CHAPTER 4

BASIC PRINCIPLES OF OFDM

4.1 BASICS OF OFDM

Orthogonal Frequency Division Multiplexing (OFDM) is very similar to the well known and widely used technique of Frequency Division Multiplexing (FDM). OFDM uses the principles of FDM to allow multiple messages to be sent over a single radio channel. It is however in a much more controlled manner, allowing an improved spectral efficiency.

A simple example of FDM is the use of different frequencies for each FM (Frequency Modulation) radio stations. All stations transmit at the same time but do not interfere with each other because they transmit using different carrier frequencies. Additionally they are bandwidth limited and are spaced sufficiently far apart in frequency so that their transmitted signals do not overlap in the frequency domain. At the receiver, each signal is individually received by using a frequency tunable band pass filter to selectively remove all the signals except for the station of interest. This filtered signal can then be demodulated to recover the original transmitted information.

OFDM is different from FDM in several ways. In conventional broadcasting each radio station transmits on a different frequency, effectively using FDM to maintain a separation between the stations. There is however no coordination or synchronization between each of these stations. With an OFDM transmission such as DAB, the information signals from multiple stations are combined into a single multiplexed stream of data. This data is then transmitted using an OFDM ensemble that is made up from a dense packing of many subcarriers. All the subcarriers within the OFDM signal are time and frequency synchronized to each other, allowing the interference between subcarriers to be carefully controlled. These multiple subcarriers overlap in the frequency domain, but do not cause Inter-Carrier Interference (ICI) due to the orthogonal nature of the modulation. Typically with FDM the transmission signals need to have a large frequency guard-band between channels to prevent interference. This lowers the overall spectral efficiency. However with OFDM the orthogonal packing of the subcarriers greatly reduces this guard band, improving the spectral efficiency.
All wireless communication systems use a modulation scheme to map the information signal to a form that can be effectively transmitted over the communications channel. A wide range of modulation schemes has been developed, with the most suitable one, depending on whether the information signal is an analogue waveform or a digital signal. Some of the common analogue modulation schemes include Frequency Modulation (FM), Amplitude Modulation (AM), Phase Modulation (PM), Single Side Band (SSB), Vestigial Side Band (VSB) and Double Side Band Suppressed Carrier (DSBSC). Common single carrier modulation schemes for digital communications include, Amplitude Shift Keying (ASK), Frequency Shift Keying (FSK), Phase Shift Keying (PSK) and Quadrature Amplitude Modulation (QAM). Each of the carriers in a FDM transmission can use an analogue or digital modulation scheme. There is no synchronization between the transmission and so one station could transmit using FM and another in digital using FSK. In a single OFDM transmission all the subcarriers are synchronized to each other, restricting the transmission to digital modulation schemes. OFDM is symbol based, and can be thought of as a large number of low bit rate carriers transmitting in parallel. All these carriers transmit in unison using synchronized time and frequency, forming a single block of spectrum. This is to ensure that the orthogonal nature of the structure is maintained. Since these multiple carriers form a single OFDM transmission, they are commonly referred to as ‘subcarriers’, with the term of ‘carrier’ reserved for describing the RF carrier mixing the signal from base band. There are several ways of looking at what make the subcarriers in an OFDM signal orthogonal and why this prevents interference between them.

4.1.1 Orthogonality

Signals are orthogonal if they are mutually independent of each other. Orthogonality is a property that allows multiple information signals to be transmitted perfectly over a common channel and detected, without interference. Loss of orthogonality results in blurring between these information signals and degradation in communications. Many common multiplexing schemes are inherently orthogonal. Time Division Multiplexing (TDM) allows transmission of multiple information signals over a single channel by assigning unique time slots to each separate information signal.
During each time slot only the signal from a single source is transmitted preventing any interference between the multiple information sources. Because of this, TDM is orthogonal in nature. In the frequency domain most FDM systems are orthogonal as each of the separate transmission signals are well spaced out in frequency preventing interference. Although these methods are orthogonal the term OFDM has been reserved for a special form of FDM. The subcarriers in an OFDM signal are spaced as close as is theoretically possible while maintaining the orthogonality between them.

OFDM achieves orthogonality [9], [31] in the frequency domain by allocating each of the separate information signals onto different subcarriers. OFDM signals are made up from a sum of sinusoids, with each corresponding to a subcarrier. The baseband frequency of each subcarrier is chosen to be an integer multiple of the inverse of the symbol time, resulting in all subcarriers having an integer number of cycles per symbol. As a consequence the subcarriers are orthogonal to each other. Sets of functions are orthogonal to each other if they match the conditions in equation (4.1). If any two different functions within the set are multiplied, and integrated over a symbol period, the result is zero, for orthogonal functions. Another way of thinking of this is that if we look at a matched receiver for one of the orthogonal functions, a subcarrier in the case of OFDM, then the receiver will only see the result for that function. The results from all other functions in the set integrate to zero, and thus have no effect.

\[
\int_{0}^{T} s_i(t)s_j(t)dt = \begin{cases} C & i = j \\ 0 & i \neq j \end{cases}
\] (4.1)

Equation (4.2) shows a set of orthogonal sinusoids, which represent the subcarriers for an unmodulated real OFDM signal.

\[
s_k(t) = \begin{cases} \sin(2\pi f_0 t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad k = 1,2,\ldots,N
\] (4.2)

where \( f_0 \) is the carrier spacing, \( N \) is the number of carriers, \( T \) is the symbol period. Since the highest frequency component is \( Nf_0 \) the transmission bandwidth is also \( Nf_0 \).

These subcarriers are orthogonal to each other because when we multiply the waveforms of any two subcarriers and integrate over the symbol period the result is zero. Multiplying the two sine waves together is the same as mixing these subcarriers. This
results in sum and difference frequency components, which will always be integer subcarrier frequencies, as the frequency of the two mixing subcarriers has integer number of cycles. Since the system is linear we can integrate the result by taking the integral of each frequency component separately then combining the results by adding the two sub-integrals. The two frequency components after the mixing have an integer number of cycles over the period and so the sub-integral of each component will be zero, as the integral of a sinusoid over an entire period is zero. Both the sub-integrals are zeros and so the resulting addition of the two will also be zero, thus we have established that the frequency components are orthogonal to each other.

4.1.2 Frequency Domain Orthogonality

Another way to view the orthogonality property of OFDM signals is to look at its spectrum. In the frequency domain each OFDM subcarrier has a sinc, $\sin(x)/x$, frequency response, as shown in Figure 4.1. This is a result of the symbol time corresponding to the inverse of the carrier spacing. As far as the receiver is concerned each OFDM symbol transmitted for a fixed time ($T_{FFT}$) with no tapering at the ends of the symbol. This symbol time corresponds to the inverse of the subcarrier spacing of $1/T_{FFT}$ Hz. This rectangular, boxcar, waveform in the time domain results in a sinc frequency response in the frequency domain. The sinc shape has a narrow main lobe, with many side-lobes that decay slowly with the magnitude of the frequency difference away from the centre. Each carrier has a peak at the centre frequency and nulls evenly spaced with a frequency gap equal to the carrier spacing. The orthogonal nature of the transmission is a result of the peak of each subcarrier corresponding to the nulls of all other subcarriers. When this signal is detected using a Discrete Fourier Transform (DFT) the spectrum is not continuous as shown in Figure 4.1 (a), but has discrete samples. The sampled spectrum is shown as ‘o’s in the figure. If the DFT is time synchronized, the frequency samples of the DFT correspond to just the peaks of the subcarriers, thus the overlapping frequency region between subcarriers does not affect the receiver. The measured peaks correspond to the nulls for all other subcarriers, resulting in orthogonality between the subcarriers.
Fig 4.1, Frequency response of the subcarriers in a 5 tone OFDM signal. 
(a) Shows the spectrum of each carrier, and the discrete frequency samples seen by an OFDM receiver. Note, each carrier is sinc, \( \text{sinc}(x)/x \), in shape. 
(b) Shows the overall combined response of the 5 subcarriers (thick black line).

4.2 OFDM GENERATION

To generate OFDM successfully the relationship between all the carriers must be carefully controlled to maintain the orthogonality of the carriers[32]. For this reason, OFDM is generated by firstly choosing the spectrum required, based on the input data, and modulation scheme used. Each carrier to be produced is assigned some data to transmit. The required amplitude and phase of the carrier is then calculated based on the modulation scheme (typically BPSK, QPSK, or QAM). The required spectrum is then converted back to its time domain signal using an Inverse Fourier Transform. In most applications, an Inverse Fast Fourier Transform (IFFT) is used. The IFFT performs the transformation very efficiently, and provides a simple way of ensuring the carrier signals produced are orthogonal.

The Fast Fourier Transform (FFT) transforms a cyclic time domain signal into its equivalent frequency spectrum. This is done by finding the equivalent waveform, generated by a sum of orthogonal sinusoidal components. The amplitude and phase of the sinusoidal components represent the frequency spectrum of the time domain signal. The IFFT performs the reverse process, transforming a spectrum (amplitude and phase of each component) into a time domain signal. An IFFT converts a number of complex data points, of length which is a power of 2, into the time domain signal of the same number of points. Each data point in frequency spectrum used for an FFT or IFFT is called a bin.
The orthogonal carriers required for the OFDM signal can be easily generated by setting the amplitude and phase of each bin, then performing the IFFT. Since each bin of an IFFT corresponds to the amplitude and phase of a set of orthogonal sinusoids, the reverse process guarantees that the carriers generated are orthogonal. Figure 4.2 shows Block diagram of an OFDM system using FFT. The signal generated is a base band, thus the signal is filtered, then stepped up in frequency before transmitting the signal.

4.2.1 Serial To Parallel Conversion
The input serial data stream is formatted into the word size required for transmission and shifted into a parallel format. The data is then transmitted in parallel by assigning each data word to one carrier in the transmission.

4.2.2 Modulation of Data
The data to be transmitted on each carrier is then mapped into a phase shift keying format. The data on each symbol is then mapped to a phase angle based on the modulation method. For example in BPSK, the phase angles are 0 and 180 degrees, whereas in QPSK the phase
angles used are 0, 90, 180, and 270 degrees. The use of phase shift keying produces a constant amplitude signal and was chosen for its simplicity and to reduce problems with amplitude fluctuations due to fading.

4.2.3 Inverse Fourier Transform

After the subcarrier modulation stage each of the data subcarriers is set to an amplitude and phase based on the data being sent and the modulation scheme; all unused subcarriers are set to zero. This sets up the OFDM signal in the frequency domain. An IFFT is then used to convert this signal to the time domain, allowing it to be transmitted. In the frequency domain, before applying the IFFT, each of the discrete samples of the IFFT [33] corresponds to an individual subcarrier. Most of the subcarriers are modulated with data. The other subcarriers are unmodulated and set to zero amplitude. These zero subcarriers provide a frequency guard band before the nyquist frequency and effectively act as an interpolation of the signal and allows for a realistic roll off in the analog anti-aliasing reconstruction filters.

4.2.4 Guard Period

For a given system bandwidth the symbol rate for an OFDM signal is much lower than a single carrier transmission scheme. For example for a single carrier BPSK modulation, the symbol rate corresponds to the bit rate of the transmission. However for OFDM the system bandwidth is broken up into \( N \) subcarriers, resulting in a symbol rate that is \( N \) times lower than the single carrier transmission. This low symbol rate makes OFDM naturally resistant to effects of Inter-Symbol Interference (ISI) caused by multipath propagation.

Multipath propagation is caused by the radio transmission signal reflecting off objects in the propagation environment, such as walls, buildings, mountains, etc. These multiple signals arrive at the receiver at different times due to the transmission distances being different. This spreads the symbol boundaries causing energy leakage between them. The effect of ISI on an OFDM signal can be further improved by the addition of a guard period to the start of each symbol. This guard period is a cyclic copy that extends the length of the symbol waveform. Each subcarrier, in the data section of the symbol, (i.e. the OFDM
symbol with no guard period added, which is equal to the length of the IFFT size used to
generate the signal) has an integer number of cycles. Because of this, placing copies of the
symbol end-to-end results in a continuous signal, with no discontinuities. Figure 4.3 shows
the insertion of a guard period.

![Diagram of OFDM signal with guard period](image)

Figure 4.3 Addition of a guard period to an OFDM signal

The total length of the symbol is \( T_s = T_G + T_{FFT} \), where \( T_s \) is the total length of the symbol in
samples, \( T_G \) is the length of the guard period in samples, and \( T_{FFT} \) is the size of the IFFT
used to generate the OFDM signal. In addition to protecting the OFDM from ISI, the guard
period also provides protection against time-offset errors in the receiver.

4.2.5 Receiver

The receiver basically does the reverse operation to the transmitter to retrieve the
information transmitted. The guard period is removed. The FFT of each symbol is then
taken to find the original transmitted spectrum. The phase angle of each transmission
carrier is then evaluated and converted back to the data word by demodulating the received
phase. The data words are then combined back to the same word size as the original data.

4.3 MATHEMATICAL DESCRIPTION OF OFDM

With the input sequence \( \{a[k]\}, 0 \leq k \leq N-1 \), the frequency spacing \( \Delta f \) between the
different subcarriers and the symbol interval \( T_s \), the transmitted signal \( x_d(t) \) can be
expressed as:

\[
x_d(t) = \sum_{k=0}^{N-1} a[k] \cdot e^{j2\pi \Delta f k t}
\]
\[ x_a(t) = \sum_{k=0}^{N-1} a[k] e^{j2\pi k t / T_s}, 0 \leq t \leq T_s \] (4.3)

If the signal is sampled at the rate \( T_s / N \), then the above equation may be rewritten as:

\[ x_a[n] = x_a\left(\frac{n}{N} T_s\right) = \sum_{k=0}^{N-1} a[k] e^{j2\pi k n / N} \] (4.4)

If the following equation,

\[ \Delta f T_s = 1 \] (4.5)

is satisfied, then the multicarriers are orthogonal to each other and equation 4.4 can be rewritten as:

\[ x_a[n] = \sum_{k=0}^{N-1} a[k] e^{j2\pi k n / N} = N \text{IDFT}\{a[k]\} \] (4.6)

The above is just the IDFT expression of the input stream \( \{a[k]\} \) with a difference of the gain factor \( 1/N \). At the receiver, the DFT implementation to find the approximate signal \( \hat{a}[k] \) can be written as:

\[
\hat{a}[k] = \text{DFT}\{x_a[n]\} = \sum_{n=0}^{N-1} x_a[n] e^{-j2\pi n k / N}
\]

\[
= \frac{1}{N} \sum_{m=0}^{N-1} x_a[m] \sum_{n=0}^{N-1} e^{j2\pi m(n-k) / N}
\]

\[
= \frac{1}{N} \sum_{m=0}^{N-1} a[m] \sum_{n=0}^{N-1} e^{j2\pi m(n-k) / N}
\]

\[
= \frac{1}{N} \sum_{m=0}^{N-1} a[m] N \delta[m-k]
\]

\[
= a[k]
\]

Here \( \delta[m-k] \) is the delta function defined as

\[
\delta[n] = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise}
\end{cases}
\] (4.8)
4.4 PERFORMANCE OF OFDM WITH VARIOUS MODULATION SCHEMES

Here simulation results are given for an OFDM system using 64 subcarriers. The system performance with four modulation schemes, binary phase shift keying, quadrature phase shift keying, 16 quadrature amplitude modulation, and 64 quadrature amplitude modulation is presented. The cyclic prefix is 16 samples long. As seen in Figure 4.4, the binary phase shift keying raw BER obtained through simulation matches perfectly with the theoretical curve obtained by using the equation (4.9)

\[ P_e = Q \left( \frac{2E_b}{N_0} \right) \]  

(4.9)

where \( Q(\cdot) \) represents the Q function given by,

\[ Q(z) = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \]  

(4.10)

and implemented in MATLAB by the complementary error function

\[ Q(z) = \frac{1}{2} erfc \left( \frac{z}{\sqrt{2}} \right) \]  

(4.11)

Fig 4.4 BPSK BER performance of OFDM in an AWGN channel
The BER performance of OFDM system in Rayleigh environments is presented. Rayleigh fading emerges when multiple time-shifted or delayed versions of the originally transmitted signal emerge at the receiver. This phenomenon is due to the existence of various paths the signal can take before arriving at destination. These replicas interfere with one another, causing Rayleigh fading. When the delays are clearly separated, the system suffers from frequency selective Rayleigh fading. Because of the properties of OFDM, each subcarrier is considered flat.

![Figure 4.5 BER performance of OFDM in an AWGN channel](image)

As seen, our simulation results match the theoretical BPSK BER curve which is

\[
P_e = \frac{1}{2} \left( 1 - \sqrt{\frac{\bar{\beta}}{1 + \bar{\beta}}} \right)
\]

where \(\bar{\beta}\) is the average signal to noise ratio (SNR) of the channel.

### 4.5 SPECTRAL EFFICIENCY

By adding a cyclic prefix (CP) to every OFDM symbol, the residual ISI can be eliminated. This is a very attractive choice from an implementation point of view. For perfect ISI and ICI elimination the length of the cyclic prefix has to be at least the length of the channel impulse response. The drawback of the cyclic prefix is that it reduces the spectral efficiency.
of OFDM systems since it contains no user data. Let $N$ be the number of carriers in the OFDM system. If $G$ samples are added as cyclic prefix, then the spectral

\[ \frac{N}{N+G} \]

For practical systems, the length of the cyclic prefix varies from 10\% to 40\% of $N$. For example, in 802.11a standard, this is 25\% of an OFDM symbol duration, indicating a significant loss in utilization.

Figure 4.6 BPSK BER performance of OFDM over a fast Rayleigh faded channel. This is a case of fast fading. The performance of OFDM with all the four modulation schemes cited above in this same fast Rayleigh fading channel can be seen in Figure 4.7.

Figure 4.7 OFDM performance in a fast flat Rayleigh faded channel.

efficiency of the system will be $N/(N+G)$. For practical systems, the length of the cyclic prefix varies from 10\% to 40\% of $N$. For example, in 802.11a standard, this is 25\% of an OFDM symbol duration, indicating a significant loss in utilization.
4.6 TECHNIQUES TO IMPROVE THE SPECTRAL EFFICIENCY

4.6.1.1 Operator Perturbation Technique (OPT)

One of the techniques used to improve the spectral efficiency of OFDM systems is OPT. This is an iterative method to effectively approximate and invert linear and nonlinear operations. For matrices, it is also known as Jacobi-Iteration. This technique is used for the inversion of the OFDM channel matrix for ISI cancellation[34]. Even though this technique eliminates the use of cyclic prefix, there will be residual errors. These errors are due to the fact that an initialization of the OPT algorithm by the diagonal of the channel matrix can be insufficient in cases, where one or several of the diagonal elements are too close to zero. The longer the impulse response of the channel, the more diagonal elements are small. Thus OPT gets worse when the impulse response length increases. Even if the number of iterations in the OPT algorithm are increased, the result does not converge to correct solution but stays in a local minimum.

4.6.1.2 Polynomial Cancellation Coding (PCC)

PCC is another technique used in OFDM to avoid the necessity of CP. The block diagram of a PCC-OFDM communication system is shown in Fig 4.8. In this technique, independent data to be transmitted are mapped onto the weighted groups of subcarriers instead of individual subcarriers [35], [36]. In practical situations, groups of two subcarriers are used. Thus each complex value is transmitted using two subcarriers and this reduces the spectral efficiency to half. The elimination of cyclic prefix will make up for some but not all of this loss.

Fig 4.8 Block diagram of PCC-OFDM communication system
4.6.1.3 Time Domain Equalizer (TEQ)

TEQ can be combined with OFDM for improving the spectral efficiency. TEQ block is used preceding the FFT demodulator at the receiver inorder to constrain the length of the Effective Channel Impulse Response (EIR) to be shorter than the selected cyclic prefix duration[37],[38]. This permits the use of a much shorter CP that could otherwise be employed and so raises the transmission efficiency. However, the EIR obtained in the frequency domain using this algorithm has spectral nulls. Consequently, the subchannels that fall in the nulls are severely degraded due to the low SNR and subsequently, the noise is amplified. Another disadvantage of this algorithm is that it needs a reasonable amount of processing time and memory to store the incoming data during the training of TEQ.

4.6.1.4 Partial Response Coding (PRC)

Partial response coding is an efficient transmission technique exploited in literature on band limited digital communication systems. In this methodology, PRC introduces memory or correlation to the transmitted data stream in the time domain in a way that the power spectrum of the transmitted band limited signal is shaped to exhibit gradual roll off to band edges. This spectral property dramatically reduces the amount of excessive intersymbol interference at the receiver when the symbol timing is not perfectly synchronized. PRC can be adapted to shape the signal spectrum of OFDM signals with an attempt to achieve high spectral compactness[39],[40]. The idea here is to introduce correlation between the transmitted symbols and the signal transmission is realized at the nyquist rate. At the receiver, the detection process is modified in order to remove correlated interference. One of the merits of PRC is that the controlled interference can be used to shape the system spectrum. Also, this spectral shaping can make the system less sensitive to timing errors.

4.7 IMPAIRMENTS TO OFDM SIGNALS DUE TO CHANNEL EFFECTS

In this section, we introduce the impairments to OFDM signals, like Doppler shift, dispersive fading, timing and frequency offsets, sampling clock offset, and nonlinear distortion due to large peak-to-average-power ratio (PAPR) of the OFDM signal caused by the effects of various channel disturbances.
4.7.1 Interference

In a multipath environment, different versions of the transmitted symbol reach the receiver at different times. This is due to the fact that different propagation paths exist between transmitter and receiver [41]. As a result, the time dispersion stretches a particular received symbol into the one following it. This symbol overlap is called inter-symbol interference, or ISI. It is also a major factor in timing offset. One other form of interference is inter-carrier interference orICI. In OFDM, successful demodulation depends on maintaining orthogonality between the carriers. We demodulate a specific subcarrier \( N \) at its spectral peak, meaning that all the other carriers must have a corresponding zero spectra at the \( N^{th} \) center frequency (frequency domain perspective). Frequency offsets lead to this criterion not being met. This condition can seriously hinder the performance of our OFDM system. Figure 4.9 below shows that when the decision is not taken at the correct center frequency (i.e. peak) of carrier considered, adjacent carriers factor in the decision making, thus reducing the performance of the system.

4.7.2 Time-Varying Impairments

Both Doppler shift and frequency offset can be modeled as time-varying impairments. Here we first derive a general expression for the effect of the time-varying impairments and then discuss the effect of Doppler shift and frequency offset [42], respectively.

![Effect of Frequency Offset](image)

Fig 4.9 Effect of frequency offset (maintaining orthogonality)
Consider an OFDM signal,
\[ s(t) = \sum_k s_k e^{j2\pi f_k t}, \quad 0 \leq t \leq T_s \] (4.14)
where \( f_k = f_0 + k\Delta f \) and \( s_k \) is the signal transmitted over the \( k \)-th subchannel. If there is a multiplicative time-varying distortion, \( \gamma(t) \), that is caused by either frequency offset or Doppler spread, the received signal will be
\[ x(t) = \gamma(t)s(t) \] (4.15)
The demodulated signal will be
\[
X_m = \frac{1}{T_s} \int_0^{T_s} x(t)e^{-j2\pi \nu t} dt
\]
\[
= \frac{1}{T_s} \int_0^{T_s} \gamma(t) \sum_k s_k e^{j2\pi \nu t} e^{-j2\pi \nu t} dt
\]
\[
= \sum_k \left[ \frac{1}{T_s} \int_0^{T_s} \gamma(t)e^{-j2\pi (f_0-f_k)^t} dt \right] s_k
\]
\[
= a_0 s_m + \sum_{k \neq m} a_{m-k} s_k
\] (4.16)
Where \( a_l \) is defined as
\[
a_l = \frac{1}{T_s} \int_0^{T_s} \gamma(t)e^{-j2\pi l f t} dt
\] (4.17)
The second term in equation (4.16) represents the intercarrier interference. \( a_0 \) is usually a complex number, whose magnitude and phase represent the attenuation and the phase shift on the desired signal, respectively. \( a_l, s \) for \( l \neq 0 \), are complex gains of the interchannel interference (ICI). If \( \gamma(t) \) is not a constant, then \( a_l \neq 0 \) for some \( l \neq 0 \), and ICI exists.

### 4.7.3 Effect of Frequency Offset

If there is a frequency offset, \( \Delta f \), between the transmitter and receiver, then \( \gamma(t) \) in equation 4.15 is a deterministic function and can be expressed as
\[
\gamma(t) = e^{j2\pi \Delta f t} = e^{j2\pi \Delta f t}
\]
where $\alpha = \frac{\delta f}{\Delta f}$. From equation (4.17)

$$a_i = \frac{1}{T_s} \int_0^{T_s} e^{j2\pi f_0 t} e^{-j2\pi f_i t} dt$$

$$= \sin[\pi f_0] e^{-j\pi f_i t}$$

$$= -\frac{\sin(\pi \alpha)}{\pi (l - \alpha)} e^{j\pi \alpha}$$

(4.18)

and $\alpha = k_0 + \varepsilon$, where $k_0$ is an integer and $\varepsilon$ is a fractional number with $|\varepsilon| \leq \frac{1}{2}$, then

$$a_i = -\frac{\sin(\pi \varepsilon)}{\pi (l - k_0 - \varepsilon)} e^{j\pi \varepsilon}$$

(4.19)

When $\alpha \leq 1/2$ ($k_0=0$ and $\varepsilon = \alpha$, $0 < |a_i| \leq |a_0|$). The desired signal is the dominant component in the demodulated signal. However, there is also ICI since $a_i \neq 0$ for $l \neq 0$. When $\alpha$ is an integer ($k_0=0$ and $\varepsilon = 0$), $a_{k_0} = 1$, $a_i = 0$ for $l \neq k_0$ and $X_i = s_{l-k_0}$. Therefore, the frequency offset causes a simple tone shift and there is no ICI. In general, neither $k_0$, nor $\varepsilon$ is zero; consequently, tone shift, attenuation, phase shift, and ICI all exist [43]. However, the signal distortion caused by frequency offset is deterministic. Furthermore, once the frequency offset is known, its effect can be corrected.

### 4.7.4 Effect of Doppler Shift

For channels with Doppler spread, $\gamma(t)$ is can be modeled as a zero-mean and narrow-band wide-sense stationary (WSS) stochastic process. For the classical Doppler spectrum, the spectral density of $\gamma(t)$ is

$$P_j(f) = \begin{cases} \frac{1}{\pi f_d} \frac{1}{\left( 1 - \left( \frac{f}{f_d} \right)^2 \right)^{1/2}}, & \text{if } |f| < f_d \\ 0, & \text{otherwise} \end{cases}$$

(4.20)

where $f_d$ is the maximum Doppler frequency. Two extreme cases of the Doppler spectrum are the uniform and the two-path models. For these two models, the spectral densities are
\[ P_u(f) = \begin{cases} \frac{1}{2f_d} & \text{if } |f| < f_d \text{ (Uniform)} \\ 0 & \text{otherwise} \end{cases} \]  

and \[ P_f(f) = \frac{1}{2} \left[ \delta(f + f_d) + \delta(f - f_d) \right] \text{ (Two-path)} \]

respectively. The correlation function of \( \gamma(t) \), defined as \( r(\tau) = \mathbb{E}\{\gamma(t + \tau)\gamma^*(t)\} \), is easily obtained as

\[ r(\tau) = F^{-1}\{P(f)\} \]

The correlation functions for the three models given above are

\[ r_j(\tau) = J_0(2\pi f_d \tau), \quad \text{(classical)} \]
\[ r_u(\tau) = \sin(2\pi f_d \tau), \quad \text{(Uniform)} \]

and

\[ r_t(\tau) = \cos(2\pi f_d \tau), \quad \text{(Two-path)} \]

respectively, where \( J_0(x) \) is the zero-order Bessel function of the first kind. It should be noted that the two-path model corresponds to an OFDM system with a fixed frequency offset of \( f_d \) Hz.

Since \( \gamma(t) \) is a stochastic process, \( a_i \) is a random variable with zero mean and variance given by

\[ \sigma_i^2 = \Delta E\left\{ a_i^2 \right\} = \int_{-1}^{1} r(T_\tau x) (1 - |x|) e^{-i2\pi fx} dx \]  

and the total ICI power due to Doppler spread is

\[ P_{ICI} = \sum_{i=0}^{\Delta E} \left| a_i s_{m-i} \right|^2 \]

\[ = \int_{-1}^{1} (1 - |x|)(1 - r(T_\tau x)) dx \]

\[ = 1 - \int_{-f_{T_\tau}}^{f_{T_\tau}} P(f) \sin^{2}(fT_\tau) df \]

Once the time-domain correlation or the Doppler spectral density of the time-varying channel is known, the ICI power can be calculated [44]. For the classical model,
\[ P_{IC} = 1 - \int_{-1}^{1} (1 - |x|) J_0(2\pi f_d T_s x) dx \]  

(4.26)

For uniform and two path models,

\[ P_{IC} = 1 - \frac{1 - \cos(2\pi f_d T_s) - 2\pi f_d T_s \sin(2\pi f_d T_s)}{2(\pi f_d T_s)^2} \]  

(4.27)

and \[ P_{IC} = 1 - \sin c^2(f_d T_s) \] respectively, where

\[ S_i(x) = \pi \int_{0}^{x} \frac{\sin(\pi u)}{\pi u} du = \pi \int_{0}^{x} \sin c(u) du \]

Using the expressions derived above, the ICI power can be exactly calculated. However, the exact expressions are complicated and do not easily provide much insight. Furthermore, in many instances, the exact time-domain correlation or power spectrum is not available.

Though Doppler shift has only minor effect on the ICI of OFDM signals when \( f_d T_s \) is very small. It does make the channel parameters to vary from one OFDM block to another. Therefore, when channel parameters are used for coherent detection or adaptive antenna arrays in mobile wireless systems, channel tracking is still essential in most environments.

### 4.7.5 Effect of Sampling Clock Offset

In practice, the sampling clock at the receiver is often different from that at the transmitter. The sampling clock difference will degrade the performance of the systems [45]. At the receiver, the OFDM signal can be simply demodulated by performing DFT to the samples if the continuous signal \( s(t) = \sum_{k=0}^{N-1} s_k e^{j2\pi f_k t} \) is sampled at a sampling interval of \( T_{sa} = \frac{T_s}{N} \).

However, if the sampling interval at the receiver is \( T_{sa} = T_{sa} + \beta T_{sa} \), then the samples will be \( \{s(nT_{sa})\}_{n=0}^{N-1} \). If the DFT is still used for OFDM demodulation, then

\[ X_m = \frac{1}{N} \sum_{n=0}^{N-1} s(nT_{sa}) e^{-j2\pi m n / N} \]

\[ = a_{m,m} s_m + \sum_{k \neq m} a_{m,k} s_k \]  

(4.28)
where \( a_{m,k} = \frac{\sin[\pi(k-m+\beta k)]}{N \sin\left(\frac{\pi}{N}(k-m+\beta k)\right)} e^{j\frac{N-1}{N}(\tau-m+\beta k)} \)

From equation (4.28), the demodulated signal, \( X_m \) consists of the desired symbol component \( s_m \) and ICI. The desired symbol is modified by

\[
a_{m,m} = \frac{\sin(\pi \beta m)}{N \sin\left(\frac{\pi}{N} \beta m\right)} e^{j\frac{N-1}{N} \beta m}
\]

(4.29)

which is a subchannel dependent phase rotation.

![Diagram](a) \( e(t) \) is a part of the next block

![Diagram](b) \( e(t) \) is a part of guard interval

![Diagram](c) \( e(t) \) is a part of cyclic extension

Fig 4.10 Effect of timing offset in OFDM signal
4.7.6 Effect of Timing Offset

The effect of timing offset is shown in the figure above. In that case the observed signal will be

\[
\tilde{s}(t, \tau) = \begin{cases} 
  s(t + \tau), & \text{if } 0 \leq t \leq T_s - \tau, \\
  e(t - T_s + \tau), & \text{if } T_s - \tau \leq t \leq T_s 
\end{cases}
\]  

(4.30)

where \( \tau \) represents the timing offset and \( e(t) \) represents the interference due to the timing offset. For OFDM systems without a guard interval or cyclic extension, \( e(t) \) is a part of next OFDM block, as shown in Fig. 4.10 (a). For OFDM systems with a guard interval larger than \( \tau \), \( e(t) = 0 \), as shown in Fig. 4.10 (b). For OFDM systems with a cyclic extension period larger than \( \tau \), \( e(t) = s(t) \), as shown in Fig. 4.10 (c). For OFDM systems with a cyclic extension period or guard interval less than \( \tau \), \( e(t) \) is a mixture of the above two or three signals [29].

When there is a timing offset, the demodulated signal at the receiver is

\[
X_m = \frac{1}{T_s} \int_0^{T_s} \tilde{s}(t, \tau) e^{-j2\pi n t} dt 
\]

\[
= \frac{1}{T_s} \int_0^{T_s-\tau} s(t + \tau) e^{-j2\pi n t} dt + \frac{1}{T_s} \int_{T_s-\tau}^{T_s} e(t - T_s + \tau) e^{-j2\pi n t} dt 
\]

\[
= \frac{1}{T_s} \int_0^{T_s} s(t) e^{-j2\pi n t} dt + \frac{1}{T_s} \int_0^{\tau} e(t) e^{-j2\pi n (t-\tau)} dt 
\]

\[
= \frac{1}{T_s} \int_0^{T_s} s(t) e^{-j2\pi n t} dt - \frac{1}{T_s} \int_0^{\tau} s(t) e^{-j2\pi n (t-\tau)} dt + \frac{1}{T_s} \int_0^{\tau} e(t) e^{-j2\pi n (t-\tau)} dt 
\]

\[
= s_m e^{j2\pi n \tau} + \frac{1}{T_s} \int_0^{\tau} [e(t) - s(t)] e^{-j2\pi n (t-\tau)} dt 
\]

(4.31)

From equation (4.31), timing offset introduces a phase shift to the desired symbol component and an additive interferer depending on whether a cyclic extension or a null interval is used. When the system has no guard interval or cyclic extension, then \( e(t) \) is a part of the next OFDM block, which is independent of \( s(t) \). Therefore, the resulting average interference power is the summation of the powers

\[
\frac{1}{T_s} \int_0^{\tau} s(t) e^{j2\pi n (t-\tau)} dt \quad \text{and} \quad \frac{1}{T_s} \int_0^{\tau} e(t) e^{j2\pi n (t-\tau)} dt. 
\]
When the system has a guard interval that is larger than \( \tau \), then \( e(t) = 0 \), only single term remains. However, if a proper cyclic extension is used then \( e(t) = s(t) \) and there is no interference. Therefore, a proper cyclic extension can effectively cancel additive interference caused by timing offset.

### 4.7.7 Effect of Delay Spread

For a channel with multipath delay spread, the received signal is a summation of the transmitted signal with different (complex) gains and delays, as shown in Fig. 4.11. Here we assume that the transmitted signal has a proper cyclic suffix extension and the length of the cyclic extension, \( T_g \), is larger than the delay span or channel length, \( T_h \), of the multipath fading channel [30]. Furthermore, we assume that the starting time of integration/observation is between \( T_h \) and \( T_g \), that is, \( T_h \leq \tau \leq T_g \).

Let the gain and delay of each path be \( \gamma_i \) and \( \tau_i \) respectively. In this case, the received signal can be expressed as

\[
x(t) = \sum_{i} \gamma_i \tilde{s}(t, \tau - \tau_i)
\]  

Therefore the demodulated signal at the receiver is

\[
X_m = \frac{1}{T_s} \int_{0}^{T_s} r(t) e^{-j2\pi f_d \omega t} dt
= \sum_{i} \gamma_i \frac{1}{T_s} \int_{0}^{T_s} \tilde{s}(t, \tau - \tau_i) e^{-j2\pi f_d \omega t} dt
\]  

Let the gain and delay of each path be \( \gamma_i \) and \( \tau_i \) respectively.

![Fig 4.11 Effect of delay spread on OFDM signal](image-url)
When \( \tau_{\text{max}} \leq \tau \leq T_g \), from equation (4.31),
\[
\frac{1}{T_s} \int_0^T \tilde{x}(t, \tau - \tau_i) e^{-j2\pi\nu t} dt = s_m e^{j2\pi\nu \tau_i}
\]

Hence, \( X_m = \sum_i \gamma_i s_m e^{j2\pi\nu \tau_i} \)

\[
= s_m e^{j2\pi\nu \tau_i} \sum_i \gamma_i e^{-j2\pi\nu \tau_i} = H(f_m) e^{j2\pi\nu \tau_i} s_m
\]

where \( H(f) \) is the frequency response of the multipath channel defined as
\[
H(f) = \sum_i \gamma_i e^{-j2\pi\nu_i t}
\]

From (4.34), the received symbol is the original symbol with a phase shift determined by the timing offset, and multiplicative distortion determined by the frequency response at each subchannel, which makes signal detection very simple and is also a crucial difference between OFDM and single-carrier modulation. For single-carrier modulation, delay spread or frequency selectivity of wireless channels will cause ISI, which makes signal detection very complicated.

### 4.7.8 System Nonlinearity

Nonlinear devices in wireless systems will distort OFDM signals. For an OFDM signal with \( N \) subchannels the peak power can be as large as \( N^2 \) while the average power is \( N \) when \( E|s_i|^2 = 1 \); consequently, the largest PAPR will be \( \text{PAPR} = N \). For an OFDM signal with 128 subchannels, \( \text{PAPR}=21 \text{ dB} \), while it is about 6 dB for single carrier modulation.

When an OFDM signal is passed through a nonlinear device, such as a transmitter power amplifier, it will suffer significant nonlinear distortion, which generates spectral spreading and in-band noise. Figure 4.12 demonstrates inband distortion and spectral spread due to the nonlinearity of an amplifier. If the amplifier is modeled as a 3-dB clipper, then it will cause about 14 dB inband noise, about 22 dB adjacent channel interference. Adjacent channel interference or spectral spread can be mitigated by a clipping-and-filtering algorithm. However, inband noise still degrades the system performance.
Fig 4.12 (a) Spectral spread and (b) inband distortion caused by clipping.