3.1 Introduction:

There have been extensive investigations on gravitational collapse in order to understand the nature of the singularities, in past thirty years. The study of gravitational collapse of spherically symmetric spacetimes led to many examples of naked singularities as presented by Eardley and Smar (1979), Christodoulou (1984), Newman (1986), Patil and Thool (2005), Patil (2003), Papapetrou (1985), Waugh and Lake (1986), Dwivedi and Joshi (1989), Patil and Thool (2006), Ori and Piran (1987). Most of these works concentrate on the aim to obtain light rays emanated from the central singularity and escaping the Schwarzschildian trapped surface at least locally. In order to investigate the nature of singularity, global behaviour of radial null geodesics must be studied in full generality. The cosmic censorship hypothesis has been one of the most active
areas of research in general relativity for more than thirty years. Penrose (1969) proposed cosmic censorship hypothesis stating that singularities formed in gravitational collapse of physically reasonable matter may not be observed. There are two versions of this cosmic censorship hypothesis. The weak version states that all singularities formed in gravitational collapse are hidden behind the event horizon of the gravity and are invisible to the distant observer from infinity. On the other hand, the strong version asserts that no singularities are visible.

Many workers have attempted to reformulate this hypothesis, but neither proof nor mathematical formulation for this hypothesis is available so far as shown by Shapiro and Teukolsky (1991), Chiba, Nakamura, Nakao and Sasaki (1994). Much work has been done on cosmic censorship hypothesis in spherically symmetric spacetimes. Almost all researches on this issue till date, are limited to the collapse in spherically symmetric spacetimes. Of course, spherical symmetry is a characteristic feature of many solutions of the Einstein field equations of general relativity, but to prove or disprove the cosmic censorship hypothesis, it is essential to investigate the gravitational collapse in non-spherical case as well.
In this chapter, we have given a brief guide to the issue of naked singularities in non-spherically symmetric spacetimes.

One of the hot topics in astrophysics of late has been the possibility of the discovery of strange stars i.e. stars composed of u, d and s quark matter. A quark star, or strange star is a hypothetical type of star composed of quark matter or strange matter. It is theorized that when the neutron degenerate matter which makes up a neutron star is put under sufficient pressure due to the star's gravity, the individual neutron breaks down into their constituent quarks-up quarks and down quarks. Some of these quarks may then become strange quarks and form strange matter. The star which then becomes is known as a quark star or strange star. The broader meaning of strange matter is just quark matter that contains three flavors quarks-up, down and strange. In this definition, there is a critical pressure and an associated critical density, and when nuclear matter is compressed beyond this density, the protons and neutrons dissociate into quarks, giving strange matter. Many research works on the strange quark matter have appeared so far, explaining the formation and properties of strange stars as described by Farhi and Olinto (1986), Chakraborty (1991), Cheng and Dai (1996), Chen and
Xu (2006), Alcock and Farhi (1985). The strange quark matter is given by the equation of state

\[ p = \frac{\rho - 4B}{3}, \]

where \( B \) be the difference between the energy density of the perturbative and non-perturbative quantum chromodynamic vacuum, known as the bag constant, and \( p \) are energy density and thermodynamic pressure of the quark matter respectively. In view of Harko and Cheng (2000), the typical value of the bag constant is of the order of \( 10^{15} \text{ g/cm}^3 \) and \( \rho \approx 5 \times 10^{15} \text{ g/cm}^3 \). In view of eq. (3.1), it is obvious that quark matter will always satisfy the energy conditions \( \rho \geq p \geq 0 \). Harko and Cheng (2000) have presented the gravitational collapse of strange matter and analysed the condition for formation of a naked singularity in the spherically symmetric Vaidya like spacetime. It has been studied that depending on the initial distribution of density and velocity and on the nature of collapsing matter, either a black hole or a naked singularity is formed.

In this chapter, we have reported that the results of Harko and Cheng (2000) get modified for non-spherically symmetric i.e. for
plane symmetric and cylindrically symmetric spacetimes. In section 3.2 we have obtained the general solution for strange quark matter in view of equation of stat (3.1) in non-spherically symmetric spacetimes. In section (3.3) we have analysed the equations of the outgoing radial null geodesics in order to obtain the nature of singularity, and in section 3.4 we have presented the concluding remarks.

3.2 Gravitational Collapse of Strange quark matter in Non-spherical Cases:

In view of Harko and Cheng (2000), Lemos (1999), Lemos (1998), one may obtain the metric describing the radial collapse of charged strange quark matter in toroidal, cylindrical and planar spacetimes as

\[ ds^2 = -\left(\alpha^2 r^2 - \frac{qm(u, r)}{r}\right)du^2 + 2dudr + r^2(d\theta^2 + d\varphi^2), \]

where \( u \) is an advanced Eddington time coordinate and

\[ \alpha = \sqrt{-\Lambda/3}, \]
r as the radial coordinate with

\[ 0 < r < \infty, \]  

and \( m(u,r) \) be the mass function showing gravitational mass inside the sphere of radius \( r \). Lemos (1999) presented that coordinates \( \theta, \phi \) give the two-dimensional zero curvature space described by the two-dimensional commutative Lie group \( G_2 \) isometries. In view of Lemos (1999), Lemos (1998), one may describe the topology of two-dimensional space. Topology of toroidal model has \( S \times S \), cylindrical model has \( R \times S \) and planar symmetrical model has \( R \times R \) and range of \( \theta \) and \( \phi \) reads,

(i) Toroidal model

\[ 0 \leq \theta \leq 2\pi \]
\[ 0 \leq \phi < 2\pi \]

(ii) Cylindrical model

\[ -\infty < \theta < \infty \]
\[ 0 < \phi < 2\pi \]

(iii) Planar model

\[ -\infty < \theta < \infty \]
\[ -\infty < \phi < \infty. \]
The parameter q has different values, depending upon the topology of the two-dimensional space. Hence, for

(i) Toroidal model \( m(u,r) \) be the mass and

\[
q = \frac{2}{\pi}
\]

(ii) Cylindrical model \( m(u,r) \) as the mass per unit length, and

\[
q = \frac{4}{\alpha}
\]

(iii) Planar model \( m(u,r) \) be the mass per unit area, and

\[
q = \frac{2}{\alpha^2}
\]

Lemos (1998) has presented that the values of q are taken from Arnowitt-Deser-Misner masses of the corresponding black holes. In view of Wang and Wu (1999), the energy momentum tensor assumes the form

\[
T_{ik} = T_{ik}^{(n)} + T_{ik}^{(m)} + E_{ik},
\]

where

\[
T_{ik}^{(n)} = \sigma(u,r) \ell_i \ell_k,
\]
being the component of the matter field that moves along the null hypersurface

\( u = \text{constant}. \)  

\( T_{ik}^{(m)} = (\rho + p)(\ell_i n_k + \ell_k n_i) + pg_{ik}, \)

representing the energy momentum tensor of the strange quark matter

\( E_{ik} = \frac{1}{4} \left( F_{ij} F_{jk}^\prime - \frac{1}{4} g_{ik} F_{mn} F_{mn} \right) \)

as the electromagnetic contribution.

Let us take the null vector \( \ell_i, n_k \) such that

\( \ell_i = \delta^0_i, \)

\( n_i = \frac{1}{2} \left( \alpha^2 r^2 - \frac{q m (u, r)}{r} \right) \delta^0_i - \delta^1_i, \)

\( \ell_k \ell^k = n_k n^k = 0, \)

\( \ell_k n^k = -1. \)

The Maxwell equations read as
\[ (3.20) \quad \frac{\partial^F_{ik}}{\partial x^j} + \frac{\partial^F_{ji}}{\partial x^i} + \frac{\partial^F_{ij}}{\partial x^j} = 0, \]

\[ (3.21) \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} F^{ik} \right) = -4\pi J^i. \]

Let us select the vector potential as

\[ (3.22) \quad A_i = \frac{e(u)}{r} \delta^{(i)}_{(i)} \]

without loss of generality, and where \( e(u) \) be an arbitrary integrable function. In view of eqs. \((3.20)\) - \((3.21)\), the nonvanishing components of \( F_{ik} \) are

\[ (3.23) \quad F_{ru} = -F_{ur} = \frac{e(u)}{r^2}, \]

and therefore

\[ (3.24) \quad E^k_i = \frac{e^2(u)}{r^4} \text{diag} \left( -1,1,-1,1 \right). \]

The Einstein field equations for the given energy momentum tensor, assume the form

\[ (3.25) \quad \sigma = \frac{q m}{8\pi r^2}, \]
Wang and Wu (1999), Harko and Cheng (2000) have presented the energy conditions for the above type of fluids as:

(i) The weak and strong energy conditions:

\[(3.28) \quad \sigma > 0, \quad \rho \geq 0, \quad p \geq 0.\]

(ii) The dominant energy conditions:

\[\sigma > 0, \quad \rho \geq p \geq 0.\]

In view of the bag equation of state and also from eqs. (3.26) - (3.27), one obtains

\[(3.30) \quad 3qm''r^2 + 2qm'r = 64\pi Br^2 + 24\alpha^2 r^3 - \frac{q^2 e^2(u)}{r}.\]

The general solution of eq. (3.30) may be obtained as

\[(3.31) \quad qm(u, r) = qg(u) + qh(u)r^{1/3}\]

\[+ Ar^3 - \frac{q^2 e^2(u)}{4r}.\]
where,

(3.32) \[ A = (8\pi B/3) + \alpha^2, \]

and \( g(u) \) and \( h(u) \) are two arbitrary functions.

Hence, one obtains the solution to the Einstein equations for collapsing strange quark matter in toroidal (cylindrical or planar) spacetime

(3.33) \[ ds^2 = -\left(\alpha^2 r^2 - \frac{gq(u)h(u)}{r^{2/3}}Ar^2 + \frac{q^2e^2(u)}{4r^2}\right) \]

\[ du^2 + 2dudr + r^2 (d\theta^2 + d\phi^2). \]

In view of the mass function (3.31), we get

(3.34) \[ \sigma = \frac{1}{8\pi r^2} \left( qg(u) + gh(u)r^{1/3} - \frac{q^2e(u)e(u)}{2r} \right), \]

(3.35) \[ \rho = \frac{1}{8\pi r^2} \left( \frac{1}{3} qh(u) r^{-2/3} + 8\pi Br^2 - \frac{3}{4r^2} q^2 e^2(u) \right), \]

(3.36) \[ p = \frac{1}{16\pi r} \left( \frac{2}{9} qh(u) r^{-5/3} - 16\pi Br - \frac{q^2 e^2(u)}{2r^3} \right). \]
It is to be noted here that with suitable choice of \( g(u) \) and \( h(u) \), weak and strong energy conditions are satisfied. Because of bag equation of state, one always has

\[
\rho \geq p \geq 0. \tag{3.37}
\]

### 3.3 The Structure of the Collapse:

In order to study the structure of collapse i.e. nature of the singularity, one has to consider the radial null geodesics defined by

\[
ds^2 = 0, \tag{3.38}
\]

along with

\[
\dot{\theta} = \dot{\phi} = 0. \tag{3.39}
\]

In view of eqs. (3.38) - (3.39), the eq. (3.33) assumes the form

\[
\frac{dr}{du} = \frac{1}{2} \left( \alpha^2 r^2 - \frac{g(u)}{r} \frac{q(u) h(u)}{r^{2/3}} - Ar^2 + \frac{q^2 e^2(u)}{4r^2} \right). \tag{3.40}
\]

The above equation, in general, does not give analytical solution. However, if
then this equation becomes homogeneous and may be solved in terms of elementary functions. Hence, let us select

(3.44) \( qg(u) = au \),
\( qh(u) = bu^{2/3} \),
\( q^2 e^2 (u) = d^2 u^2 \),
for some
\( a > 0, \quad b > 0, \quad d \geq 0 \).

In view of the above, the eq. (3.40), becomes

(3.48) \[
\frac{du}{dr} = \frac{2}{\left( \alpha^2 r^2 - \frac{au}{r} - \frac{bu^{2/3}}{r^{2/3}} - Ar^2 + \frac{d^2 u^2}{4r^2} \right)}.
\]

It is obvious that the eq. (3.48), has singularity at

(3.49) \( u = 0, \quad r = 0 \).
In order to investigate the nature of singularity one has to analyse the outgoing radial null geodesics terminating at the singularity in the past. In doing we follow the technique presented by Joshi (1993). Let us consider

\[(3.50) \quad X_0 = \lim X = \lim u / r \]

\[u \to 0 \quad u \to 0 \]

\[r \to 0 \quad r \to 0 \]

Hence, we get

\[(3.51) \quad X_0 = \lim \frac{du}{dr} = \frac{8}{-4aX_0^2 - 4bX^{2/3} + d^2X_0^2} \]

\[u \to 0 \]

\[r \to 0 \]

or,

\[(3.52) \quad d^2X_0^3 - 4aX_0^2 - 4bX_0^{5/2} - 8 = 0. \]

The equation (3.52) describes the nature of singularity. If the eq. (3.52) gives a real and positive root, then there exist future directed radial null geodesics originating from \(r = 0, v = 0\), and in
this situation the singularity becomes naked. If eq. (3.52) has no real and positive roots, then the singularity be covered and the collapse proceeds to form a black hole.

Let us put

\[(3.53) \quad X_0 = y^3\]

Hence, we obtain

\[(3.54) \quad d^2 y^9 - 4ay^6 - 4aby^5 - 8 = 0.\]

In order to obtain the nature of the root of eq. (3.54), the following rule from the theory of equations may be quite useful:

"Every equation of odd degree has at least one real root whose sign is opposite to that of its last term, the coefficient of the first term being positive."

The equation (3.54) has at least one positive root. In particular if

\[(3.55) \quad d = 0.1\]
\[(3.56) \quad a = 0.1\]
\[(3.57) \quad b = 0.01\]
then

\begin{equation}
y = 1.2916. \tag{3.58}
\end{equation}

Hence

\begin{equation}
X_0 = y^3 \tag{3.59}
\end{equation}

\begin{equation}
= 2.1545. \nonumber
\end{equation}

This positive root ensures that the singularity be naked.

Now, let us consider the gravitational collapse of uncharged

\begin{equation}
q^2 e^2 (u) = 0, \tag{3.60}
\end{equation}

strange quark matter. Hence, for uncharged case obtain

\begin{equation}
4aX_0^2 + 4bX_0^{5/3} + 8 = 0. \tag{3.61}
\end{equation}

Again for \(X_0 = y^3\), we get

\begin{equation}
4ay^6 + 4by^3 + 8 = 0. \tag{3.62}
\end{equation}

It is obvious from eq. (3.62) that all the coefficients are positive, hence, there is no positive root of the eq. (3.62) in view of the theory of equations. Therefore outgoing radial null geodesics
having definite tangent at the singularity in the past are not present, showing that in this singularity is not naked. Thus, the gravitational collapse of uncharge strange quark matter leads to form black holes.

3.4 Concluding Remarks:

We have presented the non-spherical gravitational collapse of the strange quark null fluid and obtained the nature of the singularities in non-spherically symmetric spacetimes. We have studied the visibility of naked singularities, investing the behaviour of radial null geodesics in the non-spherical gravitational collapse of strange quark matter. To analyse the nature of the root of any even or odd degree, the following rule from the theory of equations as:

"Every equation of odd degree has at least one real root whose sign is opposite to that of its last term, the coefficient of the first term being positive".

It is shown that charge plays a crucial role in the non-spherical gravitational collapse of strange quark matter. It is found that the non-spherical gravitational collapse of charged strange quark matter leads to a naked singularity where as the gravitational collapse of neutral strange quark matter proceeds to form a black hole. Hence,
non-spherical gravitational collapse of charged strange quark matter contradicts the cosmic-sensorship hypothesis, where as collapse of neutral strange quark matter accepts it.