CHAPTER II

On The Del Relation In

Join-Semilattices.

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Chapter - II

1. Introduction:

This chapter is a continuation of our attempts to extend the theory of symmetric lattices to join-semilattices. In a lattice \( L \) with 0 the \( \triangledown \)-relation is well-known. For \( a, b \in L \), \( a \triangledown b \) means that

\[(x \triangledown a) \land b = x \land b \text{ for every } x \text{ in } L.\]

It is known that this \( \triangledown \)-relation plays an important role in the direct sum decomposition of a lattice \( L \) with 0. Janowitz [2] investigated \( \triangledown \)-relation in a lattice with 0 which resulted in the revelation of the structure of atomistic lattices with the covering property. In fact, he succeeded in obtaining characterizations of \( \triangledown \)-symmetry, \( \triangledown \)-continuity for some special types of lattices called finite-statistisch lattices with the covering property. Janowitz and Cote [4] studied standard elements, finite distributive atomistic lattices in the context of \( \triangledown \)-relation. In fact, they succeeded in giving an affirmative answer to the following open problem of Maeda and Maeda [5].
Problem: Is there an Ac-lattice where $a \vee b$ does not imply $b \vee a$?

In what follows we endeavour to extend the above considerations to a more general setting of join-semilattice. In doing so we introduce several novel notions such as $a \bowtie b$ (for elements $a, b$ of a join-semilattice), $\triangledown$-standard elements, property (A), property (B) etc.

In this chapter we define the $\triangledown$-relation for elements of a join-semilattice and obtain several properties of $\triangledown$-relation. The connection between $\triangledown$-relation in a given join-semilattice and its completion by cuts is explored. It is shown that in an $S\triangledown^*$ join-semilattice $a \bowtie b$ is equivalent to $a \bowtie b$. We also succeed in obtaining direct sum decomposition of a join-semilattice $L$ with $0$. Further we investigate in detail the $\triangledown$-relation in atomistic join-semilattices. Several characterizations of $a \bowtie b$ are obtained. The notions of subspaces of $\neg\neg(L)$ and the lattice of all subspaces for an atomistic join-semilattice $L$, denoted by $L(\neg\neg(L))$, are discussed quite in detail. We also introduce the concepts of statistisch and finite-statistisch join-semilattices. Several properties of $\triangledown$-relation in the context of statistisch and finite-statistisch lattices are obtained. Several properties involving central cover in $L(\neg\neg(L))$ and $\triangledown$-relation are obtained. We also
show that properties (A) and (B) (stated explicitly at a later stage) are equivalent in an atomistic join-semilattice with the finite covering property. We also introduce the novel notion of distributive pairs in a join-semilattice, that further facilitates the introduction of concepts of standard and \( \triangledown \)-standard elements in a join-semilattice. Several characterizations of \( \triangledown \)-standard elements in an atomistic join-semilattice are given. For an atomistic join-semilattice \( L \) with finite-covering property we relate \( \triangledown \)-standard elements of \( L \) with central cover in \( L(\triangledown\neg L(L)) \). The concept of finite-distributive join-semilattice is introduced and their several properties are investigated. A few counter examples are also given.

The detailed discussion, however, is postponed to respective sections. Whenever, there is no explicit mention, by \( L \) we shall always mean a join-semilattice, as was done in the first chapter.
2. Fundamental properties of the \( \sqcap \)-relation.

We begin this section by introducing the \( \sqcap \)-relation in a join-semilattice as follows:

**Definition (2.1)** Let \( L \) be a join-semilattice, and let \( a, b \in L \). We write \( a \sqcap b \) when

\[(\sqcap) \quad y \leq x \lor a \text{ and } y \leq b \text{ together imply } y \leq x.\]

For a subset \( S \) of \( L \) we denote by \( S \uparrow \) the set of \( a \in L \) such that \( a \sqcap b \) for all \( b \in S \).

Next is a close relative of \( \sqcap \)-relation.

We write \( a \sqapprox b \) when

\[(\sqapprox) \quad y \leq x \lor a, \quad y \leq x \lor b \text{ imply } y \leq x.\]

Here are immediate consequences of the above definition:

**Remark (2.1):** (i) If \( L \) has \( 0 \), then \( a \sqcap b \) implies \( a \land b = 0 \). In fact, if \( y \) is a lower bound of \( \{a, b\} \), then \( y = 0 \) since \( y \leq c \lor a, \quad y \leq b \). Moreover, it is evident that \( a \sqcap 0 \) and \( 0 \sqcup a \) hold for every \( a \) in \( L \).
(ii) If \( a \uparrow \uparrow a \) holds for some \( a \) in \( L \), then \( L \) has 0 and \( a = 0 \). In fact, \( a \leq x \) for every \( x \in L \) since \( a \leq x \uparrow a \) and \( a \leq a \).

We now show that \( \uparrow \) is more general than \( \uparrow \uparrow \).

**Proposition (2.2):** Let \( L \) be a join-semilattice. Then \( a \uparrow a \) implies \( a \uparrow b \) for \( a, b \in L \).

**Proof:** Suppose \( a \uparrow a \) holds. Let \( y \leq x \uparrow a \) and \( y \leq b \).

Then \( a \uparrow b \) together with \( y \leq x \uparrow a \), \( y \leq x \uparrow b \) implies that \( y \leq x \). Thus \( a \uparrow b \) holds.

q.e.d.

As expected, we see that this concept of \( \uparrow \)-relation coincides with the known concept of \( \uparrow \)-relation in a lattice.

**Proposition (2.3):** If \( L \) is a lattice, then \( a \uparrow b \) if and only if \( (x \uparrow a) \wedge b = x \wedge b \) for every \( x \) in \( L \).

**Proof:** Assume \( a \uparrow b \). For \( x \) in \( L \) we put \( y = (x \uparrow a) \wedge b \).

Then \( y \leq x \uparrow a \), \( y \leq b \) which together with \( a \uparrow b \) implies that \( y \leq x \). Hence \( y \leq x \wedge b \leq (x \uparrow a) \wedge b \leq y \). Thus \( y = x \wedge b \) i.e. \( (x \uparrow a) \wedge b = x \wedge b \).

Conversely, suppose that \( (x \uparrow a) \wedge b = x \wedge b \) for every \( x \) in \( L \).

Let \( y \leq x \uparrow a \), \( y \leq b \). Then \( y \leq (x \uparrow a) \wedge b = (x \wedge b \). Thus \( y \leq x \wedge b \). Hence \( y \leq x \wedge b \).

q.e.d.
In a like manner we also obtain a characterization of $a \supset b$ in a lattice.

**Proposition (2.4):** If $L$ is a lattice, then $a \supset b$ if and only if $x = (x \vee a) \land (x \vee b)$ for every $x$ in $L$.

**Proof:** Assume $a \supset b$. Put $y = (x \vee a) \land (x \vee b)$ for $x$ in $L$. Then $y \leq x \vee a$ and $y \leq x \vee b$. Hence by $a \supset b$,

\[ y \leq x \leq (x \vee a) \land (x \vee b) = y. \]

Thus $y = (x \vee a) \land (x \vee b) = x$.

Conversely, suppose that for every $x$ in $L$, $x = (x \vee a) \land (x \vee b)$ holds. Let $y \leq x \vee a$, $y \leq x \vee b$ then $y \leq (x \vee a) \land (x \vee b) = x$.

Thus $y \leq x$. Hence $a \supset b$.

q.e.d.

After looking at the definition of $a \vee b$ in a join-semilattice from many angles, we set ourselves with obtaining properties involving $\vee$-relation.

**Lemma (2.5):** In a join-semilattice $L$,

(i) $a \vee b$ and $b \leq x \vee a$ imply $b \leq x$.

(ii) $a \supseteq b$, $a_1 \leq a$ and $b_1 \leq b$ imply $a_1 \supseteq b_1$.

(iii) $a \supseteq b$, and $a_2 \supseteq b$ imply $(a_1 \vee a_2) \supseteq b$.

(iv) $S \supseteq$ is an ideal for every subset $S$ of $L$.

**Proof:** (i) If $b \leq x \vee a$ and $b \leq b$ then by $a \vee b$ we have $b \leq x$. 


(ii) Assume \( a \vee b, \ a_1 \leq a \) and \( b_1 \leq b \). Let \( y \leq x \vee a_1, \ y \leq b_1 \). Then \( y \leq x v a_1, \ y \leq b_1 \). Therefore by \( a \vee b, \ y \leq x \). Thus \( a_1 \vee b_1 \).

(iii) If \( y \leq x v a_1 v a_2, \ y \leq b \) then by \( a_2 \n v b \) we have \( y \leq x v a_1 v a_2 \). Again by \( a_1 \n v b \) we get \( y \leq x \). Thus \( (a_1 v a_2) v b \).

(iv) Let \( S \) be a subset of \( L \). Then from (ii) and (iii) \( S \n v \) is an ideal.

q.e.d.

The concept of completion by cuts of a lattice can be pulled down mutatis mutandis to a poset. In that direction we have

**Definition (2.2)**: Let \( L \) be a poset. The completion by cuts, \( \overline{L} = \{ X \subseteq L; X = X^u \} \) of \( L \) can be defined by the same way as (12.3) of Maeda and Maeda [5]; where \( X^u \) (respectively \( X^l \)) is the set of all upper (respectively lower) bounds of \( X \). Denote by \( J_a \) the set \( \{ x \in L; x \leq a \} \).

Next, we have

**Proposition (2.6)**: Let \( L \) be a poset.

(i) For every \( a \) in \( L \), \( J_a \) belongs to \( \overline{L} \).

(ii) The mapping \( a \rightarrow J_a \) is a one-to-one and isotone mapping of \( L \) into the complete lattice \( \overline{L} \), which preserves any existing meets and joins.
Proof: (i) Since \((J_a)^U = \{x \in L; x \geq a\}\), we have
\((J_a)^U = J_a\). Hence \(J_a\) belongs to \(\bar{L}\) for every \(a\) in \(L\).

(ii) If \(J_a = J_b\) then \(\{x \in L; x \leq a\} = \{x \in L; x \leq b\}\). Hence
\(a = b\), and the mapping is one-to-one. If \(a \leq b\) then
clearly \(J_a \subseteq J_b\). Therefore the mapping is isotone, hence
an isomorphism. Further, if \(a, b \in L\) and \(a \land b\) exists then
\[J_a \land b = \{x \in L; x \leq a \land b\} = \{x \in L; x \leq a\} \cap \{x \in L; x \leq b\}\]

Thus the mapping preserves meets.

For \(a, b \in L\) if \(a \lor b\) exists then clearly \(x \not\in J_a \cup J_b\)
implies \(x \not\in J_a\) or \(x \not\in J_b\). That is \(x \not\leq a\) or \(x \not\leq b\). Hence
\(x \not\leq a \lor b\). Thus \(x \not\in J_a \lor J_b\) and \(J_a \lor J_b \subseteq J_a \lor b\). If \(J_a \leq x\)
and \(J_b \leq x\) in \(\bar{L}\) then for every \(y\) in \(X^U\) we have
\(a \leq y\) and \(b \leq y\). Hence \(a \lor b \leq y\). Therefore \(a \lor b \in X^U = X\)
and thus \(J_{a \lor b} \leq X\), which proves that \(J_{a \lor b} = J_a \lor J_b\) in \(\bar{L}\)
and the mapping preserves joins.

q.e.d.

In the next result we connect equivalence of the
relations \(\lor\) and \(\sim\) in \(L\) with \(\lor\) and \(\sim\) in \(\bar{L}\), the
completion by cuts of \(L\).
Proposition (2.7): Let a and b be elements of a join-semilattice \( L \).

(i) \( a \vee b \text{ in } L \) if and only if \( J_a \vee J_b \text{ in } \bar{L} \).

(ii) \( a \wedge b \text{ in } L \) if and only if \( J_a \wedge J_b \text{ in } \bar{L} \).

Proof: (i) Assume \( a \vee b \). To prove \( J_a \vee J_b \), it suffices to show that \((X \vee J_a) \wedge J_b \leq X\) for every \( X \) in \( \bar{L} \). For any \( x \in X^u \) we have \( x \vee a \in (X \vee J_a)^u \). If \( y \in (X \vee J_a) \wedge J_b \), then since \( y \in (X \vee J_a)^u \) and \( y \in J_b \), we have \( y \leq x \vee a \) and \( y \leq b \), whence \( y \leq x \). Hence \( y \in X^u = X \). Thus \( J_a \vee J_b \) holds.

Conversely, suppose \( J_a \vee J_b \) holds. Let \( y \leq x \vee a \), \( y \leq b \) for \( x \) in \( L \). Hence \( y \in J_{x \vee a} = J_x \vee J_a \) and \( y \in J_b \). Therefore \( y \in (J_x \vee J_a) \wedge J_b = J_x \wedge J_b = J_x \wedge b \).

Thus \( y \leq x \wedge b \leq x \) and hence \( a \vee b \) holds.

(ii) Suppose \( a \wedge b \). To prove \( J_a \wedge J_b \), it is sufficient if we prove

\((X \vee J_a) \wedge (X \vee J_b) \leq X\) for every \( X \) in \( \bar{L} \). For any \( x \in X^u \) we have \( x \vee a \in (X \vee J_a)^u \). If \( y \in (X \vee J_a) \wedge (X \vee J_b) \), then since \( y \in (X \vee J_a)^u \) and \( y \in (X \vee J_b)^u \), we have \( y \leq x \vee a \) and \( y \leq x \vee b \). Hence by \( a \wedge b \), we have \( y \leq x \) and thus \( y \in X^u = X \).
Conversely, if \( J_a \subseteq J_b \). Let \( y \leq x \vee a \) and \( y \leq x \vee b \) then \( y \in J_{x \vee a} \) and \( y \in J_{x \vee b} \). That is \( y \in J_x \vee J_a \) and \( y \in J_x \vee J_b \). Hence, \( y \in (J_x \vee J_a) \wedge (J_x \vee J_b) = J_x \). Thus \( y \leq x \), which proves \( a \leq b \).

q.e.d.

Very naturally the well known concept of a dual section semicomplemented (i.e. \( \text{SSC}^* \)) lattice can be pulled down to that of \( \text{SSC}^* \) join-semilattice.

**Definition (2.3)** : A join-semilattice \( L \) with \( 1 \) is called \( \text{SSC}^* \) when \( L \) satisfies the following condition:

If \( a < b \) then there exists \( c \) such that \( a \leq c < 1 \) and \( cvb = 1 \).

Next, we characterize \( \text{SSC}^* \) join-semilattice \( L \) in terms of \( \text{SSC}^* \) lattice \( \overline{L} \).

**Lemma (2.8)**: A join-semilattice \( L \) with \( 1 \) is \( \text{SSC}^* \) if and only if \( \overline{L} \) is \( \text{SSC}^* \).

**Proof**: (i) Assume that \( L \) is \( \text{SSC}^* \), and let \( A < B \) in \( \overline{L} \). We take \( b \in B \) with \( b \notin A \). Then, there exists \( a \in A^u \) such that \( b \downarrow a \), since \( b \not\in A^u = A \). Since \( a < avb \), there exists \( c \in L \) such that \( a \leq c < 1 \) and \( cva \vee b = 1 \). Evidently, \( A \leq J_c < L \). Moreover, \( (J_c \vee B)^u = \{1\} \) since \( x \geq c \) and \( x \geq b \) together imply \( x = 1 \). Hence, \( J_c \vee B = (J_c \vee B)^u = L \). Thus, \( L \) is \( \text{SSC}^* \).
(ii) Assume that $\overline{L}$ is $SSC^*$ and let $a < b$ in $L$.

Since $J_a < J_b$, there exists $X \in \overline{L}$ such that $J_a \leq X < L$
and $X \uparrow J_b = L$. $X \uparrow L$ implies $X^\uparrow \uparrow \uparrow \{1\}$ and hence there
is $c \in X^\uparrow \uparrow \uparrow \{1\}$ with $c < 1$. Evidently we have $a \leq c$ and $cvb = 1$.

q.e.d.

As a nice generalization of the notion of $SSC^*$ the
notion of dually relative semicomplemented lattice is known
in the literature. It can be trivially pulled down to a
join-semilattice. In fact, we have

Definition (2.4): A join-semilattice $L$ is called $RSC^*$
when $L$ satisfies the following condition:

If $a < b \leq d$ then there exists $c$ such that $a \leq c < d$
and $cvb = d$.

Now we cite an example of a lattice $L$ which is $RSC^*$
but its completion by cuts $\overline{L}$ is neither $RSC^*$ nor $SSC^*$.

Example (2.9): Let $X$ be an infinite set and let $X_1, X_2$
be disjoint infinite subsets of $X$ with $X_1 \cup X_2 = X$.
Consider $L$ to be the lattice of all finite subsets of
$X$. Clearly $L$ is $RSC^*$ but its completion by cuts $\overline{L}$,
which is isomorphic with lattice $L$ with $1$ is neither
$RSC^*$ nor $SSC^*$. 
Now we establish an equivalence of relations $\triangledown$ and $\sim$ in $RSC^*$ or $SSC^*$ join-semilattices.

**Theorem (2.10):** Let $L$ be a join-semilattice and $a, b \in L$. If $L$ is $RSC^*$ or $SSC^*$, then the following statements are equivalent.

(i) $a \triangledown b$.

(ii) $a \sim b$.

(iii) $b \leq x \triangledown a$ implies $b \leq x$.

**Proof:** (ii) implies (i) follows from Proposition (2.2).

(i) implies (iii): Suppose $a \triangledown b$ and $b \leq x \triangledown a$. Since $b \leq b$, we have $b \leq x$.

(iii) implies (ii): First we assume that $L$ is $RSC^*$. Let $y \leq x \triangledown a$ and $y \leq x \triangledown b$. If $y \not\leq x$ then since $x \leq x \triangledown y \leq x \triangledown b$, there would exist $d$ such that $x \leq d \leq x \triangledown b$ and $d \triangledown xy = x \triangledown b$. Then $b \leq d \triangledown x \triangledown y \leq d \triangledown x \triangledown a = d \triangledown a$ and hence $b \leq d$ by (iii). This implies $x \triangledown b \leq d \leq x \triangledown b$, a contradiction. Hence $y \leq x$. Now assume that $L$ is $SSC^*$. Let $y \leq x \triangledown a$, $y \leq x \triangledown b$. If $y \not\leq x$, then $x \leq x \triangledown y$. Hence by definition of $SSC^*$, there exists $d$ such that $x \leq d \leq 1$ and $d \triangledown xy = 1$. Then $b \leq 1 = d \triangledown xy \leq d \triangledown x \triangledown a = d \triangledown a$ and hence $b \leq d$ by (iii). This implies $y \leq x \triangledown b \leq d$, whence $d = d \triangledown xy = 1$, a contradiction. Thus $y \leq x$.

q.e.d.
In the same vein, one has

**Corollary (2.11):** Assume that a join-semilattice is RSC or SSC.

(i) \( L \) is \( \triangledown \)-symmetric.

(ii) If \( a_i \triangledown b \) for every \( i \in I \) and \( \bigvee_{i \in I} a_i \) exists then \( \bigvee_{i \in I} a_i \triangledown b \).

**Proof:** (i) From the equivalence of (i) and (ii) of the last theorem clearly \( L \) is \( \triangledown \)-symmetric.

(ii) Assume \( a_i \triangledown b \) for every \( i \in I \) and assume that \( \bigvee_{i \in I} a_i = a \) exists. If \( a \leq x \triangledown b \), then we have \( a_i \leq x \triangledown b \) for every \( i \). Hence, since \( L \) is \( \triangledown \)-symmetric, \( b \triangledown a_i \) implies \( a_i \leq x \) for every \( i \) by Theorem (2.10) (iii). Therefore, \( \bigvee_{i \in I} a_i \leq x \), i.e. \( a \leq x \). Thus \( a \triangledown b \) holds, i.e. \( \bigvee_{i \in I} a_i \triangledown b \) holds.

q.e.d.

We shall connect our present considerations with those in chapter-I. Recall the concept of dual modular pair, \((a, b)^*\) that was introduced in chapter-I. Recall also the definition of modular join-semilattice, as \( L \) is modular if and only if \((a, b)^*\) holds for every pair \( a, b \in L \). Here is the connection between \( \triangledown \)-relation and dual modular pair.
Proposition (2.12): In a join-semilattice L, if $a \sqcup b$ and if $(b, x) \leq M$ for every $x$ in L, then $b \sqcup a$.

Proof: Assume that $y \leq x \sqcup b$ and $y \leq a$. Since $x \leq x \sqcup y \leq x \sqcup b$, by $(b, x) \leq M$* there exists an element $b_1 \leq b$ such that $x \sqcup b_1 = x \sqcup y$. Then by $b_1 \leq x \sqcup y \leq x v a$ and $b_1 \leq b$. Hence by $a \sqcup b$, $b_1 \leq x$. Therefore $y \leq x \sqcup b_1 = x$. Thus $b \sqcup a$ holds.

q.e.d.

As an immediate consequence of the above proposition, we have

Corollary (2.13): Any modular join-semilattice is $\sqcup$-symmetric, that is $a \sqcup b$ implies $b \sqcup a$.

Proposition (2.14): Let $S_1, \ldots, S_n$ $(n \geq 2)$ be non-empty subsets of a join-semilattice L and assume that they satisfy the following two conditions:

1. $\{a_1 \vee \ldots \vee a_n \mid a_i \in S_i \text{ for } i = 1, \ldots, n\} = L$,
2. if $i \neq j$ then $S_i \subseteq S_j$.

Then, L has O, $S_1$ are ideals of L, and for $a \in L$ the expression $a = a_1 \vee \ldots \vee a_n$ is unique. Moreover, the product join-semilattice $S_1 \times \ldots \times S_n$ is isomorphic with L by the mapping $\phi((a_1, \ldots, a_n)) = a_1 \vee \ldots \vee a_n$. 

Proof: (I) Let \( a \in S_1 \). By (1), \( a = a_1 \vee \ldots \vee a_n \) where \( a_i \in S_1 \). We have \( a \nless a_n \) by (2) for \( n \geq 2 \). Therefore \( a_n \nless a_n \) since \( a_n \nless a \). Thus, \( L \) has \( 0 \) by Remark (2.1)(ii).

(II) Let \( b \nless a \in S_1 \). By (1), \( b = b_1 \vee \ldots \vee b_n \) where \( b_i \in S_1 \). We have \( a \nless S_j \) for \( j \geq 2 \). Thus \( a \nless b_1 \) for every \( i \geq 2 \). But \( b_1 \nless a \), hence \( b_1 \nless b_1 \) for every \( i \geq 2 \). Thus \( b_1 = 0 \) for \( i \geq 2 \). Hence \( b = b_1 \in S_1 \). Next, suppose \( a, b \in S_1 \) and \( a \nless b \), therefore \( c_i \nless c_i \). Hence \( a \nless c_i \) for \( i \geq 2 \). Thus \( c_i = 0 \) for \( i \geq 2 \). This shows that \( S_1 \) is an ideal of \( L \). Similarly, we can show that \( S_i \) are ideals.

(III) By (1), the mapping \( \phi \) is onto.

If \( (a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n) \) with \( a_i, b_i \in S_1 \) then \( a_i \preceq b_i \) for every \( i \). Hence \( a_1 \vee \ldots \vee a_n \preceq b_1 \vee \ldots \vee b_n \).

Thus \( \phi \) is isometric.

If \( a_1 \vee \ldots \vee a_n \preceq b_1 \vee \ldots \vee b_n \) where \( a_i, b_i \in S_1 \) then

since \( b_1 \nless a_1 \) for \( i \geq 2 \), we have \( b_2 \vee \ldots \vee b_n \nless a_1 \). Now \( a_1 \preceq b_1 \vee \ldots \vee b_n \) and \( a_1 \nless a_1 \) hence \( a_1 \preceq b_1 \). Similarly, we can show that \( a_i \preceq b_i \) for every \( i \). Therefore \( \phi \) is one-to-one and \( \phi^{-1} \) is isometric. Thus \( \phi \) is an isomorphism.

Moreover, the expression \( a = a_1 \vee \ldots \vee a_n \) is unique. In fact, if \( a = a_1 \vee \ldots \vee a_n = b_1 \vee \ldots \vee b_n \) where \( a_i, b_i \in S_1 \) (i=1,2,\ldots,n).
Hence $b_2 \vee \ldots \vee b_n \in S_1$, whence $b_2 \vee \ldots \vee b_n \vee a_1$. Now $a_1 \leq b_1 \vee b_2 \vee \ldots \vee b_n$ and $a_1 \leq a_1$ imply $a_1 \leq b_1$. Similarly, we have $b_1 \leq a_1$ and hence $a_1 = b_1$. In general, we can prove $a_i = b_i$ for every $i$. Thus the expression is unique.

_q.e.d._

It may happen that $L$ may be the direct sum of ideals but $\overline{L}$ may not be so. We produce an example in support of this. The example given below is the same as Example 2.9, given before.

**Example (2.15):**

Let $X$ be an infinite set, and let $X_1, X_2$ be disjoint infinite subsets of $X$ with $X_1 \cup X_2 = X$. Let $L$ be the lattice of all finite subsets of $X$. If we put $S_1 = \{ F \subseteq L ; F \subseteq X_1 \}$, then evidently $L$ is the direct sum of $S_1$ and $S_2$. But, since $\overline{L}$ is isomorphic with the lattice adding 1 to $L$, $\overline{L}$ has no direct sum decomposition.

3. _$\vee$-relation in atomistic join-semilattices._

We now study _$\vee$-relation_ in the context of atomistic join-semilattices. We begin by introducing the concept of subperspectivity which is well stretched in lattice theory.
Definition (3.1). Let \( L \) be a join-semilattice with \( 0 \), and let \( a, b \in L \). We say that \( b \) is subperspective to \( a \) when there exists \( x \in L \) such that \( b \leq a \vee x \) and \( b \wedge x = 0 \) (i.e. \( 0 \) is the only lower bound of \( \{ b, x \} \)). \( a \) and \( b \) are called perspective, denoted by \( a \sim b \), if there exists an element \( x \in L \) such that \( a \vee x = b \vee x \) and \( a \wedge x = b \wedge x = 0 \).

Let us list some connections between \( \vee \)-relation and subperspectiveity:

Lemma (3.1): Let \( L \) be a join-semilattice with \( 0 \).

(i) If \( b \) is subperspective to \( a \) and if \( a \vee b \) then \( b \leq 0 \).

(ii) An atom \( p \) of \( L \) is subperspective to \( a \) if and only if \( a \vee p \) does not hold.

Proof: (i) If \( b \leq a \vee x \) and \( b \wedge x = 0 \), then by \( a \vee b \), \( b \leq x \). Thus \( b = b \wedge x = 0 \).

(iii) Assume that an atom \( p \) is subperspective to \( a \). If \( a \vee p \) holds then by (i) \( p = 0 \), a contradiction. Hence \( a \vee p \) does not hold. Conversely, if \( a \vee p \) does not hold, then there exist \( x, y \in L \) such that \( y \leq a \vee x \), \( y \leq p \) and \( y \not\leq x \). Since \( 0 \leq y \leq p \) and \( p \) is an atom, we have \( y = p \). Hence \( p \leq a \vee x \) and \( p \wedge x = 0 \). Thus \( p \) is subperspective to \( a \).

q.e.d.
In fact, if the underlying join-semilattice happens to be atomistic, one is led to the following characterization.

**Proposition (3.2):** Let \( L \) be an atomistic join-semilattice and \( a, b \in L \). The following statements are equivalent.

(i) \( a \lor b \).

(ii) \( a \lor p \) for any atom \( p \) with \( p \leq b \).

(iii) If \( b_1 \leq b \) and \( b_1 \) is subperspective to \( a \) then \( b_1 = 0 \).

(iv) If an atom \( p \) is subperspective to \( a \) then \( p \nmid b \).

**Proof:** (i) implies (iii): By Lemma (2.5) (ii), \( b_1 \leq b \) together with (i) implies that \( a \lor b_1 \). Hence by Lemma (3.1) (i) \( b_1 \) subperspective to \( a \) and \( a \lor b_1 \) imply \( b_1 = 0 \).

(iii) implies (iv): Suppose \( p \) is an atom subperspective to \( a \). If \( p \leq b \) then by (iii) \( p = 0 \), a contradiction. Hence \( p \nmid b \).

(iv) implies (i): Assume that \( y \leq x \lor a, y \leq b \). If \( p \) is an atom with \( p \leq y \) then by (iv), \( p \) is not subperspective to \( a \). Hence by Lemma (3.1) (ii) \( a \lor p \) holds. Thus \( p \leq x \lor a, p \leq p \) implies that \( p \leq x \). Since \( L \) is atomistic, we have \( y \leq x \). Therefore \( a \lor b \) holds.
(ii) implies (iv). Let \( p \) be an atom that is subperspective to \( a \). Then by Lemma (3.1)(ii) \( a \triangleright p \) does not hold. Hence by (ii) \( p \nmid b \), since otherwise, \( p \preceq b \) will imply that \( a \triangleright p \), a contradiction.

(iv) implies (ii). Suppose \( p \preceq b \) and \( y \preceq x\forall a, y \preceq p \).

From (iv), \( p \preceq b \) implies that \( p \) is not subperspective to \( a \). Since \( p \) is an atom, \( y \preceq p \) implies either \( y = 0 \) or \( y = p \). If \( y = 0 \) then clearly \( y \preceq x \). If \( y = p \) then \( p \preceq x\forall a \). If \( p \nmid x \) then \( p \wedge x = 0 \) and \( p \) is subperspective to \( a \), a contradiction. Thus \( p \preceq x \) i.e. \( y \preceq x \) and \( a \triangleright p \) holds.

q.e.d.

There is a well-known embedding of an atomistic lattice onto its lattice of subspaces. This embedding can be used as a tool in the study of atomistic lattices with the finite covering property; see for example Janowitz [2]. What we wish to do now is to extend the study of subspaces to join-semilattices.

Definition (3.2): Let \( L \) be an atomistic join-semilattice. A subset \( \omega \) of the atom space \( \mathfrak{A}(L) \) of \( L \) is called a subspace when it satisfies the following condition:

If \( p \in \mathfrak{A}(L), q_i \in \omega \ (i=1,\ldots,n) \) and \( p \preceq q_1 \vee \ldots \vee q_n \)
then \( p \in \omega \).
The study of subspaces very naturally leads us to the investigation of \( \vee \)-relation in certain special types of join-semilattices called finite-statistisch join-semilattices. In that direction we begin with the following.

**Definition (3.3):** Let \( \{ a_\delta; \delta \in D \} \) be a family of elements of a complete lattice \( L \), where \( D \) is a directed set. We write \( a_\delta \uparrow a \) when \( \delta_1 \leq \delta_2 \) implies \( a_{\delta_1} \leq a_{\delta_2} \) and \( a = v(a_\delta; \delta \in D) \).

A complete lattice \( L \) is called an upper continuous (or \( \wedge \)-continuous) lattice if in \( L \) \( a_\delta \uparrow a \) implies \( a_\delta \wedge b \uparrow a \wedge b \) for every \( b \).

**Definition (3.4):** An atomistic complete lattice \( L \) is called a compactly atomistic lattice when \( L \) satisfies the following condition:

If \( p \) is an atom and \( S \) is a set of atoms in \( L \) such that \( p \leq v(q; q \in S) \) then there exists a finite subset \( \{ q_1, \ldots, q_n \} \) of \( S \) such that \( p \leq q_1 \wedge \ldots \wedge q_n \).

The next proposition is also crucial for our later discussions.

**Proposition (3.3)(a):** Let \( L \) be an atomistic join-semilattice. The set \( L(\wedge(L)) \) of all subspaces of
\( \cap (L) \) is a compactly atomistic lattice ordered by set-inclusion. The meet \( \bigwedge_{\alpha} \omega(a_{\alpha}) \) is the intersection of \( \omega(a_{\alpha}) \) and the join \( \omega_1 \vee \omega_2 \) is equal to the set
\[
\{ p \in \cap (L) ; p \leq q_1 \ldots \vee q_m \vee r_1 \ldots \vee r_n , q_i \in \omega_1 , r_j \in \omega_2 \}
\]
For any \( a \in L \), the set \( \omega(a) = \{ p \in \cap (L) ; p \leq a \} \) is a subspace of \( \cap (L) \) and the mapping \( a \rightarrow \omega(a) \) is a one-to-one isotone mapping of \( L \) into \( L(\cap(L)) \). This mapping satisfies the following properties:

1. \( \omega(0) = 0 \) (\( \omega(1) = 1 \) if \( L \) has 1).
2. \( \omega(\text{avb}) \geq \omega(a) \vee \omega(b) \) for all \( a, b \in L \) and equality holds if both \( a \) and \( b \) are finite elements.
3. \( F(L(\cap(L))) \) is isomorphic to \( F(L) \), where \( F(L) \) is the set of finite elements of \( L \).

**Proof:** First we shall prove that \( L(\cap(L)) \) is compactly atomistic. Since \( L(\cap(L)) \) satisfies the three conditions in Theorem (15.3) of Maeda and Maeda [5], \( L(\cap(L)) \) forms an upper continuous lattice. Further, any singleton set is an atom of \( L(\cap(L)) \), \( L(\cap(L)) \) is atomistic. Hence by Lemma (7.13) of Maeda and Maeda [5], \( L(\cap(L)) \) is compactly atomistic lattice.

Suppose \( \omega_1 , \omega_2 \in L(\cap(L)) \). Then evidently
\[
\omega_1 \vee \omega_2 = \{ p \in \cap(L) ; p \leq q_1 \ldots \vee q_m r_1 \ldots \vee r_n , q_i \in \omega_1 , r_j \in \omega_2 \}
\]
Secondly, we show that for every $a$ in $L$, $\omega(a)$ is a subspace and the mapping $a \rightarrow \omega(a)$ is one-to-one and isotone. We have $\omega(a) = \{ p \in \bigwedge(L); p \leq a \}$ for every $a$ in $L$. Let $p$ be an atom in $L$ and $q_i \in \omega(a)$ such that $p \leq q_1 \vee \ldots \vee q_m$. Then $q_i \leq a$ for every \( i = 1, 2, \ldots, n \). Hence $q_1 \vee \ldots \vee q_m \leq a$. Thus $p \leq a$ and $p \in \omega(a)$. Therefore $\omega(a)$ is a subspace of $\bigwedge(L)$ for $a \in L$.

If $a \leq b$ then clearly $\omega(a) \subseteq \omega(b)$. Hence the mapping $a \rightarrow \omega(a)$ of $L$ into $L(\neg\neg(L))$ is order preserving. Since $L$ is atomistic, this mapping is one-to-one. Now $\omega(0) = \{ p \in \bigwedge(L); p \leq 0 \}$ = 0 of $L(\neg\neg(L))$. If $L$ has 1 then $\omega(1) = \{ p \in \bigwedge(L); p \leq 1 \}$ = $L = 1$ of $L(\neg\neg(L))$.

Now we shall prove $\bigwedge_{\alpha} \omega(a_\alpha) = \omega(\bigwedge_{\alpha} a_\alpha)$.

If $p \in \bigwedge_{\alpha} \omega(a_\alpha)$ then $p \in \omega(a_\alpha)$ for every $\alpha$.

Hence $p \leq a_\alpha$ for every $\alpha$. Therefore $p \leq \bigwedge_{\alpha} a_\alpha$. Thus $p \in \omega(\bigwedge_{\alpha} a_\alpha)$. If $p \in \omega(\bigwedge_{\alpha} a_\alpha)$ then $p \leq \bigwedge_{\alpha} a_\alpha$. Hence $p \leq a_\alpha$ for every $\alpha$. Thus $p \in \omega(a_\alpha)$ for every $\alpha$ which implies that $p \in \bigwedge_{\alpha} \omega(a_\alpha)$. Now to prove (2).

Suppose $p \in \omega(a) \vee \omega(b)$. Then $p \leq (p_1 \vee \ldots \vee p_n) \vee (q_1 \vee \ldots \vee q_m)$ where $p_i \in \omega(a)$ and $q_j \in \omega(b)$ ($i = 1, \ldots, n$, $j = 1, \ldots, m$). Since $p_i \leq a$ for
each \( i \) and \( q_j \leq p \) for every \( j \) we have \((p_1v\ldots vp_n) \leq a\) and \((q_1v\ldots q_m) \leq b\). Thus \( p \leq avb \). Hence \( p \in \omega(ab) \). Now suppose that \( a \) and \( b \) are finite. Then \( a = r_1v\ldots vr_n \) for some atoms \( r_i \) and \( b = s_1v\ldots vs_m \) for some \( s_i \in \mathcal{A}(I) \). Let \( p \in \omega(\mathcal{A} \cup \mathcal{B}) \). Then \( p \leq ab = (r_1v\ldots vr_n)v(s_1v\ldots vs_m) \) where \( r_i \in \omega(a) \) and \( s_j \in \omega(b) \). Thus \( p \in \omega(a) \cup \omega(b) \) and hence equality holds.

Now we prove that \( F(L(-\omega(L))) \) is isomorphic to \( F(L) \) by the mapping \( a \to \omega(a) \). For this, we first prove that \( \omega \) in \( L(-\omega(L)) \) is an atom if and only if \( \omega = \omega(p) \) for some atom \( p \) of \( L \). Suppose \( p \) is an atom in \( L \). Then \( \omega = \omega(p) = \{ p \} \) is an atom in \( L(-\omega(L)) \). Conversely, if \( \omega \) is an atom of \( L(-\omega(L)) \) then \( \omega \) is a singleton set. Hence \( \omega = \{ p \} = \omega(p) \) for some atom \( p \) of \( L \).

The mapping \( a \to \omega(a) \) is isomone since for \( a \leq b \) in \( F(L) \) we have \( \omega(a) \leq \omega(b) \) in \( F(L(-\omega(L))) \). Also, the mapping \( a \to \omega(a) \) is one-to-one since \( L \) is atomistic. We now show that the mapping is onto. Let \( \omega \in F(L(-\omega(L))) \). Then

\[
\omega = \omega(p_1v\ldots vp_n) = \omega(p_1v\ldots vp_n) \text{ by (2)} = \omega(a) \text{ where } p_1v\ldots vp_n = af(L).
\]
Therefore the mapping $a \rightarrow \omega(a)$ is an isomorphism between $F(L)$ and $F(L(-\Omega(L)))$.

q.e.d.

Recall the following known concept.

**Definition (3.5):** A compactly atomistic M-symmetric lattice is called a matroid lattice.

If the atomistic join-semilattice $L$ has the finite covering property then the lattice of subspaces of $\Omega(L)$ assumes a rich structure. In fact we have

**Proposition (3.3)(b):** Let $L$ be an atomistic join-semilattice with the finite covering property, then $L(-\Omega(L))$ is a matroid lattice and it is isomorphic to the lattice of all ideals of $F(L)$.

**Proof:** Suppose that $L$ is an atomistic join-semilattice with the finite covering property. Since $L(-\Omega(L))$ is a compactly atomistic lattice, it is sufficient if we show that $L(-\Omega(L))$ has the covering property. Suppose $\omega(p) \uparrow \omega$ in $L(-\Omega(L))$ and let $\omega < \omega_1 \leq \omega \lor \omega(p)$. Take an atom $q$ in $\omega_1$ with $q \not\in \omega$. Then $q \in \omega \lor \omega(p)$.

Hence there exist $r_1, \ldots, r_n$ in $\omega$ such that $q \leq r_1 \lor \ldots \lor r_n \lor p$. Since $p \not\in \omega$ we have $p \downarrow r_1 \lor \ldots \lor r_n$. Therefore by the finite covering property of $L$ we have
\begin{align*}
r_1 \ldots v_r &< p v_1 \ldots v_r \quad \text{Now } q \not\in \omega \text{ hence } \\
r_1 \ldots v_r &< q v_1 \ldots v_r \leq p v_1 \ldots v_r \quad \text{which implies } \\
q v_1 \ldots v_r = p v_1 \ldots v_r. \quad \text{Thus } p \leq q v_1 \ldots v_r, \text{ i.e. } \\
p \in \omega. \quad \text{We have, therefore, } \omega' = \omega \vee \omega(p) \text{ and } \\
L(\neg_\omega(L)) \text{ is a matroid lattice.}
\end{align*}

In the second part we shall prove that \( L(\neg_\omega(L)) \) is isomorphic to the lattice of ideals of \( F(L) \). Note that \( F(L) \) is a subsemilattice of \( L \).

For any \( \omega \in L(\neg_\omega(L)) \), let \( \phi(\omega) \) be the ideal of \( F(L) \) generated by all atoms in \( \omega \). For any ideal \( J \) of \( F(L) \) put \( \psi(J) = J \cap \neg_\omega(L) \). Then

\[ \omega \subset \phi(\omega) \cap \neg_\omega(L) = \psi(\phi(\omega)). \]

Conversely, if \( p \in \phi(\omega) \cap \neg_\omega(L) \), then by the definition of \( \phi(\omega) \) there exist \( q_1, \ldots, q_n \in \omega \) such that \( p \leq q_1 \ldots v_n \). Since \( \omega \) is a subspace of \( \neg_\omega(L) \), we have \( p \in \omega \). Hence \( \omega = \psi(\phi(\omega)) \).

Next, \( \phi(\psi(J)) = \phi(J \cap \neg_\omega(L)) \subset J \)

evidently. Since any element of \( J \) is the join of a finite number of atoms in \( J \),

we have \( \phi(\psi(J)) = J \). Therefore \( \phi \) and \( \psi \) are mutually inverse one-to-one mappings between \( L(\neg_\omega(L)) \) and the set of ideals of \( F(L) \). Also \( \phi \) and \( \psi \) preserve the order. Hence they are isomorphisms.

q.e.d.
We now introduce the concepts of "statisch" and "finite-statisch" join-semilattices. It may be seen that they are obvious extensions of the known concepts of statisch and finite-statisch lattices; see Wilke [9], Janowitz [3].

**Definition (3.6):** An atomistic join-semilattice $L$ is called **statisch** when $p \leq a \lor b$ ($p \in \bigwedge(L)$) implies $p \leq a_1 \lor b_1$ with $a_1, b_1 \in \mathcal{F}(L), a_1 \leq a, b_1 \leq b$.

$L$ is called **finite-statisch** when $p \leq a \lor q$ ($p, q \in \bigwedge(L)$) implies $p \leq a_1 \lor q$ with $a_1 \in \mathcal{F}(L), a_1 \leq a$.

We begin with a nice characterization of a statisch join-semilattice $L$ involving subspaces of $\bigwedge(L)$. It is an extension of a similar characterization of statisch lattices obtained by Janowitz [2].

**Proposition (3.4):** An atomistic join-semilattice $L$ is statisch if and only if

$$\omega(a \lor b) = \omega(a) \lor \omega(b) \text{ for all } a, b \in L.$$  

**Proof:** Suppose that $\omega(a \lor b) = \omega(a) \lor \omega(b)$ for all $a, b \in L$.

Let $p \leq a \lor b$. Then $p \in \omega(a \lor b) = \omega(a) \lor \omega(b)$ by assumption.

Hence $p \leq (q_1 \lor \ldots \lor q_n) \lor (r_1 \lor \ldots \lor r_m)$ for some $q_i \in \omega(a)$ and $r_j \in \omega(b)$. Suppose

$q_1 \lor \ldots \lor q_n = a_1$ and $r_1 \lor \ldots \lor r_m = b_1$.
Then clearly $a_1$, $b_1$ are finite elements and $a_1 \leq a$, $b_1 \leq b$ with $p \leq a_1 \lor b_1$. Thus $L$ is statisch.

Conversely, suppose that $L$ is statisch and $a$, $b \in L$. The inequality $\omega(a) \lor \omega(b) \leq \omega(a \lor b)$ is proved in Proposition (3.3)(a). Hence suppose $p \in \omega(a \lor b)$ then $p \leq a \lor b$. Since $L$ is statisch there are finite elements $a_1 \leq a$, $b_1 \leq b$ such that $p \leq a_1 \lor b_1$. Let $a_1 = p_1 \lor \ldots \lor p_n$ and $b_1 = q_1 \lor \ldots \lor q_m$.

Then $p_i \leq a$ for every $i$ and $q_j \leq b$ for every $j$. Therefore $p \in \omega(a) \lor \omega(b)$. Thus we have

$$\omega(a) \lor \omega(b) = \omega(a \lor b).$$

q.e.d.

After accomplishing a characterization of statisch join-semilattices, we take up similar consideration for finite-statisch join-semilattices. Such a characterization for finite-statisch lattices was earlier given by Janowitz [2].

**Proposition (3.5):** An atomistic join-semilattice $L$ is finite-statisch if and only if

$$\omega(a \lor b) = \omega(a) \lor \omega(b)$$

for $a \in F(L)$ and $b \in L$.

**Proof:** Suppose first that for $a \in F(L)$ and $b \in L$,

$$\omega(a \lor b) = \omega(a) \lor \omega(b)$$

holds. Let $p \leq a \lor q$ where $p, q \in \land(L)$.

Now $q$ is finite, hence by assumption,

$$p \in \omega(a \lor q) = \omega(a) \lor \omega(q).$$

Therefore

$$p \leq (p_1 \lor \ldots \lor p_n) \lor q \text{ where } p_i \leq a(i=1,\ldots,n).$$
Put \( p_1 \vee \ldots \vee p_n = a_1 \). Then \( a_1 \leq a \) and \( a \in \mathcal{F}(L) \). With \( p \leq a_1 \vee q \). Hence \( L \) is finite-statisch. Conversely, assume that \( L \) is finite-statisch and \( a \in \mathcal{F}(L) \), \( b \in L \). The inequality \( \omega(a) \vee \omega(b) \leq \omega(a \vee b) \) is trivial. We have to prove the reverse inequality. First we prove it for an atom \( q \) and an element \( b \in L \). Let \( p \leq q \vee b \). Since \( L \) is finite-statisch, there exists a finite element \( b_1 \leq b \) such that \( p \leq q \vee b_1 \), i.e. \( p \in \omega(q) \vee \omega(b) \). Thus \( \omega(q \vee b) = \omega(q) \vee \omega(b) \).

Now suppose that \( a \in \mathcal{F}(L) \). Then

\[
a = p_1 \vee \ldots \vee p_n \quad \text{for some atoms} \quad p_1, \ldots, p_n.
\]

Hence

\[
\omega(a \vee b) = \omega(p_1 \vee (p_2 \vee \ldots \vee p_n \vee b))
\]

\[
= \omega(p_1) \vee \omega(p_2 \vee (p_3 \vee \ldots \vee p_n \vee b))
\]

\[
= \ldots \ldots
\]

\[
= \omega(p_1) \vee \omega(p_2) \vee \ldots \vee \omega(p_n) \vee \omega(b)
\]

\[
= \omega(p_1 \vee \ldots \vee p_n) \vee \omega(b) \quad \text{by proposition (3.3)(a)}.
\]

i.e. \( \omega(a \vee b) = \omega(a) \vee \omega(b) \) which completes the proof.

q.e.d.

As stipulated earlier we shall use the study built up so far as a tool for investigating \( \vee \)-relation in some special types of join-semilattices.
Lemma (3.6): Let \( L \) be an atomistic join-semilattice, and let \( a, b \in L \).

(i) \( a \lor b \) in \( L \) implies \( \omega(a) \lor \omega(b) \) in \( L(\cap(L)) \)
and \( a \lor b \) implies \( \omega(a) \lor \omega(b) \).

(ii) If \( L \) is statisch, then \( \omega(a) \lor \omega(b) \) implies \( a \lor b \),
and \( \omega(a) \lor \omega(b) \) implies \( a \lor b \).

(iii) If \( L \) is finite-statistisch and if \( a \in \mathcal{F}(L) \), then
\( \omega(a) \lor \omega(b) \) implies \( a \lor b \).

Proof: (i) Assume \( a \lor b \) and let
\( \omega \in L(\cap(L)) \). Trivially,
\[ \omega \land \omega(b) < (\omega \lor \omega(a)) \lor \omega(b). \]

Now suppose \( p \in (\omega \lor \omega(a)) \lor \omega(b) \). Then there exist
\( q_1 \in \omega \) \((i=1, \ldots, n)\) such that \( p \leq q_1 \lor \ldots \lor q_n \lor a \). Also
\( p \leq b \). Hence by \( a \lor b \) we have \( p \leq q_1 \lor \ldots \lor q_n \lor a \). \( \text{i.e. } p \in \omega \).
Thus \( p \in \omega \lor \omega(b) \), which proves that
\( \omega(a) \lor \omega(b) \) in \( L(\cap(L)) \).

Next suppose that \( a \lor b \) in \( L \) and \( \omega \in L(\cap(L)) \).
Evidently, \( \omega \leq (\omega \lor \omega(a)) \land (\omega \lor \omega(b)) \). Let \( p \) be an
atom and \( p \in (\omega \lor \omega(a)) \land (\omega \lor \omega(b)) \). Then \( p \notin \omega \lor \omega(a) \)
and \( p \notin \omega \lor \omega(b) \). Hence there exist \( p_i \in \omega \) \((i=1, \ldots, n)\)
and \( q_j \in \omega \) \((j=1, \ldots, m)\) such that
\[ p \leq p_1 \ldots p_n \text{ and } p \leq q_1 \ldots q_m \text{.} \]

Thus \[ p \leq (p_1 \ldots p_n q_1 \ldots q_m) \text{ and } p \leq (p_1 \ldots p_n q_1 \ldots q_m) \text{.} \]

Therefore, by \( a \not\leq b \), we have \[ p \leq p_1 \ldots p_n q_1 \ldots q_m \text{ i.e. } p \not\in \omega \text{, which proves } \omega(a) \nabla \omega(b) \text{.} \]

(ii) Assume that \( L \) is statisch and \( \omega(a) \nabla \omega(b) \) holds.

Let \( y \leq x \vee a \) and \( y \leq b \). Then

\[ \omega(y) \leq \omega(x \vee a) \wedge \omega(b) = (\omega(x) \vee \omega(a)) \wedge \omega(b) \]

by Proposition (3.4), and by \( \omega(a) \nabla \omega(b) \), we have

\[ \omega(y) \leq \omega(x) \wedge \omega(b) \leq \omega(x) \text{.} \]

Thus \( y \leq x \) and we have \( a \nabla b \). Now suppose that \( \omega(a) \nabla \omega(b) \) and let

\[ y \leq x \vee a \text{, } y \leq x \vee b \text{.} \]

Then by Proposition (3.4),

\[ \omega(y) \leq \omega(x \vee a) = \omega(x) \vee \omega(a) \text{ and } \]

\[ \omega(y) \leq \omega(x \vee b) = \omega(x) \vee \omega(b) \text{.} \]

Thus \( \omega(y) \leq (\omega(x) \vee \omega(a)) \wedge (\omega(x) \vee \omega(b)) \) which together with \( \omega(a) \nabla \omega(b) \) implies that \( \omega(y) \leq \omega(x) \text{, i.e. } y \leq x \text{ and we get } a \not\leq b \).

(iii) Now assume that \( L \) is finite-statistisch. Let \( a \not\in F(L) \) and \( \omega(a) \nabla \omega(b) \). If \( y \leq x \vee a \), \( y \leq b \) then \( \omega(y) \leq \omega(b) \).
and $\omega(y) \leq \omega(x) \lor \omega(a)$ by Proposition (3.5).
Hence $\omega(y) \leq (\omega(x) \lor \omega(a)) \land \omega(b) = \omega(x) \land \omega(b) \leq \omega(x)$.
Thus $\omega(y) \leq \omega(x)$ and hence $y \leq x$ which proves $a \lor b$.

q.e.d.

Our considerations subsume as a special case the following result of Janowitz [2].

Result: Let $L$ be an atomistic lattice, and let $a, b \in L$.
Then:

1. $a \lor b$ in $L$ implies $\omega(a) \lor \omega(b)$ in $L(\neg \neg (L))$.
2. If $L$ is statisch, then $\omega(a) \lor \omega(b)$ in $L(\neg \neg (L))$
implies $a \lor b$ in $L$.
3. If $L$ is finite-statistisch and if $a$ is finite, then
$\omega(a) \lor \omega(b)$ in $L(\neg \neg (L))$ implies $a \lor b$ in $L$.

Let us now introduce two novel notions namely what we shall call Property (A) and Property (A_f).

Definition (3.7): For an atomistic join-semilattice $L$, we consider the following two properties.

(A) $\omega(a) \lor \omega(b) \Rightarrow a \lor b$.

(A_f) $a \in F(L)$, $\omega(a) \lor \omega(b) \Rightarrow a \lor b$.

By Lemma (3.6), if $L$ is statisch (respectively finite-statistisch) then $L$ has the Property (A) (respectively Property (A_f)).
First we obtain characterizations of Property \((A_f)\).

For that we shall need the following lemma.

**Lemma (3.7)**: Let \( L \) be an atomistic join-semilattice and let \( p, q \) be atoms. \( p \) is subperspective to \( q \) in \( F(L) \) if and only if \( \omega(p) \) is subperspective to \( \omega(q) \) in \( L(\underline{\omega}(L)) \).

**Proof**: Suppose \( p \) is subperspective to \( q \) in \( F(L) \), then there is \( x \in F(L) \) such that \( p \leq q \vee x \) and \( p \wedge x = 0 \).

Hence \( \omega(p) \leq \omega(q \vee x) = \omega(q) \vee \omega(x) \) by Proposition 3.3(a)(2), and \( \omega(p) \wedge \omega(x) = 0 \). Thus \( \omega(p) \) is subperspective to \( \omega(q) \) in \( L(\underline{\omega}(L)) \). Conversely, suppose that \( \omega(p) \) is subperspective to \( \omega(q) \) in \( L(\underline{\omega}(L)) \). Then there exists \( \omega \) in \( L(\underline{\omega}(L)) \) such that \( \omega(p) \leq \omega(q) \vee \omega \) and \( \omega(p) \wedge \omega = 0 \).

Since \( p \notin \omega(q) \vee \omega \), there exist \( r_1 \in \omega \) such that \( p \leq q \vee r_1 \vee \ldots \vee r_n \). Putting \( x = r_1 \vee \ldots \vee r_n \), we have \( x \in F(L) \) and \( x \notin \omega \). Since \( p \notin \omega \), we have \( p \wedge x = 0 \). Thus \( p \) is subperspective to \( q \) in \( F(L) \).

**q.e.d.**

Now the stipulated characterizations of the Property \((A_f)\)

**Theorem (3.8)**: For an atomistic join-semilattice \( L \), each of the following three statements is equivalent to the Property \((A_f)\).
(A_f1) \( p \in \Omega(L) \), \( \omega(p) \vee \omega(b) \) implies \( p \vee b \).

(A_f2) \( p, q \in \Omega(L) \), \( \omega(p) \vee \omega(q) \) implies \( p \vee q \).

(A_f3) If \( p, q \in \Omega(L) \) and if \( p \) is subperspective to \( q \) then \( p \) is subperspective to \( q \) in \( F(L) \).

**Proof:** (A_f) implies (A_f1):

Since every atom is in \( F(L) \), the proof is obvious.

(A_f1) implies (A_f2) is clear.

(A_f2) implies (A_f3):

Suppose \( p, q \in \Omega(L) \) and \( p \) is subperspective to \( q \). Then by Lemma (3.1) (ii), \( q \vee p \) does not hold and hence by (A_f2) \( \omega(q) \vee \omega(p) \) does not hold. Therefore \( \omega(p) \) is subperspective to \( \omega(q) \) and by Lemma (3.7) we have \( p \) subperspective to \( q \) in \( F(L) \).

(A_f3) implies (A_f2): Suppose \( p, q \in \Omega(L) \) and \( \omega(p) \vee \omega(q) \). By Lemma (3.1)(ii), \( \omega(q) \) is not subperspective to \( \omega(p) \). Hence by Lemma (3.7), \( q \) is not subperspective to \( p \) and again by Lemma (3.1) (ii), \( p \vee q \) holds.

Finally we shall show that (A_f2) implies (A_f).

Suppose at \( F(L) \) and \( \omega(a) \vee \omega(b) \). Then \( a = p_1 \vee \ldots \vee p_n \) \( (p_1 \in \Omega(L)) \). If \( q \in \Omega(L) \) and \( q \leq b \) then we have \( \omega(p_1) \vee \omega(q) \) which, together with (A_f2), implies that \( p_1 \vee q \) for every \( i \). Hence \( a \vee q \) by Lemma (2.5) (iii), and we have \( a \vee b \) by Proposition 3.2.

q.e.d.
There is a very interesting consequence of the result that we just proved. The lemma given below is a generalization of Lemma (11.1) of Maeda and Maeda [5].

Lemma (3.9): Let $p$ and $q$ be atoms of a join-semilattice $L$ with the covering property. The following three statements are equivalent.

1) $p \bowtie q$

2) $q$ is subperspective to $p$

3) $p \lor q$ does not hold.

Proof: The equivalence of (2) and (3) follows from Lemma (3.1)(ii).

(1) implies (2) is trivial from definitions.

(2) implies (1): Suppose $q$ is subperspective to $p$. Then there exists an element $x$ in $L$ such that $q \leq p \lor x$ and $q \land x = 0$. We also have $p \land x = 0$; for otherwise, $q \leq x \lor p = x$, a contradiction to $q \land x = 0$. Therefore by the covering property we have $x \leq x \lor p$, which together with $x \leq x \lor q \leq x \lor p$ implies that $x \lor p = x \lor q$. Hence $p \bowtie q$.

q.e.d.

Janowitz [2] introduced a nice notion called $p$-compatible $\Box$-lattice in order to build up a structure theory for certain classes of lattices. We generalize the said concept to $\Box$-join-semilattices.
Definition (3.8): A join-semilattice $L$ with $0$ is called $p$-compatible if for atoms $p, q, p \sqcup q$ in $L$ implies $p \sim q$ in $F(L)$.

After building up so much jargon, we now state the consequence of Theorem (3.8).

Lemma (3.10): Suppose that $L$ is an AC-join-semilattice. Then the following statements are equivalent:

1) $L$ has the property $(A_f^3)$

2) $L$ is $p$-compatible.

Proof: If $L$ is $p$-compatible and if for atoms $p, q$ we have $p$ subperspective to $q$ then, by $p$-compatibility, $p$ is subperspective to $q$ in $F(L)$. This proves $(A_f^3)$.

Conversely, if $(A_f^3)$ holds then by Lemma (3.9) $L$ is $p$-compatible.

q.e.d.

Let us also brush up some of the known concepts.

Firstly,

Definition (3.9). An element $z$ of a lattice $L$ with $0$ and $1$ is called a central element when there exist two lattices $L_1$ and $L_2$ and an isomorphism between $L$ and the direct product $L_1 \times L_2$ such that $z$ corresponds to the element $[1, 0_2] \in L_1 \times L_2$. Clearly $0$ and $1$ are
central elements. The set of all central elements of $L$ is called the center of $L$ and is denoted by $Z(L)$.

The next two concepts are also highly well known.

**Definition (3.10):** A complete lattice $L$ is called a $Z$-lattice if $L$ satisfies the following conditions:

(i) The center $Z(L)$ is a complete sublattice of $L$.

(ii) If $z_i \in Z(L)$ for every $i \in I$, then $\bigwedge \{z_i; i \in I\} \wedge a = \bigwedge \{z_i \wedge a; i \in I\}$ for every $a \in L$.

**Definition (3.11):** Let $a$ be an element of a $Z$-lattice $L$. It follows from (i) of the above definition that there exists a unique least central element $z$ such that $a \leq z$. We call it the central cover of $a$ and denote it by $e(a)$.

Let us discuss these concepts in the context of $L(\neg \neg (L))$.

Let $L$ be an atomic join-semilattice. It can be seen that $L(\neg \neg (L))$ is a $Z$-lattice. And as such we can define the central cover $e(\omega)$ for every $\omega \in L(\neg (L))$.

Let us consider the following two properties.

(B) $e(\omega(a)) \wedge e(\omega(b)) = 0$ implies $a \vee b$.

(E') $a \neq F(L)$, $e(\omega(a)) \wedge e(\omega(b)) = 0$ implies $a \vee b$. 
It is possible to obtain the following

\textbf{Lemma (3.11)}: \( e(\omega_1) \land e(\omega_2) = 0 \implies \omega_1 \triangledown \omega_2 \implies \omega_1 \lor \omega_2 \)

for \( \omega_1, \omega_2 \) in \( \mathcal{L}(\Omega(\mathcal{L})) \).

We obtain now several equivalent results involving central covers of subspaces of \( \mathcal{L} \) and 'del' relation.

\textbf{Lemma (3.12)}. Let \( a \) be an element of an atomistic join-semilattice \( \mathcal{L} \). The following statements for \( a \in \mathcal{L} \) are equivalent.

(i) \( e(\omega(a)) \land e(\omega(x)) = 0 \) implies \( a \triangledown x \).

(ii) \( p \in \Omega(\mathcal{L}), e(\omega(a)) \land e(\omega(p)) = 0 \) implies \( a \triangledown p \).

(iii) \( \omega(\mathcal{a} \triangledown x) \leq e(\omega(a)) \lor \omega(x) \) for every \( x \) in \( \mathcal{L} \).

(iv) If \( p \in \Omega(\mathcal{L}) \) is subperspective to \( a \) then \( p \) belongs to \( e(\omega(a)) \).

\textbf{Proof}: (i) implies (ii) is obvious.

(ii) implies (iii): Let \( p \in \omega(\mathcal{a} \triangledown x) \). If \( p \) is not in \( e(\omega(a)) \) then,

\[ e(\omega(a)) \land e(\omega(p)) = e(e(\omega(a)) \land \omega(p)) = 0 \]

by Lemma (5.11) of Maeda and Maeda [5], and hence \( a \triangledown p \) by (ii). Since \( p \leq \mathcal{a} \triangledown x \) we have \( p \leq x \). Hence \( p \in e(\omega(a)) \lor \omega(x) \).
(iii) implies (i): Assume \( e(\omega(a)) \land e(\omega(x)) = 0 \)
and let \( u \leq y \lor a, u \leq x \). By (iii), we have
\[
\omega(u) \leq \omega(ya) \land \omega(x) \leq (e(\omega(a)) \lor \omega(y)) \land \omega(x)
= \omega(y) \land \omega(x) \leq \omega(y).
\]
Hence \( u \leq y \). Therefore \( a \nabla x \). Thus (i), (ii) and (iii) are equivalent.

(ii) implies (iv): Suppose \( p \notin \neg(L) \) and \( p \) is sub- 
perspective to \( a \). Hence by Lemma (3.1) (ii), \( a \nabla p \) does not
hold. Hence by (ii), \( e(\omega(a)) \land e(\omega(p)) \neq 0 \). Hence \( p \)
is in \( e(\omega(a)) \).

(iv) implies (ii): Suppose \( a \nabla p \) does not hold. Then
by Lemma (3.1)(ii), \( p \) is subperspective to \( a \). Hence
by (iv) \( p \) belongs to \( e(\omega(a)) \). Therefore
\( e(\omega(a)) \land e(\omega(p)) \neq 0 \). Hence (ii) is proved.

q.e.d.

This lemma is helpful in obtaining several
characterizations of Property \(-\text{(B)}\) that we introduced
earlier. Simultaneously we also succeed in obtaining several
equivalent formulations of the related Property \((B_f)\).

Theorem (3.13): Let \( L \) be an atomistic join-semilattice.

(i) Each of the following four statements is equivalent
to \( (B) \).
$$(B_1) \ p \not\in \Omega(L), \ e(\omega(a)) \land e(\omega(p)) = 0 \ implies \ a \not\triangleright p.$$  

$$(B_2) \ \omega(a \lor b) \leq e(\omega(a)) \lor \omega(b) \ for \ all \ a, b \ in \ L.$$  

$$(B_3) \ For \ any \ a \ in L, \ if \ an \ atom \ p \ is \ subperspective \ to \ a \ then \ p \ belongs \ to \ e(\omega(a)).$$  

$$(B_4) \ e(\omega(a)) \land e(\omega(b)) = 0 \ implies \ a \not\triangleright b.$$  

$$(B_5) \ Each \ of \ the \ following \ seven \ statements \ is \ equivalent \ to \ (B_4).$$  

$$(B_6) \ p \not\in \Omega(L), \ e(\omega(p)) \land e(\omega(b)) = 0 \ implies \ p \not\triangleright b.$$  

$$(B_7) \ a \not\in F(L), \ q \not\in \Omega(L), \ e(\omega(a)) \land e(\omega(q)) = 0 \ implies \ a \not\triangleright q.$$  

$$(B_8) \ For \ atoms \ p, q, \ e(\omega(p)) \land e(\omega(q)) = 0 \ implies \ p \not\triangleright q.$$  

$$(B_9) \ \omega(a \lor b) \leq e(\omega(a)) \lor \omega(b) \ for \ finite \ element \ a \ and \ b \ in \ L.$$  

$$(B_{10}) \ \omega(p \lor b) \leq e(\omega(p)) \lor \omega(b) \ for \ an \ atom \ p \ and \ b \ in \ L.$$  

$$(B_{11}) \ For \ a \ finite \ element \ a, \ an \ atom \ q \ is \ subperspective \ to \ a \ then \ q \ belongs \ to \ e(\omega(a)).$$  

$$(B_{12}) \ For \ p, q \ atoms, \ if \ q \ is \ subperspective \ to \ p \ then \ q \ belongs \ to \ e(\omega(p)).$$
Proof: (i) \((B_1), (B_2), (B_3)\) and \((B_4)\) are equivalent by Lemma (3.4.3). \((\overline{B}) \Rightarrow (B)\) is obvious since \(a \overline{\lor} b \Rightarrow a \lor b\).

\((B_2) \Rightarrow (\overline{B}).\) Assume \(e(\omega(a)) \land e(\omega(b)) = 0\) and let \(y \leq x \lor a, y \leq x \lor b\). By \((B_2)\) we have

\[
\omega(y) \leq \omega(x \lor a) \land \omega(x \lor b) \leq (e(\omega(a)) \lor \omega(x)) \land (e(\omega(b)) \lor \omega(x))
\]

\[
= (e(\omega(a)) \land e(\omega(b))) \lor \omega(x) = \omega(x),
\]

whence \(y \leq x\). Therefore \(a \overline{\lor} b\).

(ii) \((B_f), (B_{f2}), (B_{f4})\) and \((B_{f6})\) are equivalent by Lemma (3.12), and moreover \((B_{f1}), (B_{f3}), (B_{f5})\) and \((B_{f7})\) are equivalent. \((B_{f4}) \Rightarrow (B_{f5})\) is obvious.

\((B_{f5})\) implies \((B_{f4})\): Suppose \(a \in \mathcal{F}(L)\). Then \(a = p_1 \lor \ldots \lor p_n\) for some \(p_i \in \mathcal{M}(L)\). By \((B_{f5})\) we have

\[
\omega(a \lor b) \leq e(\omega(p_1)) \lor \omega(p_2 \lor \ldots \lor p_n \lor b)
\]

\[
\leq \ldots \leq e(\omega(p_1)) \lor \ldots \lor e(\omega(p_n)) \lor \omega(b)
\]

\[
= e(\omega(a)) \lor \omega(b).
\]

q.e.d.

In a \(\overline{\lor}\)-symmetric atomistic join-semilattice it follows that Property \((B)\) is general than the Property \((B_f)\).

Corollary (3.14): If an atomistic join-semilattice \(L\) is \(\overline{\lor}\)-symmetric and has the Property \((B_f)\) then \(L\) has the Property \((B)\).
Proof: Since $L$ satisfies $(B_f)$ and is $\vee$-symmetric, it satisfies $(A_f)$.

q.e.d.

The equivalence respectively of $(A)$ and $(B)$, and of $(A_f)$ and $(B_f)$ is discussed in the following:

Theorem (3.15): Let $L$ be an atomistic join-semilattice with the finite covering property

(i) For $L$, $(A)$ and $(B)$ are equivalent, and so are $(A_f)$ and $(B_f)$.

(ii) $L$ has $(A)$ if and only if $L$ is $\vee$-symmetric and has $(A_f)$.

Proof: (i) As we have proved in Proposition (3.3)(b), $L(\omega(L))$ is a matroid lattice. By Theorem (13.5) of Maeda and Maeda [5], we have $e(\omega_1) \wedge e(\omega_2) = 0$ if and only if $\omega_1 \vee \omega_2 = (\omega_1 \wedge \omega_2)^L(\omega(L))$. Hence if $(A)$ holds then $e(\omega(a)) \wedge e(\omega(b)) = 0$ implies $\omega(a) \vee \omega(b) \Rightarrow a \vee b$, which is $(B)$. Similarly, if $(B)$ holds and $\omega(a) \vee \omega(b)$ then $e(\omega(a)) \wedge e(\omega(b)) = 0 \Rightarrow a \vee b$ which proves $(A)$. On the same lines we can prove that $(A_f)$ and $(B_f)$ are equivalent.

(ii) If $L$ is $\vee$-symmetric and has $(A_f)$ then $L$ has $(B)$ by corollary (3.14) and hence by (i) $L$ has $(A)$. 
conversely, if \( L \) has (A) then \( L \) is \( \vee \)-symmetric since \( L( -\wedge(L) ) \) is \( \vee \)-symmetric by Theorem (13.5) of Maeda and Maeda \([5]\).
q.e.d.

In passing we also list the following rather interesting consequence.

**Lemma (3.16)**: Let \( L \) be an atomistic join-semilattice with the finite covering property. If \( L \) has (A) then the following five statements for \( a, b \) in \( L \) are equivalent.

(i) \( a \vee b \), (ii) \( a \wedge b \),

(iii) \( \omega(a) \vee \omega(b) \), (iv) \( \omega(a) \wedge \omega(b) \),

(v) \( e(\omega(a)) \wedge e(\omega(b)) = 0 \).

**Proof**: Combining Proposition (2.2), Lemma (3.6), Theorem (3.13), Theorem (3.15) we get the proof.
q.e.d.

As a consequence of our considerations, we finally observe the following

**Result (3.17)**: Let \( L \) be an atomistic join-semilattice with the finite covering property, and let \( \alpha \in L \), \( p \in -\wedge(L) \).

The following statements are equivalent.

(i) \( p \in e(\omega(a)) \).

(ii) \( p \) is subperspective in \( P(L) \) to some atom under \( a \).
(iii) $p$ is perspective in $F(L)$ to some atom under $a$

Proof: Since $F(L)$ has the covering property,
(ii) and (iii) are equivalent by Lemma (3.9).

(i) implies (ii): Suppose $p \in e(\omega(a))$. Then by
Theorem (13.5) of Maeda and Maeda [5], $e(p) \wedge e(\omega(a)) \neq 0$
implies that there exists some atom $q$ in $e(\omega(a))$ such
that $\omega(p) \sim \omega(q)$. Hence there is a subspace $\omega$ such that

$$\omega(p) \vee \omega = \omega(q) \vee \omega \text{ and } \omega(p) \wedge \omega = \omega(q) \wedge \omega = 0.$$ 

Then $p \in \omega(q) \vee \omega$. Therefore, there are atoms $r_1, \ldots, r_n$
in $\omega$ such that $p \leq q \vee r_1 \vee \ldots \vee r_n$. Put $x = r_1 \vee \ldots \vee r_n$. Then
$x$ is a finite element of $L$, $p \leq qx$ and $p \wedge x = 0 = q \wedge x$.
Since $L$ has the finite covering property, $x < px \leq qx$
implies $px = qx$. Thus $p$ is subperspective to $q$ in
$F(L)$, so $p$ is subperspective in $F(L)$ to some atom under $a$.

(ii) implies (i): Suppose $p, q$ are atoms, $q \leq a$. If $p$
is subperspective to $q$ in $F(L)$, then for some $x$ in $F(L)$,
$p \leq qx$ and $p \wedge x = 0$. Then

$$\omega(p) \leq \omega(q \vee x) = \omega(q) \vee \omega(x) \text{ and } \omega(p \wedge x) = \omega(p) \wedge \omega(x) = 0.$$ 

Hence $\omega(p)$ is subperspective to $\omega(q)$ in $L(\wedge(L))$.

q.e.d.
4. $\vee$-standard elements and standard elements.

The concept of distributive triple is quite well known in the literature. Let us have the concept of distributive pair which can be introduced as follows.

Definition (4.1). Let $L$ be a join-semilattice. A pair of elements $a, b$ of $L$ is called distributive and is denoted by $(a, b)_D$ when

$$(D): x \leq avb \text{ implies the existence of } a_1, b_1 \text{ in } L \text{ such that } a_1 \leq a, b_1 \leq b \text{ and } a_1v b_1 = x.$$  

We relate distributive pair with dual modular pair in the following.

Proposition (4.1): Let $L$ be a join-semilattice and $a, b$ in $L$. Then $(a, b)_D$ implies $(a, b)_M^*.$

Proof: Suppose $(a, b)_D$ holds. Let $b \leq x \leq avb$. Then by $(a, b)_D$, $x \leq avb$ implies that there exist $a_1, b_1$ in $L$ such that $a_1 \leq a, b_1 \leq b$ and $a_1v b_1 = x$. Put $d = a_1$. Then $d \leq a$ and

$$dvb = a_1vb = a_1v b_1v b = xvb = x.$$  

Hence $(a, b)_M^*$ holds. q.e.d.
The notion of standard ideals goes back to Grätzer and Schmidt [1]. This has been abstracted to standard elements in lattice. By using distributive pair, we can introduce the interesting concept of standard elements in join-semilattices.

**Definition (4.2):** An element \( s \) of a join-semilattice \( L \) is called standard if \( (s,x) \in D \) for every \( x \) in \( L \).

Next, we have

**Proposition (4.2):** If \( L \) is a lattice then (D) is equivalent to the following condition:

\[
(a \lor b) \land x = (a \land x) \lor (b \land x) \quad \text{for every } x \text{ in } L.
\]

**Proof:** Suppose first that (D) holds. Let \( y = (a \lor b) \land x \).

Then \( y \leq a \lor b \), \( y \leq x \). Hence by (D) there exist \( a_1 \leq a \), \( b_1 \leq b \) such that \( y = a_1 \lor b_1 \). Now \( a_1 \leq y \), \( b_1 \leq y \). Hence \( b_1 \leq x \), \( a_1 \leq x \). Therefore \( a_1 \leq a \land x \), \( b_1 \leq b \land x \). Thus

\[
(a \lor b) \land x = a_1 \lor b_1 \leq (a \land x) \lor (b \land x) \leq (a \lor b) \land x
\]

and we get \( (a \lor b) \land x = (a \land x) \lor (b \land x) \). Conversely, suppose that the given condition holds. Let \( x \leq a \lor b \). Then

\[
x \land (a \lor b) = x.
\]

Hence by assumption

\[
x = x \land (a \lor b) = (x \land a) \lor (x \land b).
\]

Put \( a_1 = a \land x \), \( b_1 = b \land x \) so that \( a_1 \leq a \), \( b_1 \leq b \)

and \( x = a_1 \lor b_1 \). Thus (D) holds.

\( \text{q.e.d.} \)
Hence in a lattice, our definition of standardness coincides with that in a join-semilattice.

Let us also introduce a related concept.

**Definition (4.3).** An element $s$ of a join-semilattice $L$ with 0 is called $\vee$-standard when $s \land a = 0$ implies $s \vee a$.

It is observed that the join of $\vee$-standard elements is $\vee$-standard.

**Proposition (4.3):** If $s_1$ and $s_2$ are $\vee$-standard then so is $s_1 \vee s_2$.

**Proof:** Suppose $s_1$ and $s_2$ are $\vee$-standard and $(s_1 \vee s_2) \land a = 0$. Then $s_1 \land a = 0$ and $s_2 \land a = 0$. Hence $s_1 \vee a$ and $s_2 \vee a$ which imply $(s_1 \vee s_2) \vee a$.

q.e.d.

Following is a known concept in lattices. However, we give it in the context of join-semilattices.

**Definition (4.4):** A join-semilattice $L$ with 0 is called **SSC** when it satisfies the following condition:

If $a < b$ in $L$ then there exists $c \in L$ such that $0 < c \leq b$ and $c \land a = 0$.

We state sufficient condition for an atomistic join-semilattice to be **SSC**.
**Proposition (4.4):** If a join-semilattice \( L \) is atomistic then it is SSC.

**Proof:** Suppose \( a \preceq b \). Then by definition of atomistic join-semilattice there exists an atom \( p \) in \( L \) such that \( p \preceq b \) and \( p \nless a \). Then \( p \land a = 0 \) and \( 0 < p \preceq b \). Thus \( L \) is SSC.

q.e.d.

In fact it turns out that every standard element is \( \lor \)–standard. For atomistic join-semilattice both these concepts coincide.

**Lemma (4.5):** If an element \( s \) of a join-semilattice \( L \) with \( 0 \) is standard then \( s \) is \( \lor \)–standard. The converse is true in the case that \( L \) is a lattice and is SSC (especially, \( L \) is an atomistic lattice).

**Proof:** (i) Assume that \( L \) is a join-semilattice with \( 0 \) and let \( s \in L \) be a standard element. Suppose \( s \land a = 0 \). If \( y \preceq x \lor s \lor a \) then by \( (s, x)D \) there exist elements \( x_1, s_1 \) in \( L \) such that \( s_1 \preceq s \), \( x_1 \preceq x \) and \( s_1 \lor x_1 = y \). Since \( s_1 \preceq y \preceq a \) and \( s_1 \preceq s \) we have \( s_1 \preceq s \land a = 0 \). Thus \( s_1 = 0 \). Hence \( y = x_1 \preceq x \). This proves that \( s \) is \( \lor \)–standard.

(ii) Now suppose that \( L \) is SSC lattice and let \( s \) be \( \lor \)–standard. If \( s \) were not standard, there would exist \( a, b \) in \( L \) such that \( (s \lor a) \land b > (s \land b) \lor (a \land b) \). But
since $L$ is SSC, there exists an element $c$ in $L$ such that $0 < c < (s \wedge a) \wedge b$ and $c \wedge (s \wedge b) \vee (a \wedge b) = 0$.

Now $c < b$ implies that $s \wedge c = (s \wedge b) \wedge c = 0$ and hence $s \nless c$. Now $(a \vee s) \wedge c = a \wedge c = (a \wedge b) \wedge c = 0$. But $c < a \vee s$.

Therefore $c = 0$, a contradiction. Hence $s$ must be standard.

q.e.d.

Let us discuss the concept of $\nabla$-standardness in the context of $\hat{L}$, the completion by cuts of $L$.

Remark (4.6): An element $s$ of a join-semilattice $L$ with $0$ is $\nabla$-standard if and only if $J_s$ is $\nabla$-standard in $\hat{L}$.

Proof: Assume that $s$ is $\nabla$-standard and we shall show that $J_s$ is $\nabla$-standard, that is, $J_s \wedge A = \{0\}$ implies $(X \vee J_s) \wedge A \leq X$. If $a \in (X \vee J_s) \wedge A$, then $s \wedge a = 0$ since $J_s \wedge A = 0$. Hence $s \nless a$. For any $x \in X$, we have $x \vee s$ is in $(X \vee J_s)^u$ and $a \in (X \vee J_s)^u$ implies $a \leq x \vee s$. But we have $s \nless a$, which gives $a \leq x$, and $a \in X^u$ implies $x \in X$. Therefore $(X \vee J_s) \wedge A \leq X$. Also $(X \vee J_s) \wedge A \leq A$. Thus $(X \vee J_s) \wedge A \leq X \wedge A$.

The reverse inequality is obvious. This proves that $J_s$ is $\nabla$-standard.

Conversely suppose that $J_s$ is $\nabla$-standard.

Let $s \wedge a = 0$. Then $J_s \wedge a = J_s \wedge J_a = 0$. Therefore $J_s \nless J_a$ in $\hat{L}$. Hence by Proposition (2.7)(1), $s \nless a$ in $L$.

q.e.d.
We obtain several characterizations of $\vee$-standard elements.

**Proposition (4.7):** Let $s$ be an element of an atomistic join-semilattice $L$. The following statements are equivalent.

(i) $s$ is $\vee$-standard.

(ii) If $p$ is an atom and $p \nleq s$ then $s \vee p$.

(iii) If $p$ is an atom, $p$ subperspective to $s$, then $p \leq s$.

(iv) For every $x$ in $L$, $\omega(s \vee x) = \omega(s) \cup \omega(x)$

(i.e. $p \leq s \vee x$ implies $p \leq s$ or $p \leq x$).

**Proof:** (i) implies (ii): Suppose $s$ is $\vee$-standard and $p$ is an atom with $p \nleq s$. Then $s \land p = 0$. Hence by (i), $s \vee p$ holds.

(ii) implies (iii): Suppose $p$ is an atom and $p$ is subperspective to $s$. Then by Lemma (3.1) (i), $s \vee p$ does not hold. Therefore by (ii), $p \leq s$.

(iii) implies (ii): Suppose $p$ is an atom and $p \nleq s$. Then by (iii), $p$ is not subperspective to $s$, and by Lemma (3.1) (ii), $s \vee p$ holds.

(iii) implies (iv): Suppose $p \leq s \vee x$. If $p \nleq \omega(x)$, then $p \land x = 0$. Now $p \leq s \vee x$ and $p \land x = 0$. That is $p$ is subperspective to $s$. Hence, by (iii), $p \leq s$, i.e. $p \in \omega(s)$. Thus we have proved $\omega(s \vee x) \subseteq \omega(s) \cup \omega(x)$. The reverse inequality is obvious, and we have (iv).
(iv). implies (i): Suppose \( s \land a = 0 \) and let \( y \leq x \vee s \), \( y \leq a \). If \( p \leq y \) then \( p \leq x \vee s \), i.e. \( p \in \omega(x \vee s) = \omega(s) \cup \omega(x) \) by (iv). Since \( p \leq y \leq a \) we have \( p \nsubseteq s \). Therefore \( p \in \omega(x) \), i.e. \( p \leq x \). Since \( L \) is atomistic, this proves that \( y \leq x \) and hence \( s \nleq a \) holds.

q.e.d.

We explore, now, the relationship of \( \nleq \) -standard elements in \( L \) with its respective subspace \( \omega(s) \).

**Proposition (4.8):** Let \( s \) be an element of an atomistic join-semilattice \( L \).

(i) If \( s \) is \( \nleq \) -standard then \( \omega(s) \) is standard in \( L(\nleq L) \). If \( L \) has the finite covering property and if \( s \) is \( \nleq \) -standard then \( \omega(s) \) is central in \( L(\nleq L) \).

(ii) If \( L \) has the Property (A) (respectively (B)) and if \( \omega(s) \) is standard (respectively central) in \( L(\nleq L) \) then \( s \) is \( \nleq \) -standard.

**Proof:** (i) Suppose \( s \) is \( \nleq \) -standard. If \( \omega(p) \nsubseteq \omega(s) \), then since \( p \nsubseteq s \) we have \( s \nleq p \). Hence, \( \omega(s) \nleq \omega(p) \) by Lemma (3.6)(i). Therefore, \( \omega(s) \) is \( \nleq \) -standard by Proposition (4.7), whence \( \omega(s) \) is standard by Lemma (4.5). If \( L \) has the finite covering property, then \( L(\nleq L) \) is a matroid lattice. We can prove that any standard element \( a \) of a matroid lattice is central. In fact, if
p \nmid a$, then $p$ is not subperspective to $a$ and hence $p \nmid e(a)$. by Theorem (13.5) of Maeda and Maeda [5]. Thus $a = e(a)$.

(ii) If $L$ has (A) (respectively (B)) and $\omega(s)$ is standard (respectively central), then for any $p \nmid s$, we have $\omega(s) \lor \omega(p)$ (respectively $e(\omega(s)) \land e(\omega(p)) = 0$). Hence $s \lor p$ and thus $s$ is $\lor$-standard.

q.e.d.

Here is a nice characterization which substantially generalizes a similar result of Maeda and Maeda [5] for lattices. This result involves $P$-relation that we discussed in Chapter-I.

Theorem (4.9) : Let $L$ be an atomistic join-semilattice. The following statements are equivalent.

(i) Every atom of $L$ is $\lor$-standard.

(ii) Every finite element of $L$ is $\lor$-standard.

(iii) If $p, q$ are atoms and $p$ is subperspective to $q$ then $p = q$.

(iv) If $p$ is an atom, $a \in F(L)$ and $p$ is subperspective to $a$ then $p \leq a$.

(v) $(p, x)P$ for every atom $p$ and $0 \neq x \in L$ and

$\omega(p, x) = \{p, q\}$ for every $p, q$ in $F(L)$ with $p \nmid q$.

(this means that every line contains exactly two points).
Proof: (ii) implies (i) is obvious.

(i) implies (ii): Suppose \( a \) is a finite element of \( L \), then \( a = p_1 \lor \ldots \lor p_n \) for some atoms \( p_1, \ldots, p_n \). Suppose \( a \land x = 0 \). Since \( p_i \leq a \) for every \( i = 1, \ldots, n \), we have, \( p_i \land x = 0 \). Hence by (i), \( p_i \lor x \) for every \( i \). Therefore by Lemma (2.5) (iii), we have \( a \lor x \). This proves (i) is equivalent to (ii).

(i) implies (iii): Suppose \( p, q \) are atoms and \( p \) is subperspective to \( q \). By (i), \( q \) is \( \lor \)-standard. Hence by Proposition (4.7), we have \( p \leq q \). But \( p, q \) are atoms. Hence \( p = q \).

(iii) implies (i): Suppose \( q \) is an atom. If \( p \) is atom and \( p \) is subperspective to \( q \), then by (iii), \( p = q \). Hence by Proposition (4.7), \( q \) is \( \lor \)-standard. Thus every atom is \( \lor \)-standard and hence (i) and (iii) are equivalent.

(ii) implies (iv): Suppose \( p \) is an atom and \( a \) is a finite element of \( L \) such that \( p \) is subperspective to \( a \). Then by (ii), \( a \) is \( \lor \)-standard. Hence by Proposition (4.7), \( p \leq a \).

(iv) implies (ii): Suppose \( a \) is a finite element of \( L \). We shall use the equivalent conditions of Proposition (4.7). Let \( p \) be an atom and \( p \) subperspective to \( a \). Then by (iv) we have \( p \leq a \). Thus \( a \) is \( \lor \)-standard and we have proved the equivalence of (ii) and (iv).
(iii) implies (v): Suppose $p$ is an atom and $0 \not\in x \in L$. We shall prove $(p, x)P$. Let $q \leq p v x$. If $q \nmid x$ then $q \land x = 0$. Thus $q$ is subperspective to $p$. Hence by (iii), $p = q$. Taking an atom $r$ with $r \leq x$, we have $q \leq p v r$ and we have $(p, x)P$. If $q \leq x$ then we may choose $r = q$ and thus we get $(p, x)P$. Next, suppose that $p \nmid q$. Then by Proposition (4.7), we have

$$\omega(p v q) = \omega(p) \cup \omega(q) = \{p, q\}.$$  

(v) implies (iii): Let $p, q$ be atoms with $p$ subperspective to $q$. Then for some $x$ in $L$ we have $p \leq q v x$, $p \land x = 0$. If $x = 0$, then we have $p = q$. If $x \neq 0$ then by (v), $(q, x)P$ holds. Hence there exists atom $r$ such that $r \leq x$ and $p \leq q v r$. Then, $q \nmid r$, since otherwise, $p \leq r \leq x$, a contradiction. Therefore, by (v), $p \nmid \omega(q v r) = \{q, r\}$. But $p \nmid r$. Hence $p = q$ and we get (iii).

q.e.d.

Recall that a complemented distributive lattice is called a Boolean lattice.

If every atom of an atomistic join-semilattice happens to be $\vee$-standard then we get several interesting consequences.
Theorem (4.10) i Let $L$ be an atomistic join-semilattice where every atom is $\vee$-standard. Then

(i) $L$ is finite-statisch.
(ii) $L$ has the covering property.
(iii) If $a$ is a finite element of $L$, then $\omega(a)$ is a finite set and $(a, x)^*$ for every $x$ in $L$.
(iv) $\mathcal{L}(\omega^{-1}(L))$ is a Boolean lattice formed by all subsets of $\omega^{-1}(L)$.
(v) An element $a$ in $L$ is $\vee$-standard if and only if every element which covers $a$ is $\vee$-standard.
(vi) If $L$ has 1 and if $a \vee b = 1$ for some finite element $b$ then $a$ is $\vee$-standard. (especially, every dual-atom of $L$ is $\vee$-standard).

Proof: (i) If $p \leq a \vee q$ where $p$, $q$ are atoms, Then by assumption $q$ is $\vee$-standard. Hence by Proposition (4.7) (iv), we have

$$p \leq \omega(p \vee q) = \omega(a) \cup \omega(q) \leq \omega(a) \vee \omega(q).$$

Therefore, there are atoms $p_1, \ldots, p_n$ such that $p_1 \leq a$ and $p \leq p_1 \vee \ldots \vee p_n \vee q$. Let $p_1 \vee \ldots \vee p_n = a_1$. Then $a_1$ is finite with $a_1 \leq a$. Thus $p \leq a_1 \vee q$. Hence $L$ is finite-statisch.
(iii) Suppose \( a \) is finite, then \( a = p_1 \vee \ldots \vee p_n \) for some atoms \( p_1, \ldots, p_n \) in \( L \). Suppose all \( p_i \) are distinct. By assumption every \( p_i \) is \( \bigvee \)-standard. Hence by Proposition (4.7), we have

\[
\omega(a) = \omega(p_1) \cup \omega(p_2 \vee \ldots \vee p_n) = \ldots = \omega(p_1) \cup \ldots \cup \omega(p_n)
\]

Thus \( \omega(a) \) is a finite set.

Now we shall prove that \((a, x)^* \) for every \( x \in L \). Suppose \( x < y \leq a \Rightarrow x \), then putting \( \omega = \{ p \in \mathcal{P}(L) \mid p \leq y, p \not\subseteq x \} \), we have \( \omega \subseteq \omega(a) \). Because \( p \leq y \) and \( p \not\subseteq x \) imply that \( p \not\subseteq a \vee x \) or \( p \not\subseteq a \vee x \Rightarrow y \). Hence \( p \not\subseteq \omega(a) \). Thus, \( \omega \) is a finite set and \( d = \bigvee(p; p \not\subseteq \omega) \) exists. Evidently, \( d < a \) and \( d \vee x < y \). If \( p \) is an atom and \( p \leq y \), then we have \( p \leq a \vee x \), i.e., \( p \not\subseteq \omega(a \vee x) = \omega(a) \cup \omega(x) \). Hence if \( p \not\subseteq x \) then \( p \not\subseteq \omega \) and hence \( p \leq d \). Thus we have \( p \leq x \) or \( p \leq d \) and hence \( p \leq d \vee x \). Therefore \( d \vee x = y \) since \( L \) is atomistic.

(ii) Since the covering property is equivalent to \((p, x)^* \) for every atom \( p \) and every \( x \) in \( L \), the proof is clear.

(iv) Let \( \omega \) be an arbitrary subset of \( \mathcal{M}(L) \). Let \( p \) be an atom with \( p \leq q_1 \vee \ldots \vee q_n \) where \( q_i \in \omega(i=1, \ldots, n) \) (we may assume that \( q_i \) are different). Since \( p \not\subseteq \omega(q_1 \vee \ldots \vee q_n) = \{ q_1, \ldots, q_n \} \), \( p = q_i \) for some \( i \) and hence \( p \not\subseteq \omega \). Therefore \( \omega \in \mathcal{L}(L) \).
Suppose \( a \) is in \( L \), \( a \) is \( \triangledown \)-standard and let \( b \) be an element which covers \( a \). Since \( L \) is atomistic and \( b \) covers \( a \), \( b = avp \) for some atom \( p \). But \( a \) and \( p \) are \( \triangledown \)-standard, hence \( b \) is also \( \triangledown \)-standard. Conversely, assume that every element which covers \( a \) is \( \triangledown \)-standard. By Proposition (4.7), it is sufficient if we prove that for every \( x \) in \( L \), \( \omega(ax) = \omega(a) \cup \omega(x) \). If \( x \leq a \) then this clearly holds. If \( x \not\leq a \) then there exists an atom \( p \) such that \( p \leq x \) and \( p \not\leq a \). Then, by (ii), \( avp \) covers \( a \). Since \( avp \) and \( p \) are \( \triangledown \)-standard, we have

\[
\omega(avn) = \omega(avnvx) = \omega(avn) \cup \omega(x) = \omega(a) \cup \omega(p) \cup \omega(x) = \omega(a) \cup \omega(x).
\]

(vi) Suppose \( b \) is a finite element and \( avb = 1 \). Then, by Theorem (5.11) of Chapter-I, the interval \([a,1]\) is an \( \triangledown \)-join-semilattice and the length \( n(a) \) of \([a,1]\) is finite. If \( n(a) = 1 \) (i.e., \( a \) is a dual atom) then only the element \( 1 \) covers \( a \), and hence \( a \) is \( \triangledown \)-standard by (v). If \( c \) covers \( a \), then evidently \( n(a) = n(c) - 1 \). If we assume that \( n(a) = k \) implies \( a \) is \( \triangledown \)-standard, then by (v), \( n(a) = k+1 \) implies \( a \) is \( \triangledown \)-standard. Therefore, by induction, \( a \) is \( \triangledown \)-standard for any \( n(a) \).

q.e.d.
These considerations very naturally lead us to the equivalence of \( \bigvee \)-symmetry, statischness, Property (A) and Property (B).

Theorem (4.11): Let \( L \) be an atomistic join-semilattice where every atom is \( \bigvee \)-standard. The following statements are equivalent.

(i) \( L \) is \( \bigvee \)-symmetric.

(ii) Every element of \( L \) is \( \bigvee \)-standard.

(iii) \( L \) is statischn.

(iv) \( L \) has the Property (A).

(v) \( L \) has the Property (B).

Proof: Since every atom of \( L \) is \( \bigvee \)-standard, by Theorem (4.10), \( L \) has the covering property, and hence the finite covering property. Therefore, by Theorem (3.15) (iv) and (v) are equivalent. Further, by the same Theorem, (iv) and (i) are equivalent. Thus (i), (iv) and (v) are equivalent.

(i) implies (ii): Let \( a \in L \). If \( p \in \mathcal{A}(\bar{a}) \) and \( p \nmid a \), then since \( p \) is \( \bigvee \)-standard, we have \( p \bigvee a \). Hence by (i), \( a \bigvee p \) holds. Therefore by Proposition (4.7), \( a \) is \( \bigvee \)-standard.
(ii) implies (iii): Suppose \( p \) is an atom and \( p \leq a \lor b \). Then by (ii), \( a \) is \( \lor \)-standard and hence by Proposition (4.7), \( p \leq \omega(a_\lor b) = \omega(a) \lor \omega(b) \). Thus \( p \leq \omega(a) \) or \( p \leq \omega(b) \).

Without loss of generality, assume that \( p \leq \omega(a) \). Then there exist atoms \( p_1, \ldots, p_n \leq a \) such that \( p \leq p_1 \lor \cdots \lor p_n = a \), say. Then \( p \leq a \lor b \) where \( b \) is any finite element and \( b_1 \leq b \).

Hence \( L \) is stastisch.

(iii) implies (iv): Suppose \( L \) is stastisch. Then by Proposition (3.6), \( \omega(a) \lor \omega(b) \) implies \( a \lor b \). Thus \( L \) has the Property (A).

Q.E.D.

It can be seen easily that join of standard elements in a join-semilattice \( L \) with 0 is standard.

We now introduce one more concept.

Definition (4.5): A join-semilattice \( L \) with 0 is called finite-distributive when every finite element of \( L \) is standard.

Next, we have

Proposition (4.12): If \( L \) is a join-semilattice with 0, then the following statements are equivalent.

1. \( L \) is finite-distributive.

2. Every atom of \( L \) is standard.
Proof: (i) implies (ii) is obvious.

(ii) implies (i): Suppose \( a \) is a finite element of \( L \), then \( a = p_1 \lor \ldots \lor p_n \) for some atoms \( p_1, \ldots, p_n \). Since every atom is standard, \( p_i \) is standard for every \( i \). Hence by proposition (4.3), \( p_1 \lor \ldots \lor p_n = a \) is standard.

q.e.d.

The following result also seems to be of interest.

**Theorem (4.15):** Let \( L \) be a finite-distributive atomistic join-semilattice.

(i) An element \( a \in L \) is standard if and only if every element which covers \( a \) is standard.

(ii) If \( L \) has \( 1 \) and if \( a \lor b = 1 \) for some \( b \) in \( F(L) \), then \( a \) is standard.

Proof: (i) Assume that \( a \) is standard and \( b \) covers \( a \). Then \( b = a \lor p \) for some atom \( p \). Now \( a \) and \( p \) are standard. Hence \( a \lor p = b \) is also standard.

Conversely, suppose that every element which covers \( a \) is standard. We shall show that \( (a, x) D \) for every \( x \) in \( L \). If \( x \lesssim a \), then clearly \( (a, x) D \) holds. Hence suppose that \( x \not\lesssim a \). Since \( L \) is atomistic, there exists an atom \( p \) such that \( p \lesssim x \) and \( p \not\lesssim a \). By assumption, \( p \) and \( a \lor p \) are standard. Hence \( (a, p) D \) and \( (a \lor p, x) D \) hold. So, we have, \( (a, x) D \), since \( x = p \lor x \).
(ii) Suppose \( L \) has 1 and \( ab = 1 \) for some finite element \( b \). Then \([a, 1]\) is an \( A_c \)-join-semi-lattice by Theorem (3.11) of Chapter I, and the length \( n(a) \) of \([a, 1]\) is finite. If \( n(a) = 1 \) (i.e., \( a \) is a dual atom) then only the element 1 covers \( a \). Hence \( a \) is standard by (i). If \( c \) covers \( a \), then \( n(a) = n(c) + 1 \). Hence, if we assume that \( n(a) = k \) implies that \( a \) is standard then, by (i) \( n(a) = k+1 \) will imply that \( a \) is standard. Therefore, by induction, \( a \) is standard for any \( n(a) \).

\[ q.e.d. \]

As a remark we enlarge the list of equivalent statements given in Theorem (4.9) in the following:

**Remark (4.14)**: If \( L \) is an atomistic lattice, then the statements (i) to (v) in Theorem (4.9) are equivalent to the following statements.

**(ii)'** \( L \) is finite-distributive.

**(v)'** \( L \) is finite-modular (that is, \( (a, b) \) for every \( a \) in \( L \) and \( b \) in \( F(L) \), and \( \omega(p \cdot q) = \{ p, q \} \) for all atoms \( p, q \) with \( p \parallel q \).

**Proof:** By Lemma (4.5), if \( L \) is atomistic lattice then an element \( a \) is standard if and only if it is \( \omega \)-standard. Hence (ii) and (ii)' are equivalent.
Now (v) of Theorem (4.9) implies that $L$ has the covering property. Hence by Theorem 1 of Maeda [6], $L$ is finite-modular. (v) implies (v) is obvious.

$q.e.d.$

Before we complete this chapter, we produce a counter example which shows that there is an atomistic join-semilattice which satisfies (i) to (v) of Theorem (4.9) but does not satisfy (ii) and (v)' of the above remark.

**Example (4.15):** Let $X$ be an infinite set, and put

$$L = \{ X, \phi \}, \text{ all finite subsets of } X \} \cup \{ X - \{ p \}, p \in X \}.$$ 

$L$ is an atomistic join-semilattice ordered by set inclusion. $A \wedge B = A \cup B$ for all $A, B$ in $L$. Moreover, $\Omega(L) = X$ and $\omega(A) = A$ for every $A$ in $L$. Hence,

$$\omega(A \wedge B) = A \wedge B = A \cup B = \omega(A) \cup \omega(B),$$

and hence every $A$ in $L$ is $\vee$-standard. But, we can show that if $F$ is a finite subset of $X$ and $p \in F$ then $(X - \{ p \}, F^\star)$ does not hold. In fact, taking $q \in X - F$, we have

$$F \subseteq X - \{ q \} \subseteq X = (X - \{ p \}, F)^\star.$$ 

If there were $A \in L$ such that $A \subseteq X - \{ p \}$ and $A \wedge B = X - \{ q \}$, then $A$ would be a finite subset since $A \subseteq X - \{ p, q \}$, and hence $A \wedge F$ is finite, a contradiction. Therefore, $L$ is not finite-modular.

Moreover, since $(X - \{ p \}, F)$ does not hold for $p \in F$, every element of $L$ except $X$ and $\phi$ is not standard.
Lastly, we have the following result as a corollary of Theorem (4.11).

**Corollary (4.16):** Let \( L \) be a finite-distributive atomistic lattice. The following statements are equivalent.

(i) \( L \) is \( \triangledown \)-symmetric.

(ii) \( L \) is distributive.

(iii) \( L \) is statisch.

(iv) \( L \) has the Property (A).

(v) \( L \) has the Property (B).
REFERENCES


