1. Introduction

In this chapter we adapt these theorems to the calculus of variations. The theorems of Noether (1918) describe a relationship between the invariance of the action integral with respect to given groups of transformations and some identities satisfied by Euler-Lagrange expressions. There are two types of Noether theorems. The first theorem deals with transformations depending on scalar parameters and the second theorem deals with transformations depending on functions. They can be roughly stated as follows (Logan, 1977):

(i) If the action $W$ is div-invariant under an $r$-parameter continuous group of transformations of the variables, then there result $r$ identities between Euler-Lagrange expressions $\frac{\partial}{\partial \lambda_k}$ and quantities which can be written as divergences.

(ii) If the action is div-invariant under a group of transformations which depend upon $r$ arbitrary functions and their derivatives up to some order $q$, there exist $r$
identities between the Euler-Lagrange expressions \( \mathcal{L}_k \) and their derivatives up to order \( q \).

In this chapter we adapt these theorems to the variational principle discussed in chapter 3 and examine some applications.

2. Noether's first theorem and Galilean group of transformations:

We consider a class of transformations depending on scalar parameters.

Let 
\[
W = \int \text{d}V \, L(\mathbf{x}, \mathbf{p}(\mathbf{x}), \mathbf{s}(\mathbf{x}))
\]
be defined on a suitable function space. We consider the infinitesimal transformations defined by

\[
\Delta x^\alpha = x^\alpha + \delta x^\alpha, \quad \alpha = 0,1,2,3,
\]
where \( \delta x^\alpha \) are functions of \( \mathbf{x}, \mathbf{p}(\mathbf{x}), \mathbf{s}(\mathbf{x}) \) and their derivatives.

Then the functional \( W \) is transformed to

\[
\tilde{W} = W + \Delta_0 W,
\]
\[ L(x, P(X), s(X)), \]

where \( \tilde{V} \) is the transformed region of \( V \).

Now we have,

\[ \Delta_{\tilde{V}} L = \int_{\tilde{V}} dV \left[ \frac{\partial L}{\partial \alpha} \delta_{\tilde{p}} \alpha + \frac{\partial L}{\partial s} \delta_{\tilde{s}} \alpha + \delta_{\alpha} (L \Delta x^\alpha) \right], \quad (4.5) \]

where

\[ \delta_{\tilde{p}} \alpha = \delta_{\beta} (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta), \quad (4.6) \]

and

\[ \delta_{\tilde{s}} \alpha = \delta_{\beta} (s^\beta \Delta x^\alpha - s^\alpha \Delta x^\beta). \quad (4.7) \]

Definition (4.8):

(a) The functional \( W \) is said to be 'hydromechanically invariant up to a divergence (div-invariant)' with respect to the transformations (4.2) if there exists a vector \( (C^\alpha) \) such that \( \Delta_{\tilde{V}} W = \delta_{\alpha} C^\alpha \) identically in \( V \).

(b) If \( C^\alpha = 0 \) in (a) so that \( \Delta_{\tilde{V}} W = 0 \), then \( W \) is said to be 'absolutely invariant' with respect to the transformations (4.2).
Noether's first theorem can be adapted to our variational principle as follows:

\[ \Delta_{\alpha} W = \int \delta W \left[ \partial_{\beta} \left( \partial_{\alpha} \Delta x^{\beta} \right) - \partial_{\alpha} \Delta x^{\beta} \right]. \]  

(4.12)

Theorem (4.9):

If the functional \( W \) is div-invariant with respect to the transformations (4.2), depending on \( r \) arbitrary parameters, then exactly \( r \) linearly independent linear forms of the Euler-Lagrange expressions \( \gamma_{\alpha} \) are divergences, provided the variations of the field variables are restricted to hydromechanical variations.

Proof:

By the hypothesis of the theorem, within the infinitesimals of first order,

\[ \Delta x^{\alpha} = g^{\alpha}_{m} \varepsilon^{m} \]  

(4.10)

and

\[ C^{\alpha} = C^{\alpha}_{m} \varepsilon^{m}, \quad m = 1, 2, 3, \ldots, r, \]  

(4.11)

where \( \varepsilon^{1}, \varepsilon^{2}, \ldots, \varepsilon^{r} \) are infinitesimal scalar parameters and \( g^{\alpha}_{m}, C^{\alpha}_{m} \) are given functions.
From equation (3.24) we have

\[ \Delta_0^W = \int_V dV \left[ \partial_\beta (T_\alpha^B \Delta x^\alpha) - \gamma_\alpha \Delta x^\alpha \right]. \]  \hspace{1cm} (4.12)

Substituting from the equations (4.10) and (4.11) in the equation (4.12), we get

\[ \varepsilon^m \int_V dV \partial_\beta C_m^\beta = \varepsilon^m \int_V dV \left[ \partial_\beta (T_\alpha^B g_m^\alpha) - \gamma_\alpha g_m^\alpha \right], \]  \hspace{1cm} (4.13)

identically in \( \varepsilon^m \) and \( V \). Hence

\[ \gamma_\alpha g_m^\alpha = \partial_\beta (T_\alpha^B g_m^\alpha - C_m^\beta), \]  \hspace{1cm} (4.14)

which completes the proof.

During the motion we have \( \gamma_\alpha = 0 \), and equation (4.14) becomes

\[ \partial_\beta (T_\alpha^B g_m^\alpha - C_m^\beta) = 0. \]  \hspace{1cm} (4.15)

We apply the formula (4.15) to the case when (4.2) is the group of Galilean transformations. For this, we suppose that the action is absolutely invariant.
with respect to the transformations defined by

$$\Delta x^\alpha = a^\alpha + a^\alpha_\beta x^\beta; \quad \alpha,\beta = 0,1,2,3, \quad (4.16)$$

in which $a^\alpha$ and $a^\alpha_\beta$ are scalar parameters satisfying the conditions

$$a^0_\beta = a^\alpha_\alpha = 0, \text{ for } \alpha,\beta = 0,1,2,3 \quad (4.17)$$

and

$$a^\alpha_\beta + a^\beta_\alpha = 0, \text{ for } \alpha,\beta = 1,2,3. \quad (4.18)$$

In this case $C^\alpha = 0$ and by equation (4.15) we have the following conservation laws:

3. Noether's second theorem and generalized Helmholtz theorems

$$\delta_\alpha (T^\beta_\alpha) = 0; \quad \alpha,\beta = 0,1,2,3. \quad (4.19)$$

and

$$\delta_\alpha (M^\alpha_\beta_\gamma) = 0; \quad \alpha = 0,1,2,3, \quad \beta,\gamma = 1,2,3. \quad (4.20)$$

where

$$M^\alpha_\beta_\gamma = T^\alpha_\beta x^\gamma - T^\alpha_\gamma x^\beta. \quad (4.21)$$
Substituting in equations (4.18) and (4.19) the expressions for $T_\alpha^\beta$ given by (3.26), we get the conservation laws of energy, impulse and angular momenta respectively. If $L$ takes the usual form given by (3.10) these laws become the familiar ones:

\begin{align}
(i) & \quad \rho \frac{D}{Dt} \left( \frac{1}{2} |\vec{v}|^2 + U + \frac{1}{\rho} (P+E) \right) = \frac{\delta}{\delta t} P, \quad (4.21) \\
(ii) & \quad \rho \frac{D}{Dt} (\vec{v}) = - \nabla P, \quad (4.22) \\
(iii) & \quad \rho \frac{D}{Dt} (\vec{x} \times \vec{v}) = - \vec{x} \times \nabla P. \quad (4.23)
\end{align}

3. Noether's second theorem and generalized Helmholtz theorems

Now we state Noether's second theorem adapted to hydromechanical variations.

**Theorem (4.24):**

By the hypothesis, $W$ is div-invariant under the transformations $x \rightarrow x + \delta x$ and $\hat{v} \rightarrow \hat{v} + \delta \hat{v}$, and their derivatives up to a given order $N$, where $\delta x$ and $\delta \hat{v}$ are arbitrary functions depending essentially on $1$ arbitrary functions $\Theta^\alpha_1, \Theta^\alpha_2, \ldots, \Theta^\alpha_N$ and their derivatives up to a given order $q$.

There exist exactly $1$ linearly independent identities depending on the variations of the field variables $\hat{v}$ and $\rho$ being given functions $\Theta^\alpha_1, \Theta^\alpha_2, \ldots, \Theta^\alpha_N$.

**Proof:**

If the functional $W$ is div-invariant with respect to the transformations $x \rightarrow x + \delta x$ and $\hat{v} \rightarrow \hat{v} + \delta \hat{v}$, and their derivatives up to a given order $N$, then $W$ is a constant of motion. Therefore, $W$ is a constant of motion for the hydromechanical variations $\delta x$ and $\delta \hat{v}$, and their derivatives up to a given order $q$. This proves the theorem.
\[ \Delta x^\alpha = \Lambda_s^\alpha (\varnothing^s), \]
\[ = A_s^\alpha \varnothing^s + A_s^\alpha \lambda_1 \partial_1 \varnothing^s + A_s^\alpha \lambda_1 \lambda_2 \partial_1 \lambda_2 \varnothing^s + \ldots + A_s^\alpha \lambda_1 \lambda_2 \ldots \lambda_q \partial_1 \lambda_2 \ldots \lambda_q \varnothing^s, \]

where \( C_s^\alpha, C_s^{\alpha \lambda_1}, \ldots \) are given functions.

depending essentially on \( r \) arbitrary functions \( \varnothing^s, \)
\( s = 1, 2, \ldots, r \) and their derivatives up to a given order \( q, \)
the coefficients \( A_s^\alpha, A_s^{\alpha \lambda_1}, \ldots \) etc. being given functions,
there exist exactly \( r \) linearly independent identities
between the Euler-Lagrange expressions \( \dot{\kappa}_\alpha \) and their
derivatives, provided the variations of the field
variables \( \bar{p} \) and \( \bar{s} \) are restricted to hydromechanical
variations.

Proof:

By the hypothesis, \( W \) is div-invariant under the
transformations (4.2).

\[ \Lambda_s^\alpha (\varnothing^s) = A_s^\alpha \varnothing^s - \partial_1 \left( A_s^\alpha \lambda_1 \varnothing^s \right) \]

In definition (4.8a) we take
\[ C^\alpha = \Gamma^\alpha_s (\varphi^s) \]

\[ = \mathcal{C}_s^\alpha \varphi^s + \mathcal{C}^{\alpha_1}_s \partial_\lambda^s + \mathcal{C}^{\alpha_2}_s \partial_\lambda^2 \varphi^s + \cdots + \mathcal{C}^{\alpha_1}_s \lambda^1 \cdots \lambda^q \partial_\lambda^1 \cdots \partial_\lambda^q \varphi^s, \quad (4.26) \]

where \( \mathcal{C}_s^\alpha, \mathcal{C}^{\alpha_1}_s, \ldots \) are given functions.

Substituting from equations (4.25) and (4.26) in equation (4.12), we have

\[ \int_V dV \partial_\alpha (\Gamma^\alpha_s (\varphi^s)) \]

\[ = \int_V dV \left[ \partial_\beta (\Pi^\beta_s \Lambda^\alpha_s (\varphi^s)) - \lambda^\alpha_s \Lambda^\alpha_s (\varphi^s) \right], \quad (4.27) \]

identically in the functions \( \varphi^s \) and in the region \( V \).

Let \( \Lambda^\alpha_s(\cdot) \) be the operator adjoint to the operator \( \Lambda^\alpha_s(\cdot) \), i.e.,

\[ \Lambda^\alpha_s(\varphi^s) = \mathcal{A}_s^\alpha \varphi^s - \lambda^\alpha_1 (\mathcal{A}_s^\alpha \varphi^s) \]

\[ + \cdots + (-1)^q \lambda^1 \lambda^2 \cdots \lambda^q \mathcal{A}_s^\alpha \varphi^s. \quad (4.28) \]
Integration by parts gives

\[ \int dV \tilde{\alpha}_s(\tilde{\gamma}_a) \varphi^s = \tilde{\varphi} dV \beta \left[ \Gamma^\beta_\alpha(\varphi^s) + \tau^\beta_\alpha \wedge_s(\varphi^s) \right] \]

\[ + \tilde{\varphi} dV \lambda_1 \tilde{\gamma}_a \wedge_s(\varphi^s). \] (4.29)

As the functions \( \varphi^s \) are arbitrary, we select them so as to vanish along with their derivatives up to order \( q-1 \) on the boundary \( \partial V \) of \( V \). Therefore, from the identity (4.29), it follows that

\[ \tilde{\alpha}_s(\tilde{\gamma}_a) = 0, \] (4.30)

and these are the identities for the Euler-Lagrange expressions \( \tilde{\gamma}_a \). This completes the proof.

We apply this theorem to the group of transformations (4.2), consisting of those \( \Delta x^\alpha \) for which \( \delta_0 p^\alpha = 0 \) and \( \delta_0 s^\alpha = 0 \) (i.e., hydromechanical variations of the field variables are vanishing identically). The following theorem characterize this group of transformations.
Theorem (4.31):

Hydromechanical variations of \( p^\alpha \) and \( s^\alpha \) vanish;

\[ \Delta x^\alpha = p^\alpha \phi + \frac{u^\beta e^{\alpha \beta \lambda \mu}}{p^\nu u^\nu} \delta^\lambda \phi^\mu, \tag{4.34} \]

if and only if

\[ \phi^\mu = 0, \tag{4.35} \]

and

\[ \delta^\beta s^\alpha = \delta^\beta (s^\beta \Delta x^\alpha - s^\alpha \Delta x^\beta) = 0, \tag{4.33} \]

where \( \phi \) is an arbitrary scalar function, \( u^\beta \) is an arbitrary four vector and \( \phi^\mu \) is any vector satisfying the conditions

\[ p^\lambda (\delta^\lambda \phi^\mu - \delta^\mu \phi^\lambda) = 0 \tag{4.35} \]

and

\[ 0 = \delta^\beta s e^{\alpha \beta \lambda \mu} \delta^\lambda \phi^\mu, \tag{4.36} \]

where \( S \) being the specific entropy.
Following Drobot and Rybarski (1959, theorem 4), we have the following results.

Lemma (4.37):

\[ \delta_{\alpha} \rho^{\alpha} = 0 \text{ if and only if} \]

\[ \Delta x^{\alpha} = p^{\alpha} \phi + \frac{u_{\beta} e^{\alpha\beta\lambda\mu}}{p^{\gamma} u_{\gamma}} \delta_{\lambda} \phi_{\mu}, \]  

(4.38)

where \( \phi \) is an arbitrary scalar, \( (u_{\beta}) \) an arbitrary 4-vector and \( (\phi_{\mu}) \) is any 4-vector satisfying the conditions (4.35).

Lemma (4.39):

If \( \Delta x^{\alpha} = p^{\alpha} \phi + \frac{u_{\beta} e^{\alpha\beta\lambda\mu}}{p^{\gamma} u_{\gamma}} \delta_{\lambda} \phi_{\mu}, \)

(4.34) with the conditions (4.35). Then lemma (4.39) completes the proof.

\[ p^{\beta} \Delta x^{\alpha} - p^{\alpha} \Delta x^{\beta} = e^{\alpha\beta\lambda\mu} \delta_{\lambda} \phi_{\mu}. \]  

(4.40)

Proof of the theorem (4.31):

Let \( f^{\alpha\beta} = p^{\beta} \Delta x^{\alpha} - p^{\alpha} \Delta x^{\beta}. \)
Since $s^\alpha = p^\alpha S$, \(\delta_0 s^\alpha = 0\) which imply \(\delta_0 s^\alpha = 0\) \hspace{1cm} (4.42)

by the relation (4.43). This completes the proof.

$$\delta_0 s^\alpha = \delta_\beta (s^\beta \Delta x^\alpha - s^\alpha \Delta x^\beta),$$

As the differential operator defining $\Delta x^\alpha$ in (4.34) has the side conditions (4.35) and (4.36) we cannot apply the theorem (4.24) directly to get identities. Therefore use Lagrangian multipliers to incorporate the side conditions and derive identities (4.30) in an indirect way.

 Necessary part:

Let $\delta_0 p^\alpha = 0$ and $\delta_0 s^\alpha = 0$. Then from equation (4.43) we have

$$\delta_\beta S f^{\alpha \beta} = 0 .$$

Since $\delta_0 p^\alpha = 0$, by lemma (4.37), $\Delta x^\alpha$ takes the form (4.34) with the conditions (4.35). Then lemma (4.39) and equation (4.44) ensure the condition (4.36). This completes the proof.

Sufficient part:

Let $\Delta x^\alpha$ take the form given by (4.34), together with conditions (4.35) and (4.36). Then $\delta_0 p^\alpha = 0$ by
lemma (4.37) and \( f^{\alpha\beta} = e^{\alpha\beta\lambda\mu} \delta^{\lambda}_{\mu} \), by lemma (4.39). Thus \( \delta^{\alpha}_{\mu} p^{\alpha} = 0 \) and \( \delta^{\alpha}_{\mu} S f^{\alpha\beta} = 0 \), which imply \( \delta^{\alpha}_{\mu} s^{\alpha} = 0 \) by the relation (4.43). This completes the proof.

As the differential operator defining \( \triangle x^{\alpha} \) in (4.34) has to satisfy the side conditions (4.35) and (4.36) we cannot apply the theorem (4.24) directly to get identities like (4.30). We use Lagrangian multipliers to incorporate the side conditions and derive identities corresponding to (4.30) in an indirect way.

Since \( \delta^{\alpha}_{\mu} p^{\alpha} = 0 \) and \( \delta^{\alpha}_{\mu} s^{\alpha} = 0 \), from equations (4.12) we get

\[
\int dV \, \nabla_{\alpha} \triangle x^{\alpha} = \int dV \, \delta^{\beta}_{\alpha} [(T^{\beta}_{\alpha} - \delta^{\beta}_{\alpha} L) \triangle x^{\alpha}] .
\] (4.45)

We transform the right hand side of the equation (4.45) by Gauss formula into a hypersurface integral over the boundary \( \partial V \) of the region \( V \). On \( \partial V \) we take \( \varphi = \varphi_{\mu} = 0 \) and the vector \( u^{\beta}_{\mu} \) normal to it. Then \( \triangle x^{\alpha} = 0 \) on \( \partial V \) and equation (4.45) becomes

\[
\int dV \, \nabla_{\alpha} \triangle x^{\alpha} = \int dV \, \delta^{\beta}_{\alpha} u^{\beta}_{\mu} e^{\alpha\beta\lambda\mu} \partial^{\lambda} \varphi_{\mu} = 0 ,
\] (4.46)
provided that the side conditions (4.35) and (4.36) hold. (Here we have used the relation (3.30)). These side conditions are taken by means of the Lagrangian multipliers $\Omega^\mu$ and $\zeta^\alpha$.

Thus we have

$$
\int dV \left[ \frac{\nu^\alpha u_\beta e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu} + \Omega^\mu p^\lambda - \Omega^\lambda p^\mu \\
+ \zeta^\alpha \delta_\beta S e^{\alpha\beta\lambda\mu} \right] \delta_\lambda \phi_\mu = 0,
$$

where the functions $\phi_\mu$ are arbitrary functions provided that they vanish on the boundary $\partial V$. Integrating by parts, the last identity leads to the following conclusion: there exist vectors $\Omega^\mu$, $\zeta^\alpha$ such that

$$
\delta_\lambda \left( \frac{u_\beta}{p_\nu u_\nu} e^{\alpha\beta\lambda\mu} \zeta^\alpha \right) = \delta_\lambda \left( \Omega^\lambda p^\mu - \Omega^\mu p^\lambda \right) \\
- \delta_\lambda \left( \zeta^\alpha \delta_\beta S e^{\alpha\beta\lambda\mu} \right). \tag{4.48}
$$

These are the identities corresponding to (4.30).

As $\Omega^\mu$, $\zeta^\mu$ do not depend on the particular choice of $u_\beta$, we can choose $u_\beta$ arbitrarily. As the side conditions (4.35) are linearly dependent, one of the
Lagrangian multipliers can be chosen arbitrarily.

Putting \( u_0 = 1, u_1 = u_2 = u_3 = 0, \Omega^0 = 0 \) in the equation (4.48), we get the following identities:

\( \mu = 0: \)

\[
\text{div} \left( \tilde{\Omega} + \zeta \times \nabla S \right) = 0, \tag{4.49}
\]

where

\[
\tilde{\Omega} = \left( P^1 \Omega^1, P^2 \Omega^2, P^3 \Omega^3 \right),
\]

\[
\zeta = \left( \zeta^1, \zeta^2, \zeta^3 \right),
\]

and 'div' is the divergence operator and three dimensional vector notations are used for convenience.

\( \mu = 1, 2, 3: \)

\[
\nabla \times \left( \frac{\zeta}{\rho^0} \right) = - \frac{d}{dt} \left( \tilde{\Omega} + \zeta \times \nabla S \right)
- \nabla \times \left( \left( \tilde{\Omega} + \zeta \times \nabla S \right) \times \nabla S \right)
+ \nabla \left( \zeta^0 + \nabla \cdot \zeta \right) \times \nabla S, \tag{4.50}
\]

using the relation \( \frac{DS}{Dt} = 0. \)
Since \( \ddot{\gamma} = 0 \) during the motion, equations (4.50) reduce to

\[
\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{\nabla} \times \vec{v}) + \nabla \times (\vec{\nabla} \times \vec{v}) + \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{\iota}) = \nabla \times (\vec{\iota} \times \nabla \vec{s})
\]

The conservation of the vector lines and the strength of the vector tubes of the vector field \( \vec{\iota} = \nabla \times \vec{v} \).

\[
= \nabla \left( \vec{\iota} \cdot \vec{\nabla} \times \nabla \vec{s} \right)
\]

Note that when \( \iota \) takes the usual form the Euler-Lagrange equations \( \iota = 0 \) become the vector equation (4.32).

We shall show that these identities lead to the conditions equivalent to generalized Helmholtz theorems, when the Lagrangian takes the usual form given by (3.10).

Since the usual Lagrangian does not depend on \( s', s^2 \) and \( s^3 \) the last three conditions in (4.36) can be deleted. Then the equations (4.51) reduce to

\[
\frac{\partial}{\partial t} (\vec{\iota} \cdot \vec{\nabla}) + \nabla \times (\vec{\iota} \times \vec{\nabla} \vec{v}) = \nabla \vec{\iota} \cdot \nabla \vec{s}
\]

If we define the quantity \( \iota \), as the time integral of \( \vec{\iota} \cdot \vec{\nabla} \vec{s} \),

\[
\iota = \int_0^t \vec{\iota}(x,t)dt,
\]

using the condition \( \frac{DS}{Dt} = 0 \), we can write the equation (4.52) in the form

\[
\frac{\partial}{\partial t} \iota + \nabla \times (\iota \times \vec{\nabla} \vec{v}) = \nabla \iota \cdot \nabla \vec{s}
\]
\frac{\partial}{\partial t} (\bar{\omega} - \nabla \times \nabla S) + \nabla \times (\omega - \nabla \times \nabla S) \times \bar{v} = 0, \quad (4.54)

which is the 'Helmholtz-Zorawski criterion' (Truesdell (1954), Truesdell and Toupin (1960)) for the conservation of the vector lines and the strength of the vector tubes of the vector field \( \bar{\omega} - \nabla \times \nabla S \).

Note that when \( L \) takes the usual form the Euler-Lagrange equations \( \bar{\omega} = 0 \) become the vector equation (3.32).

Using the thermodynamic relation

\[ \frac{\nabla P}{\rho} = \nabla I - \nabla T \times S, \]

where \( I \) is the specific enthalpy and taking curl of the vector equation (3.32) we get

\[ \frac{\partial}{\partial t} (\bar{\omega}) + \nabla \times (\bar{\omega} \times \bar{v}) = \nabla T \times \nabla S. \quad (4.55) \]

Comparing the equations (4.52) and (4.55) we can easily see that

\[ \bar{\omega} = \omega \quad \text{and} \quad \omega^0 = T \]

is a solution of the equation (4.52).
In this case, $\mathcal{U} = \int_0^t T \, dt = \eta$, the thermasy defined by (2.7) and the equation (4.52) becomes the 'Helmholtz-Zorawski criterion' for the generalized vorticity vector $\vec{\omega} - \nabla \eta \times \nabla S$. Thus we have the following results:

(i) The vector lines of $\vec{\omega} - \nabla \eta \times \nabla S$ are material lines.

(ii) The strengths of the generalized vortex tubes are conserved during the motion.

These are precisely generalised Helmholtz theorems. Thus the identity (4.52) obtained as the Noether identity of the transformation group defined by (4.34, 4.35 and 4.36) corresponds to generalized Helmholtz theorems.

1. If $C_1$ and $C_2$ are any two circuits encircling a vortex tube in the same direction then the circulation around $C_1$ is equal to the circulation around $C_2$.

$$\oint_{C_1} \vec{\omega} \cdot d\vec{l} = \oint_{C_2} \vec{\omega} \cdot d\vec{l}$$

Some of the results presented in this chapter will appear in the Journal of Mathematical and Physical Sciences. (Thomas Joseph and George Mathew).