4.1 **Introduction:**

The first solution of the Einstein field equations was obtained by Schwarzschild for the interior of a star in hydrostatic equilibrium, ever since the relativistic stellar models have been considered by various workers in the field. Vaidya and Tikekar (1982) have presented a solution for a superdense star by proposing an ansatz for the geometry of the 3-surface embedded in a 4-Euclidean space. The ansatz describes a spheroidal geometry for the 3-surface, by two parameters \( \lambda \) and \( R \), for \( \lambda = 0 \) gives spherical while \( \lambda = -1 \) corresponds to flat space. Their solution corresponds to \( \lambda = 2 \), which was also presented independently by Durgapal and Bannerji (1983), without reference to the spheroidal geometry of 3-space. Tikekar (1990) presented the solution for \( \lambda = 7 \), and made a detailed investigation of the model. Maharaj and Leach (1996) have presented more solutions for a set of discrete values of \( \lambda \). The
solutions are either polynomials or products of polynomials and algebraic functions. As the parameter $\lambda$ represents a 3-space geometry it is important to look for an analytic solution valid for all possible values of $\lambda$.

With a view for obtaining the eigenfrequencies and eigenfunctions of radial modes of a massive star, one carries out the analysis as described by Shapiro and Teukolsky (1982), Cox (1980), on the basis of the Einstein field equations for a given metric.

### 4.2 The Metric and Solutions:

Let us consider the spherically symmetric metric of the form

\[ ds^2 = -e^{2\nu(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]

For the perfect fluid distribution, one may obtain in a straightforward way in view of the above mentioned ansatz

\[ e^{2\mu(r)} = \frac{1 + \lambda \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}}, \]

where pressure and density read
\begin{align}
(4.3) \quad p &= \frac{(2\nu'/r + 1/r^2)(1-r^2/R^2)}{1 + \lambda r^2/R^2} - \frac{1}{r^2} \\
(4.4) \quad \rho &= \frac{(1+\lambda)(3+\lambda r^2/R^2)}{R^2 \left(1 + \frac{\lambda r^2}{R^2}\right)^2}.
\end{align}

The matter density attains its maximum value at the centre $r = 0$ with expression

\begin{equation}
(4.5) \quad \rho_0 = \frac{3(1+\lambda)}{R^2}.
\end{equation}

It decreases radially outward from this positive maximum value and on the boundary $r = a$ of the configuration attains the minimum value $\rho_a$

\begin{equation}
(4.6) \quad \rho_a = \frac{(1+\lambda)(3+\lambda a^2/R^2)}{R^2 \left(1 + \frac{\lambda a^2}{R^2}\right)^2}.
\end{equation}

The ration $\alpha = \rho_a/\rho_0$ which describes density variation parameter has the form

\begin{equation}
(4.7) \quad \alpha = \frac{1}{3} \frac{(3+\lambda a^2/R^2)}{\left(1 + \frac{\lambda a^2}{R^2}\right)^2}.
\end{equation}
\[
= \left(1 + \lambda \frac{a^2}{3R^2}\right)\left(1 + \frac{a^2}{R^2}\right)^{-2}.
\]

The spacetime in the exterior region \( r > a \) is described by Schwarzschild metric

\[
(4.8) \quad ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2
\]

\[+ r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),
\]

where \( m \) denotes the mass of star in geometric units. The interior spacetime is jointed smoothly with the exterior spacetime it the boundary, \( r = a \) of the stellar configuration by stipulating the continuity of the fluid pressure \( p \) across the boundary. The continuity of the metric coefficient of \( dr^2 \) leads to mass-radius relation

\[
(4.9) \quad \frac{m}{a} = \frac{1}{2} \frac{a^2}{R^2} \left(\lambda + 1\right) \left(1 + \lambda \frac{a^2}{R^2}\right)^{-1}.
\]

Now the pressure isotropy condition

\[
(4.10) \quad \left(1 + \lambda - \lambda x^2\right)\psi''(x) + \lambda x \psi'(x) + \lambda (\lambda + 1) \psi = 0
\]

where
(4.11) \[ \psi = e^\nu \]

(4.12) \[ x^2 = 1 - r^2/R^2, \]

the prime denotes differentiation with respect to the argument. The eq. (4.10) is the only equation to be solved for the metric function \( \nu \). Tikekar (1990) presented a solution of eq. (4.10) for \( \lambda = 7 \), and Maharaj and Leach for a set of values of \( \lambda \)

(4.13) \[ \lambda = n^2 - 2 \]

where \( n \) be an integer.

In order to obtain the general solution, let us put

(4.14) \[ Z = \left( \frac{\lambda}{\lambda + 1} \right)^{\frac{1}{2}} x. \]

In view of the above equations, the eq. (4.10) assumes the form

(4.15) \[ (1 - Z^2)\psi''(z) + Z\psi'(z) + (\lambda + 1)\psi(z) = 0 \]

One obtains the general solution

(4.16) \[ \psi = \psi_1 + \psi_2 \]
\[ n = (\lambda + 2)^{\frac{1}{2}} \]

and \( T_i^k \) be a Gegenbauer function as presented by Morse and Feshbach (1953).

Now, it is to be noted that

\[ \psi'(z) = A_1 T_{n-1}^{-1/2}(z) + A_2 (n + 1) (n + 3) \]

\[ (1 - z^2)^{\frac{1}{2}} T_{n-1}^{\frac{1}{2}}(z). \]

If \( i \) is zero or positive integer \( n \), \( T_i^k \) be a finite polynomial in \( z \) as

\[ T_i^k(z) = D_{kn} \left(1 - z^2\right)^k \frac{d^n}{dz^n} \left(1 - z^2\right)^{n+k}, \]

where \( D_{kn} \) is given in terms of \( k \) and \( n \). For \( n = (\lambda + 2)^{\frac{1}{2}} \), one may obtain several integral values of \( (\lambda + 2)^{\frac{1}{2}} \), for example

\[ \lambda = z \]
\[ n = z \]

for \( \lambda = 7 \)

\[ n = 3 \]

we obtain

\[(4.21) \quad \psi = A_1^{3/4} (z) \]

\[= c_1 (1 - z^2)^{-3/2} \frac{d^4}{dz^4} (1 - z^2)^{5/2},\]

\[(4.22) \quad \psi_2 = A_2 (1 - z^2)^{3/2} T_{3/2}^{3/2} (z) \]

\[= C_2 \frac{d}{dz} (1 - z^2)^{5/2},\]

which is recognised as Tikekar (1990) solution. It is interesting to note that

\[(4.23) \quad nT_{n^{-\frac{1}{2}}} (z),\]

and

\[(4.24) \quad (1 - z^2)^{-\frac{1}{2}} T_{n-1}^{\frac{1}{2}} (z),\]

are Tschebyscheff polynomials. Hence for real \( z \) with
(4.25) \[ 0 < z \left( \lambda/\lambda + 1 \right)^{\frac{1}{2}}, \]

we obtain

(4.26) \[ nT_{\frac{1}{2}}^{-\frac{1}{2}}(z) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \cos \left( n \cos^{-1} z \right), \]

(4.27) \[ (1 - z^2)^{\frac{1}{2}} T_{n-1}^{\frac{1}{2}}(z) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \sin \left( n \cos^{-1} z \right). \]

In view of eqs. (4.26) - (4.27), the eq. (4.19) gives the general solution \( \psi \)

(4.28) \[ \psi = A \left[ \frac{\cos \left( (n+1)\xi + \gamma \right)}{n+1} - \frac{\cos \left( (n-1)\xi + \gamma \right)}{n-1} \right] \]

where

(4.29) \[ \xi = \cos^{-1} z, \]

and \( A, \gamma \) are constants.

### 4.3 Physical features of Spheroidal Star:

Now, let us discuss some physical features of a spheroidal star in some general way. In this model, pressure and density read
\[
\begin{align*}
(4.30) \quad p &= -\frac{1}{R^2 (1-z^2)} \left(1 + \frac{2z\psi'}{(\lambda + 1)\psi}\right), \\
(4.31) \quad \rho &= \frac{1}{R^2 (1-z)^2} \left(1 + \frac{2}{(\lambda + 1)(1-z^2)}\right),
\end{align*}
\]

where \( p \geq 0 \) requires

\[
(4.32) \quad z\psi'/\psi \leq -\frac{1}{2}(\lambda + 1)
\]

and \( \rho > 0 \) for

\[
(4.33) \quad \lambda > -1.
\]

The radius of the star 'a' is obtained by the condition

\[
(4.35) \quad p = 0
\]

at \( r = a \), and we get

\[
(4.36) \quad z_a \frac{\psi'(z_a)}{\psi(z_a)} = -\frac{1}{2}(\lambda + 1)
\]

where

\[
(4.37) \quad Z_a = \left(\frac{\lambda}{1 + \lambda}\right)^{\frac{1}{2}} \left(1 - a^2/R^2\right)^{\frac{1}{2}}.
\]
It will determine the constant $\gamma$ for a given radius, $a$, of the star. In order to obtain $p$ as finite $\psi$ should not have a zero in the range

$$Z_a \leq z \leq Z_0,$$

$$Z_0 = \left(\frac{\lambda}{1 + \lambda}\right)^{\frac{1}{2}}.$$

It is also required that both $dp/dz$ and $d\psi/dz$ should increase monotonically as $z$ increases from $z_a$ to $z_0$. Now let us consider the case

$$dp/d \rho < 1,$$

and it gives

$$\left(\frac{1}{1 - z^2}\right)\left(\frac{1}{2z} - d\right) \leq \frac{\psi'}{\psi} \leq \left(\frac{1}{1 - z^2}\right)\left(\frac{1}{2z} + d\right),$$

where

$$d = \left[4 + \frac{1}{4z^2} + (1 + \lambda)(1 - z^2)\right]^{\frac{1}{2}}.$$
we get

\[ \frac{\psi'}{\psi} \leq -\frac{1 + \lambda}{2z}. \]  

(4.43)

In view of the above constraints, we obtain the bounds

\[ \frac{1}{(1-z^2)} \left( \frac{1}{2z} - d \right) \leq \frac{\psi'}{\psi} \leq -\frac{1 + \lambda}{2z}, \]

(4.44)

and for the case of realistic model, one has

\[ \lambda > \frac{3}{17}, \]

(4.45)

which excludes the flat space, \( \lambda = -1 \), and Schwarzschild interior solution, \( \lambda = 0 \). It is because the condition

\[ dp/d\rho \ll 1. \]

(4.46)

may not be satisfied in these cases. The constraint \( \lambda > 3/17 \) gives an upper bound on possible values of

\[ \bar{b} = b/R = a/R \]

(4.47)
\[(4.48) \quad (1 - b^2) \geq \frac{\lambda^2 + 5\lambda + 12 - \left(17\lambda^2 + 82\lambda + 129\right)^{\frac{1}{2}}}{\lambda(\lambda + 5)}.\]

The mass inside the radius \(r\) reads

\[(4.49) \quad M(r) = \frac{1}{2} \int_0^r r'^2 \rho(r') \, dr'.\]

Integrating for \(r = a\) gives

\[(4.50) \quad \frac{M(a)}{a} = \frac{(1 + \lambda) a^2 / R^2}{2 \left(1 + \lambda \frac{a^2}{R^2}\right)}.\]

### 4.4 Stability of a state against radial oscillations:

In view of obtaining the eigenfrequencies and eigenfunctions of radial modes of a massive star, one carries out the analysis as shown by Shapiro and Teuklsky (1983), Cox (1980), on the basis of Einstein field equations for a metric

\[(4.51) \quad ds^2 = e^{2r(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right).\]
The adiabatic motion of the star in its n-th normal mode, where \( n = 0 \) is the fundamental mode, is described in terms of the amplitude \( u_n(r) \) with small perturbation in \( r \) as

\[
\delta r(r, t) = e^{i\nu(r)}u_n(r)e^{i\omega_n' n'/r^2},
\]

where \( \omega_n \) be the star's oscillation frequency. The latter satisfies the Sturm-Liouville's form

\[
\frac{d}{dr} \left( p(r) \frac{d u_n(r)}{dr} \right) + \left[ Q(r) + w^2 W(r) \right] u_n(r) = 0,
\]

\[
p(r) \frac{d^2}{dr^2} \left( u_n(r) \right) + \frac{dp}{dr} \frac{du_n}{dr}
\]

\[
+ \left[ Q(r) + w^2 W(r) \right] u_n(r) = 0,
\]

where functions \( p(r), Q(r) \) and \( W(r) \) are given in terms of the equilibrium configuration of the star as

\[
p(r) = e^{(\lambda+3\nu)}r^2 \Gamma p/r^2,
\]

\[
Q(r) = -4e^{(\lambda+3\nu)} \frac{1}{r^3} \frac{dp}{dr} - \frac{8\pi}{r^2} e^{(\lambda+\nu)} p(\varepsilon + p)
\]

\[
+ e^{(\lambda+3\nu)} \frac{1}{r^2 (\varepsilon + p)} \left( \frac{dp}{dr} \right)^2,
\]
(4.57) \[ W(r) = e^{3\lambda + \nu} \left( \epsilon + p \right)/r^2. \]

where \( \epsilon \) be the energy density or total mass energy and \( p \) as the pressure of the stellar equilibrium configuration as measured by a local observer. The symbol \( \Gamma \) denotes the varying adiabatic index at constant entropy as

(4.58) \[ \Gamma = \frac{(\epsilon + p)}{p} \frac{dp}{dr}. \]

The pressure gradient \( (dp/dr) \) is obtained from the Tolman-Oppenheimer-Volkoff (1934, 1939)

(4.59) \[ \frac{dm}{dr} = 4\pi r^2 \rho, \]

(4.60) \[ \frac{dp}{dr} = -\frac{\rho m}{r^2} \left( 1 + \frac{p}{r} \right) \left( 1 + \frac{4\pi pr^3}{m} \right) \left( 1 - \frac{2m}{r} \right)^{-1}, \]

(4.61) \[ \rho = \rho(r). \]

The boundary conditions are satisfied by \( u_n(r) \) as given by Kettner et al (1995)

(4.62) \[ U_n(r) \sim r^3, \]
at star's origin \( r = 0 \), and

\[
\frac{du_n(r)}{dr} = 0, \tag{4.63}
\]

at star's surface \( r = R \).

In deed, the solution of eq. (4.56), together with boundary conditions (4.62) - (4.63), gives the frequency spectrum \( w_n^2, n = 0, 1, 2, \ldots \) of normal radial modes for a given stellar model. As a result, the eigenfrequencies \( w_n^2 \) form an infinite discrete sequence i.e.

\[
w_n^2 < w_{n+1}^2. \tag{4.64}
\]

Let us obtain the space invariant. In doing so, let us first eliminate the \( (du_n(r)/dr) \) in eq. (4.54) in view of transformation,

\[
u_n(r) = Av_n(r)/(p(r))^{\frac{1}{2}}, \tag{4.65}
\]

where \( A \) as the arbitrary constant. Now, we get

\[
v''_n(r) + q_n^2(r)v_n(r) = 0, \tag{4.66}
\]

where
(4.67) \[ q_n^2(r) = \frac{P'' + 2P(2Q - P'' + 2w_n^2W)}{4p^2}. \]

Let us have the following ansatz for the solution of eq. (4.66)

(4.68) \[ v_n(r) = Nw_n(r)\sin \varphi(r), \]

where \( \varphi(r) \) as the phase function. In view of eq. (4.68) and by equating the coefficients of sine and cosine functions to zero, one obtains

(4.69) \[ w''_n + \{q_n^2(r) - \varphi'^2\}w_n = 0, \]

(4.70) \[ \varphi''w_n + 2\varphi'w_n = 0. \]

Equation (4.70) gives the phase function

(4.71) \[ \varphi(r) = k\int w_n^{-2}(r)dr + D, \]

where K and D are arbitrary constants. Again eq. (4.69) gives a nonlinear equation

(4.72) \[ w''_n(r) + q_n^2(r)w_n(r) = k^2/w_n^3(r), \]

which is known as Milne's equation. From eq. (4.66), we get
In view of eqs. (4.73) and (4.72), one obtains the space invariant
\begin{equation}
(4.74) \quad K = k^2 \left( \frac{v_n}{w_n} \right)^2 + \left( w_n v'_n - w'_n v_n \right),
\end{equation}
satisfying
\begin{equation}
(4.75) \quad \frac{dk}{dr} = 0,
\end{equation}
and valid for all \( r \) and for all \( n \).

**4.5 Concluding Remarks:**

We have presented an exact general solution of the Einstein equations for a spheroidal star in hydrostatic equilibrium satisfying all physical constraints. Since, the solution is valid for all \( \lambda > 3/17 \), we have described the model in very general way. In particular the special role played by \( \lambda \), and hence, the geometry of the 3-space, in describing the physical features of the star. It is well known that 
\begin{equation}
e^{2\mu(r)} = 1 - \left( 2M \left( \frac{r}{r} \right) \right). \text{ Hence, } 2\mu(r) = 1 + \lambda \frac{r^2}{R^2} - 1 - \frac{r^2}{R^2}
\end{equation}
gives mass function and relates with parameter \( \lambda \) with the physical
features of star. Although the assumption of hydrostatic equilibrium normally applies to a superdense star, the model allows for arbitrarily small $M/a$. In particular, a wide range of values of $\lambda$ may sustain neutron star like objects. The parameter $\lambda$ be the measure of the spheroidal character and it determines the physical properties of the star.

It is interesting to note that an invariant of space, like presented here, while appears in the work of Ermakov (1980), be the characteristic feature of a second-order ordinary differential equation in general and of Sturm-Liouville form in particular with respect to the independent variable, the same has been investigated a variety of contexts such as Kaushal (1998), Lewis (1968), (1969), Ray and Reid (1979), Roy (1988), Lee (1984). For the last four decades, an invariant has been the subject of study in context of time-dependent classical harmonic oscillator, an invariant similar to here, also in solving the Dirac equation in one dimension. The analogies in the quantum mechanics have been carried by Korsch and Laurent (1981), Lee (1984), and common basis in the Riccati equation in saught by Kaushal (1996) Kaushal and Parashar (1996) for both
Schrodinge quantum mechanics and supersymmetric quantum mechanics.

We have presented, regarding the equations governing stability of star against the radial oscillations, mathematically, a space invariant \( k \) such that \( \frac{dk}{dr} = 0 \). It is also shown that in addition to the normal boundary conditions used at the star's origin \( r = 0 \) and its surface \( r = R \), the invariance \( K \) for all \( r \) will work as a sufficiency condition in the obtaining results. Hence, this invariant is expected to play the role of a geometrical constraint.

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