but there is no change in majority of the other flow properties, i.e., velocity, magnetic field and gas pressure. This is a hydrostatic equilibrium corresponding to diverse density, entropy and temperature on the two sides of the discontinuity maintaining the constant pressure. Since there is no flow through the wave, this result is solid with entropy conservation law which following a fluid part, the entropy is proportioned. Thus, if initially adjacent elements of fluid have different entropies, this discontinuity in entropy is maintained. The case in which shock wave exists whose strength is not constant, the entropy changes across a shock depends upon its strength. Thus, fluid elements going through the shock at slightly different times have slight different entropies. For finding other solution; consider the propagation speed is non zero, i.e.,

\[
\left( \frac{\partial p}{\partial \rho} - \frac{n p}{\rho} \delta \rho \right) = 0, \tag{2.9}
\]

This corresponds to the statement that if there is a flow through the wave and if entropy is conserved along the streamline, then the entropy on both sides of the wave must be same.

2.3 INTERMEDIATE WAVES

There are three propagation waves corresponding to the three degree of freedom of motion of boundary. In this sub-division the properties and propagation speed of one of these modes is derived. The three modes can be classified as fast, intermediate and slow according to the magnetic of their speeds of propagation. Solving the equations (2.4) and (2.6), we obtain for the propagation speed

\[
c_i^2 = \frac{B^2}{\rho}, \tag{2.10}
\]
Where, the subscript $i$ indicate the intermediate wave. Substituting this result into the full set of equations, following relations for the change $s$ in flow properties across the wave are obtained.

$$\delta u_z = \pm \frac{\delta B_z}{\sqrt{\rho}},$$  \hspace{1cm} (2.11)

$$\delta u_x = \delta \rho = \delta B_y = \delta u_y = 0,$$  \hspace{1cm} (2.12)

The sign in the equation (2.11) depends upon whether the direction of propagation is parallel or antiparallel to the normal component of magnetic field. Since $\delta u_x$ is zero this is purely transverse. Since the change in magnetic field is perpendicular to the original field there is a change in magnitude. There are changes in the tangential velocity and the direction of the magnetic field. Since the wave is purely transvers, so, the wave propagation speed is the square root of the tension divided by the density. Since the tension is $B_z^2$, the intermediate wave corresponds to the changes in magnetic field only in the $z$-direction. Thus the ordinary vibrating string corresponds to two modes which have same the same propagation velocity. In the plasma, only one of these modes corresponding to the magnetic field changes perpendicular to the magnetic field gives rise to the intermediate propagation speed. If there is a change in $B_y$, then the waves give rise to some longitudinal stresses and as a result their propagation speed is modified. Such waves are then either the fast or the slow mode.

### 2.4 FAST AND SLOW WAVES

From the equations (2.1), (2.2), (2.3), (2.5) and (2.7), the propagation speed of the remaining two modes can be obtained by eliminating the quantities which change across the wave. The resulting relation of propagation speed can be written as

$$\left( c^2 - a^2 \right) \left( c^2 - b_x^2 \right) = c^2 b_y^2$$  \hspace{1cm} (2.13)

$$\left( c^2 - a^2 \right) \left( c^2 - b_y^2 \right) = a^2 b_y^2$$  \hspace{1cm} (2.14)

Where shorthand notation for the ordinary sound speed is introduced for the ordinary sound speed and for a vector whose magnitude is equal to the Alfvén speed and whose direction is parallel to the magnetic field.
the equation is bi-quadratic in the propagation speeds which correspond to tri modes each of which can propagate in two directions. Some of the properties of propagation speeds resulting from the relation are observed. Since in both of the dispersion relation the right hand sides are positive definite, it follows that the quantities in the parentheses on the left hand side must be either positive or both negative. Therefore, from the equation (2.13) one of the solution is the square of the propagation speed must be greater than both \( a^2 \) and \( b^2 \) and for other solution it must be less than both of these quantities. Since the slow speed is lower of these, so

\[ c_i^2 \leq b^2, \quad \text{and} \quad c_i^2 \leq a^2, \quad (2.16) \]

From the equation (2.12) condition on the fast propagation speed are obtained

\[ c_f^2 \geq b^2, \quad \text{and} \quad c_f^2 \geq a^2 \quad (2.17) \]

The speed of fast wave is greater than or equal to the intermediate speed, while speed of slow wave is slower than or equal to the intermediate speed.
For several values of the ratio of sound speed $a$ to Alfvén speed $b$, polar Plot showing the dependence of the propagation speeds of the three linear wave modes on the angle between the wave normal and magnetic field. Speed have been normalized with respect to $\sqrt{a^2 + b^2}$. The solution of this dispersion relation are plotted in figure for several ratios $a$ to $b$. These plots are given by Friedrichs. The magnetic field direction is taken to be horizontal. For any point on the lines, the distance from the origin is proportional to the velocity of the wave. The angle which the line connecting the origin to the point makes with the axis corresponding to the magnetic field is the direction of the wave normal relative to the magnetic field.

For the fast mode, the propagation speed is relatively insensitive to the direction of propagation. For propagation perpendicular to the magnetic field, the propagation speed is $\sqrt{a^2}$. For propagation along the magnetic field, the propagation speed is either $a$ or $b$ depending upon which is larger. The intermediate propagation speed corresponds to two circles of radius $b/2$ whose line of centres lies in the magnetic field direction and are tangent at the origin. Since the slow propagation speed is less than the intermediate speed, the slow speed is also observed inside of these circle. Thus, both the slow and intermediate speeds are zero for propagation perpendicular to the magnetic field. For propagation along the magnetic field, the slow speed is either $a$ or $b$ depending upon which is smaller. For propagation along the magnetic field, the intermediate speed is always equal to either the slow or the fast speed. Thus, for $a < b$, the intermediate and fast speeds are equal for this direction of propagation, while for $a > b$, the intermediate and slow speeds are equal.

For the cases of $a >> b$ and $a << b$, the limiting values of the solution of the dispersion relation are larger than gas pressure and less than the magnetic pressure.
For $a > b$, the fast propagation speed is greater than $b$ therefore, from the equation (2.14), we conclude that the propagation speed approaches $a$. From the equation (2.12), we conclude the propagation speed is equal to $b$. In this limit the slow speed approaches the intermediate speed. In other words as the ratio of gas pressure to magnetic pressure is increased and slow and intermediate speeds come closer together.

For $a << b$, the fast speed approaches $b$, while the slow speed approaches $\frac{ab}{b}$. Thus, the fast propagation speed corresponds the two circles whose line of centre lies in the direction of magnetic field and are tangent at the origin. If the magnitudes of $a$ and $b$ are interchanged, the fast and slow propagation speeds are unchanged. From the equation (2.14) since the equation is unchanged when $a$ and $b$ are interchanged, $b_y$ is the product of the magnitude of $b$ and sine of the angle between the magnetic field and wave normal. This conclude that the changes in flow properties across the waves are different depending upon whether $a$ and $b$ is larger.

The changes in flow properties which occur across the waves are obtained by substituting the solutions of the dispersion relation. If the characteristic speed is equal to the intermediate speed, equations (2.4) and (2.6) allow the solutions with nonzero changes in $u_z$ and $B_z$. Since, except some points, the fast and slow speeds always differ from the intermediate speed, we conclude that across fast and slow wave

$$\partial u_z = \partial B_z = 0,$$  \hspace{1cm} (2.18)

Thus, for these modes the magnetic field stays in the plane determined by the magnetic field ahead of the wave and the wave normal. Therefore, the fast and slow waves change only the magnitude of the tangential component of magnetic field but not its direction. Earlier, we concluded that intermediate wave change only the direction but not the magnitude of the tangential component of magnetic field. From the equation (2.8) the normal component of magnetic field is constant in all cases. There are finite changes in both $u_x$ and $u_y$ for the fast and slow wave. Therefore, these waves are partially longitudinal and partially transverse. The longitudinal aspect contained in the $x$-component of the momentum equation (2.2). Using the continuity
and entropy equations (2.1) and (2.7), equation can be written as
\[ c^2 = \left( \frac{\delta(p + B_y^2/2)}{\delta \rho} \right) = a^2 + \left( \frac{\delta(B_y^2/2)}{\delta \rho} \right) \]  
(2.19)

Since the propagation of speed is squared, Therefore, the change in the longitudinal stress, resulting from both the gas pressure and the magnetic pressure divided by the change in density. Since, from the equation (2.15), the fast propagation speed is greater than the sound speed. Therefore, the change in \( B_y^2 \) and the change in density both have same sign and for a slow wave the changes are of opposite sign. The transverse motion do not lead to as simply interpretable results. Using the equations (2.3), (2.5) and (2.2), we obtain the result
\[ c^2 = \frac{b_y^2}{1 - \frac{b_y \delta \rho}{\rho B_y}} \]  
(2.20)

Thus, no changes in the longitudinal velocity implies no change in density. The propagation speed approaches the intermediate speed which is a purely transverse wave. Using the equations (2.1), (2.2), (2.3) and (2.7) to examine the direction of the velocity, we obtain
\[ \frac{\partial \mathbf{u}_x}{\partial \mathbf{u}_x} = \frac{B_x}{B_y} \left( \frac{a^2}{c^2} - 1 \right) \]  
(2.21)

From the quadratic expression (2.13) or (2.14), the two roots of the dispersion relation are related by the following equations
\[ c_f^2 + c_s^2 = a^2 + b^2 \]  
(2.22)
\[ c_f^2 c_s^2 = a^2 b^2 \]  
(2.23)

Using these relation and the equation (2.22), we get
\[ \left( \frac{\partial \mathbf{u}_y}{\partial \mathbf{u}_x} \right)_f = -\left( \frac{\partial \mathbf{u}_x}{\partial \mathbf{u}_y} \right)_s \]  
(2.24)

Which states that the changes in velocity across the fast and the slow waves are perpendicular to each other and change in velocity across the intermediate wave are in the \( z \)-direction. So, we conclude that the changes in velocity across the three waves are in mutually perpendicular directions. The changes in flow properties across a large amplitude intermediate wave are obtained by summing the changes across each of the
component small step function wave, which are considered as differential elements. From the equation (2.11) across a large amplitude wave the changes in normal velocity, density and pressure will be zero. In evaluating the change in magnetic field the coordinate system is chosen such that $B_1$ is zero ahead of each small amplitude wave. The equation (2.12) states that the differential change in magnetic field is in the plane of the wave front and perpendicular to the local field. Integrating a number of such changes gives the result that the magnitude of the magnetic field is unchaned across a large amplitude intermediate wave. However, the magnetic field vector can be rotated through an arbitrary large angle about an axis perpendicular to the wave front. The change in tangential velocity across the wave is $\nabla B/\sqrt{\rho}$ in the direction of the change in magnetic field. The angle of rotation of the magnetic field and the change in tangential velocity can be large according the order of radius and the propagation speed respectively. For small amplitude fast and slow waves the magnetic field remains in the plane defined by the wave normal and the magnetic field ahead of the waves. The intermediate wave is required in flow fields in which the boundary condition require a rotation of the plane of the magnetic field. The particular case of rotation through $\pi$, the magnetic field appears in the same plane but its tangential component changes sign. Fast or slow shock waves can not change the sign of the tangential component so the intermediate wave will also appear in cases where sign changes is required by the boundary conditions.

If $a > b$ then the fast propagation speed and the changes across the fast wave reduce to those for an ordinary sound wave. The fast wave approaches a purely longitudinal wave. The slow wave becomes a purely transverse wave. Equation (2.22) shows that the change in the y-component of velocity become very large compared to the magnetic pressure. Thus, only very small change are required to balance the change in the magnetic pressure in the longitudinal velocity.

If $a < b$ the waves do not change into purely longitude and purely longitudinal and transverse. In this limit, both the gas pressure and the dynamic pressure $\rho u^2$ are small compared to the magnetic pressure. The magnetic field lines will have virtually no change in direction across the wave. The plasma is constrained to flow in a direction parallel to the magnetic field lines. In this limit the propagation speed of slow wave is equal to the sound speed multiplied by the cosine of angle between the magnetic field
and the wave propagation propagation direction. This corresponds to a sound wave travelling along the magnetic normal. Therefore the slow wave becomes purely transverse for propagation perpendicular to the magnetic field and purely longitudinal for propagation along the magnetic field. Since the velocity change across the fast wave is perpendicular to that across the slow wave, we conclude that in this limit the fast wave is purely longitudinal for propagation perpendicular to the magnetic field. While it is purely transverse for propagation along the magnetic field.

**2.5** **EIGEN VALUE ANALYSIS OF 1D IDEAL MHD**

The dynamics of magnetized plasmas can be interpreted using solution of the properties of linear and nonlinear wave. The properties of linear waves can be study using the dispersion relation $i(\omega \pm \lambda \cdot x)$, where $\omega$ is the frequency and $\lambda$ is the wave vector. The MHD wave families are fast, Alfven, slow and entropy.

1. Fast magnetosonic waves $u \mp c_f$ are longitudinal waves with variations in pressure and density. Fast magnetosonic waves are correlated with magnetic field.

2. Alfven waves $u \mp c_a$ are transverse waves with no variation in pressure and density. Alfven waves can be polarized. The sum of linear polarizations can lead to circularly polarized alfven, slow and entropy waves.

3. Slow magnetosonic waves $u \mp c_s$ are longitudinal waves with variations in pressure and density but slow magnetosonic waves are anticorrelated with magnetic field.

4. Entropy waves $u$ is a contact discontinuity with no variation in pressure and velocity. In these waves, the minus (plus) sign is applied to left-going (right-going) waves. MHD waves involve transversal motion.

The characteristic speeds are expressed as

$$c_{f,s} = \left[ \frac{1}{2} \left( \frac{B_x^2 + B_y^2 + B_z^2}{\rho} + a^2 \right) \pm \sqrt{\frac{1}{4} \left( \frac{B_x^2 + B_y^2 + B_z^2}{\rho} + a^2 \right)^2 - \frac{a^2 B_z^2}{\rho}} \right]^{1/2},$$

(2.25)

$$c_a = \sqrt{\frac{B_z^2}{\rho}},$$

(2.26)
Where \( a \) is the sound speed given by \( a = \sqrt{\gamma P / \rho} \). The slow or fast signal could be shocks or rarefaction. Since \( c_s \leq c_a \leq c_f \). This reflects that Eigen values may coincide at a special point. Hence, the systems of Ideal MHD equations are not strictly hyperbolic. There are certain cases where two or more eigen value collapse to the same value.

1. If \( B_y = 0 = B_z \), and \( a = \sqrt{\gamma P / \rho} \neq \sqrt{B_z^2 / \rho} = c_a \), In this case either both fast or slow eigen values collapse with Alfven speed.

2. If \( B_y = 0 = B_z \) and \( a = \sqrt{\gamma P / \rho} = \sqrt{B_z^2 / \rho} = c_a \). In this case Slow and Fast eigen values collapse with Alfven eigen values.

3. If \( B_x = 0 \). Both Slow and Alfven waves collapse to entropy wave.

When Eigen values collapse, there exists possibility of over compressive and under compressible Shocks. This fact makes finding solutions to the equations to ideal MHD more complicated. Today, the most important method for solving the MHD equations are numerical methods. But the development of numerical technique to
solve MHD has been slower due to complexity of the MHD flow. Grid based methods to solve MHD equations is based on explicit finite difference scheme called the total variation diminishing (TVD) scheme. In this method the conserved variables are discretized on a grid, with volume averaged values stored at cell centers. One of grid code for MHD is Athena. Athena implements a higher-order Godunov scheme. In this method, the difference in cell average values at each grid interface define set of Riemann problems. Solution of Riemann problems averaged over cell give time-evolution of cell average value. Riemann solvers available to compute the fluxes.

The electric currents transmitted in an electrolyte solution interact with the magnetic field to form Lorentz body forces that in turn, drive fluid motion. Lorentz force is the flow in the direction perpendicular to both magnetic and electric fields in conductive, solutions in the MHD system with initial velocity and initial magnetic field by keeping electric field as negligible. The current flow in a direction perpendicular to the direction of magnetic field causes the fluid to experience a force. The force is in a direction perpendicular to both the magnetic field and the current flow. However, in many situation the changes in pressure and temperature are sufficient small that the changes in density are negligible. That flow can be modelled as an incompressible flow. For gases, to determine whether flow to use compressible or incompressible fluid dynamics, the Mach number of the flow is to be evaluated. Compressible effect can be ignored at Mach numbers below approximately 0.3. For liquids, the incompressible flow depends on fluid properties and the flow condition how close to critical pressure the actual flow pressure becomes. The phenomenon of strong and weak discontinuities in a compressible fluid has been of great interest amongst the scientist and mathematician. Rankine Hugoniot developed the jump conditions for sudden changes in physical parameters such as temp pressure and velocity etc. with discontinuity. This sudden change may occur on account of sudden explosions in compressible gases, collision of clouds, movements of super sonic jets with high mach numbers. Since most of the problems occurring are non-linear are in nature, the similarity condition play an important role in solving such problems. Analytical similarity solutions as well as numerical solution exist for electrically conducting compressible flows across such discontinuities. It is therefore proposed that study of such non-linear problems nil come out in presence of conducting and non conducting medium where conditions may be adiabatic or isothermal or both.
REFERENCES
