CHAPTER I

On Modular Pairs In Semilattices

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Chapter I

1. Introduction.

There is a tremendous spert of research activity in general lattices. Several research articles that are being spinned out are in testimony of the above statement. A probable reason may be perhaps the availability of abundant open problems that remained to be tackled in this very interesting field of mathematics. Birkhoff [1] states 166 open problems, Gratzer [3] lists 193 unsolved problems, Crawley and Dilworth [2] mention. 25 open problems, Maeda and Maeda [5] invite attention to 11 open problems from theory of symmetric lattices. Besides these there are enumerable open problems scattered throughout the literature. Nonetheless, one is constrained to observe that there is a scarce literature that is being published involving symmetric lattices. It shall not be much inaccurate if one says that except for S. Maeda, M.F. Janowitz, D.J. Foulis, M. Stern, S.S. Holland there are no workers who are working in the theory of symmetric lattices.
Let $B$ be any Banach space, $L(B)$ be the lattice of all closed subspaces of $B$. It is well known that $L(B)$ is an $M$-symmetric lattice. The concept of modular pairs in lattices has far reaching interpretations. This made Prof. Birkhoff to pose the following open problem:


How should one define modular pairs in a general poset?

The study of modular pairs has been well-developed by Maeda and Maeda [5] in their book entitled "Theory of symmetric lattices". There were, however, no attempts to tackle the problem posed by Birkhoff. In an attempt to cover some ground we have succeeded in introducing and studying the concept of modular pairs in semilattices.

In section 2 of this chapter, we introduce the concept of modular pairs and dual modular pairs in meet-semilattices and join-semilattices respectively. We discuss several properties of modular pairs and dual modular pairs in semilattices. In section 3 we introduce and study the concept of atomistic join-semilattices. A nice characterization of atomistic join-semilattice is accomplished. The notions of covering property and exchange property are extended to join-semilattices with $0$. Their interrelation-ship is also explored. Moreover, we further generalize
the concept of covering property to $A$-covering property for a non-empty subset $A$ of a join-semilattice with 0.

In a like manner the concept of exchange property is generalized to $A$-exchange property. It is shown that for an atomistic join-semilattice $L$ with 0, $A$-covering property is equivalent to $A$-exchange property. Several results involving $-\mathcal{S}^n$-covering property are investigated. In an atomistic join-semilattice with finite-covering property we investigate height of finite elements of $L$. In the course of our investigation we also supply an example of an $Ac$-join-semilattice which is not a lattice. This adds weightage to our endeavours. In Section 4 we carry out the study of dual-modular pairs of subsets in atomistic join-semilattices. In fact very pleasantly we characterize $A$-covering property in terms of dual-modular pair consisting of the atom space $-\mathcal{S}$ and the underlying subset $A$ of $L$. Furthermore we discuss and study the concept of $P$-relation, $A$-pairs of subsets of a join-semilattice $L$. Several deep and interesting results involving $P$-relation are obtained.

Throughout the thesis by a poset we mean a non-empty set $P$ in which a binary relation $x \leq y$ is defined, which satisfies the following conditions for all $x, y, z$ in $P$. 
(P₁). (Reflexive): For all \( x \) in \( P \), \( x \leq x \).

(P₂). (Antisymmetry): If \( x \leq y \) and \( y \leq x \) then \( x = y \).

(P₃). (Transitivity): If \( x \leq y \) and \( y \leq z \) then \( x \leq z \).

If any two elements of a poset \( P \) have an infimum then we call \( P \) a meet-semilattice.

If in a poset \( P \) any two elements have a supremum then \( P \) is a join-semilattice.

2. Modular Pairs.

Let us recall the well known concept of modular pairs in a lattice \( L \).

**Definition (2.1).** A pair of elements \( a, b \) of a lattice \( L \) is called modular and is denoted by \( (a, b) \) \( M \) when

\[
(M) \quad (c \wedge a) \wedge b = c \wedge (a \wedge b) \quad \text{for every} \quad c \leq b.
\]

A pair \( a, b \) in \( L \) is called dual-modular, denoted by \( (a, b) \) \( M^* \), when
(M*) \((c \wedge a) \lor b = c \wedge (a \lor b)\) for every \(c \geq b\).

Following figure illustrates the concepts of \((a, b)^M\) and \((a, b)^*\) in the context of a lattice.

In this lattice \((f, h)^M\) holds because, \(a \leq h\) implies \((a \land f) \land h = f \land h = a\) and \(a \lor (f \land h) = a \lor a = a\).

But for \(b \leq f\), \((b \lor h) \land f = g \land f = f\) and \(b \lor (h \land f) = b \lor a = b + f\).

Hence \((h, f)^M\) fails.

This also shows that \((a, b)^M\) does not imply \((b, a)^M\) in a lattice.

In the above figure, we also easily see that \((b, d)^*\) holds but \((d, b)^M\) does not hold.

Let us now extend these concepts of modular and dual modular pairs to semilattices.

**Definition (2.2)** A pair of elements \(a, b\) of a meet-semilattice \(L\) is called modular and is denoted by \((a, b)^M\) when
(M₀) \( a \land b \leq c \leq b \) implies the existence of \( d \) in \( L \) such that \( d \geq a \) and \( d \land b = c \). The figure 1 (A) illustrates the concept of \((a, b)M\) in a meet-semilattice. This semilattice is a modular meet-semilattice. We shall construct a meet-semilattice which is non-modular meet-semilattice that contains a modular pair; see figure 1 (B).

\[ \text{Figure 1(A)} \]
\[ \text{Figure 1(B)} \]

In figure (B), \( e \land g \leq f \leq g \), but there is no element which is greater than \( e \) and whose meet with \( g \) is \( f \). Thus \( (e, g)M \) does not hold. However, \((a, b)M\) holds as before.

**Definition (2,3).** A pair of elements \( a, b \) of a join-semilattice \( L \) is called dual-modular and is denoted by \((a, b)^*\) when
$$(M_0^*) \ b \leq c \leq a \lor b \ \text{implies the existence of } d \ \text{in } L \ \text{such that } d \leq a \ \text{and } d \lor b = c.$$ 

The following figure 2(A) illustrates the concept of $(a,b)M^*$ in a join-semilattice. This semilattice is a modular join-semilattice.

![Figure 2(A)](image)

Consider the figure 2(B) of a join-semilattice. We have, $g \leq f \leq e \lor g$ but there is no element which is less than $e$ and whose join with $g$ is $f$. Therefore $(e,g)M^*$ does not hold. However $(a,b)M^*$ holds as before. This is a join-semilattice in which every pair is not dual-modular.

**Lemma (2.1):** In a lattice $L$, the following hold.

(i) $(M_0)$ is equivalent to $(M)$,

(ii) $(M_0)^*$ is equivalent to $(M)^*$. 


Proof: (i) Suppose \((M)\) holds in \(L\) and let \(a \wedge b \leq c \leq b\). Then \(c \leq b\) and \((M)\) together imply that 
\[c \vee (a \wedge b) = (c \vee a) \wedge b.\]
But \(a \wedge b \leq c\), hence \(c = (c \vee a) \wedge b\).
Put \(d = c \vee a \geq a\). Then we have \(c = d \wedge b\). Therefore \((M_o)\) holds. Now suppose \((M_o)\) holds and \(c \leq b\). We have 
\[a \wedge b \leq c \vee (a \wedge b) \leq b.\]
By \((M_o)\) there exists \(d\) in \(L\) such that \(d \geq a\) and 
\[c \vee (a \wedge b) = d \wedge b.\]
From this, \(d \geq c\). Thus \(d \geq a \vee c\). Finally, we have 
\[(c \vee a) \land [c \vee (a \wedge b)] = (c \vee a) \land d \wedge b\]

i.e. 
\[c \vee (a \wedge b) = (c \vee a) \wedge b.\]

(ii) Suppose \((M)^*\) holds. Let \(b \leq c \leq a \vee b\). Then by \((M)^*\) and \(b \leq c\) we have

\[c \land (a \vee b) = (c \land a) \lor b\]
i.e. \(c = (c \land a) \lor b\).

Put \(d = c \land a\). Thus there exists \(d \leq a\) with \(c = d \lor b\).
Hence \((M_o)^*\) holds. Now suppose \((M_o)^*\) holds. Let \(b \leq c\), then \(b \leq c \land (a \vee b) \leq a \vee b\). By \((M_o)^*\), there exists \(d \leq a\) such that 
\[c \land (a \vee b) = d \lor b.\]
From this we conclude \(d \leq c\).
We have then \(d \leq c \land a\) and

\[(c \land a) \lor [c \land (a \vee b)] = (c \land a) \lor d \lor b\]

i.e. \(c \land (a \lor b) = (c \land a) \lor b\). This is \((M)^*\).

q.e.d.
In the next result we express \((a, b)M\) in terms of certain surjective maps.

Remark (2.2): Let \(a\) and \(b\) be elements of a meet-semilattice \(L\).

(i) If either \(a \leq b\) or \(a \geq b\), then \((a, b)M\).

(ii) If \((a, b)M\), then \(\{a, b\}\) has an upper bound.

(iii)(A) \((a, b)M\) means that the mapping of

\[
[a, \rightarrow] = \{x \in L; a \leq x\}
\]

into \([a \land b, b]\) = \[y \in L; a \land b \leq y \leq b\] defined by \(\varphi(x) = x \land b\) is onto.

(iii)(B) If \((a, b)M\) and if \(c\) is an upper bound of \(\{a, b\}\),
then it is evident that \(\varphi\) is a mapping of \([a, c]\) onto
\([a \land b, b]\) and conversely.

(iv) If \(a, b \in [c, d]\), then it is evident that \((a, b)M\) in
\([c, d]\) if and only if \((a, b)M\) in \(L\).

Proof: (i) Suppose \(a \leq b\). Let \(a \land b \leq c \leq b\),
i.e. \(a \leq c \leq b\) (since \(a \leq b\)). So \(c = c \land b\). Choose
\(d = c \geq a\). If \(b \leq a\) then \(a \land b \leq c \leq b\) implies that
\(b \leq c \leq b\), i.e. \(b = c\). Hence \(c = a \land b\). Choose \(d = a \geq a\).
Thus \((a, b)M\) holds if either \(a \leq b\) or \(a \geq b\).
(ii) Suppose \((a, b)M\). Then for \(a \land b \leq b \leq b\) there exists \(d \geq a\) such that \(b = d \land b\). Now \(d \geq d \land b = b\). Thus \(d\) is an upper bound of \(\{a, b\}\).

(iii)(A) Suppose \(\phi\) is onto and let \(a \land b \leq c \leq b\). Then by onto-ness of \(\phi\), there exists an element \(x\) in \([a, -]\) such that \(\phi(x) = c\), i.e., \(x \land b = c\). Choose \(d = x \geq a\). This means that \((a, b)M\) holds.

Conversely, suppose that \((a, b)M\) holds and \(x \in [a \land b, b]\). Then \(a \land b \leq x \leq b\) together with \((a, b)M\) implies that there exists an element \(d \geq a\) such that \(x = d \land b\). Thus there is an element \(d\) in \([a, -]\) such that \(\phi(d) = x\). Thus \(\phi\) is onto.

(iii)(B) Suppose \(c\) is an upper bound of \(\{a, b\}\) and \(\phi: [a, c] \rightarrow [a \land b, b]\) is onto. Let \(a \land b \leq x \leq b\).

Then by onto-ness of \(\phi\) there exists an element \(d \in [a, c]\) such that \(\phi(d) = x\), i.e., \(d \land b = x\), where \(d \geq a\). Hence \((a, b)M\) holds. Conversely, suppose \((a, b)M\) holds and \(x \in [a \land b, b]\). Clearly \(a \land b \leq x \leq b\) together with \((a, b)M\) implies that there exists an element \(d \geq a\) such that \(x = d \land b\). Consider the element \(d \land c\). Now we have \(a \leq d \land c \leq c\) and \((d \land c) \land b = x \land c = x\). Thus for \(x \in [a \land b, b]\) there exists an element \(d \land c \in [a, c]\) such that \((d \land c) \land b = \phi(d \land c) = x\), and \(\phi\) is onto.
(iv) Suppose $a, b \in [c, d]$ and $(a, b)_M$ in $[c, d]$. Let $a \land b \leq x \leq b$. From this $x \leq d$ and since $a, b \in [c, d]$, $c \leq a \land b \leq x$. Thus $x \in [c, d]$ and $a \land b \leq x \leq b$. Therefore there exists an element $y$ in $[c, d]$, $y \geq a$ with $x = y \land b$, $y$ in $L$ and we have $(a, b)_M$ in $L$.

Conversely, suppose $(a, b)_M$ in $L$. Let $a \land b \leq x \leq b$ in $[c, d]$ and $x \in L$. By assumption there exists an element $y$ in $L$, $y \geq a$ such that $x = y \land b$. Consider the element $y \land d$. As $y \geq a \geq c$ and $y \land d \leq d$, $y \land d \in [c, d]$ and $y \land d \geq a$, $(y \land d) \land b = y \land (d \land b) = y \land b = x$. Thus there exists an element $y \land d$ in $[c, d]$ with $y \land d \geq a$ and $(y \land d) \land b = x$. That is $(a, b)_M$ in $[c, d]$.

q.e.d.

The dual statements on dual-modular pairs in join-semilattices are also true.

We now obtain some properties of modular pairs in a meet-semilattice.

Lemma (2.3) : Let $a, b$ and $c$ be elements of a meet-semilattice.

(i) If $(a, b)_M$ and $(a \land b, c)_M$ then $(a_1, b_1)_M$ for any $a_1 \in [a \land c, a]$. 

(ii) If $(a, b)_M$ then $(a_1, b_1)_M$ for any $a_1 \in [a \land b, a]$, $b_1 \in [a \land b, b]$. 

Proof: (i) Let \( a_1 \land b \land c \leq x \leq b \land c \). Since
\( a \land b \land c = a_1 \land b \land c \leq x \leq c \), it follows from \((a \land b, c)_M\)
that there exists \( y \geq a \land b \) such that \( y \land c = x \). Since
\( a \land b \leq y \land b \leq b \), \((a, b)_M\) implies that there exists \( d \)
such that \( d \geq a \) and \( d \land b = y \land b \). Then we have \( d \geq a_1 \)
and \( d \land b \land c = y \land b \land c = x \land b = x \). Therefore \((a_1, b \land c)_M\)
holds.

(ii) Since \( a \land b \leq b_1 \), we have \((a \land b, b_1)_M\). Thus, from (i)
we have \((a_1, b_1)_M\) for any \( a_1 \in [a \land b_1, a] = [a \land b, a] \) and
\( b_1 \in [a \land b, b] \).

q.e.d.

It is remarked that Lemma 2.3 is a generalization of
Lemma 1.5 of Maeda and Maeda [5]. By the duality we have

\textbf{Lemma (2.4):} Let \( a, b \) and \( c \) be elements
of a join-semilattice.

(i) If \((a, b)_M^*\) and \((a \lor b, c)_M^*\) then \((a_1, b \lor c)_M^*\) for any
\( a_1 \in [a, a \lor c] \).

(ii) If \((a, b)_M^*\) then \((a_1, b_1)_M^*\) for any \( a_1 \in [a, a \lor b], \)
\( b_1 \in [b, a \lor b] \).
3. Dual-modular pairs in atomistic join-semilattices and the covering property.

In this section we extend our considerations involving dual modular pairs so as to include the concepts such as covering property, exchange property. We begin with the concept of atomistic join-semilattice \( L \).

**Definition (3.1):** In a poset, we denote \( a < b \) when \( b \) covers \( a \) (that is, \( a < b \) and there is no \( x \) with \( a < x < b \)). Let \( L \) be a poset with 0. Then \( p \in L \) is called an atom when \( 0 < p \), and the set of all atoms of \( L \) is denoted by \( \mathcal{A}(L) \) or briefly by \( \mathcal{A} \). This set \( \mathcal{A}(L) \) is called an atom space or point space. If \( p \in \mathcal{A} \), then \( p \wedge x \) exists for every \( x \) in \( L \) and \( p \nleq x \) is equivalent to \( p \wedge x = 0 \) \( (p \nleq x \) is equivalent to \( p \wedge x = p \). \( L \) is called atomistic when every non-zero element \( a \) of \( L \) is the least upper bound of \( \mathcal{A}_a = \{ p \in \mathcal{A}; p \leq a \} \). Next result is a nice characterization of an atomistic join-semilattice.

**Proposition (3.1):** A join-semilattice \( L \) is atomistic if and only if \( a \nleq b \) implies the existence of an atom \( p \) such that \( p \leq a \) and \( p \nleq b \).
Proof: Suppose $L$ is atomistic and $a \nmid b$. Then
\[ a = \bigvee \{ p \in \mathcal{P} : p \leq a \} . \]
Hence there exists an atom $p$ such that $p \leq a$ but $p \nmid b$. Conversely, suppose that $L$
 satisfies the condition. Let $a$ be a non-zero element of $L$
 and $S$ be the set of atoms contained in $a$, i.e. $S = \bigcap_{a}$. If $a \nmid \bigvee S$
 then there exists an upper bound $b$ of $S$
 such that $a \nmid b$. Hence by assumption there exists an atom $p$
 such that $p \leq a$ but $p \nmid b$. $p \leq a$ implies $p \in S$
 and $b$ is an upper bound of $S$, which is a contradiction.
Hence $a = \bigvee S$. Thus $L$ is atomistic.

q.e.d.

This characterization may remind one of a characterization of an atomistic lattice.

Let us now connect the dual-modular pairs with the covering relation in the following.

Lemma (3,2): Let $a$ and $b$ be elements of a

join-semilattice.

(i) If $\{a, b\}$ has a lower bound and if $b \leq a \wedge b$ then

\[ (a, b)M . \]

(ii) If $a \wedge b$ exists and $a \wedge b \leq a$ and if $(a, b)M^*$ then

$\ \ \ \ \ \ \ b \leq a \vee b$. \]
Proof: (i) Let $b \leq c \leq a \lor b$. Then $c = b$ or $c = a \lor b$ by assumption. Taking a lower bound $e$ of $\{a, b\}$, we put $d = e$ if $c = b$ and put $d = a$ if $c = a \lor b$. Then $d \leq a$ and $d \lor b = c$.

(ii) We have $b \leq a \lor b$ since $a \land b \leq a$. If $b \leq c \leq a \lor b$ then it follows from $(a, b)^*$ that there exists $d$ such that $d \leq a$ and $d \lor b = c$. Since $a \land b \leq d \lor (a \land b) \leq a$, it follows from $a \land b \leq a$ that $d \lor (a \land b)$ is equal to either $a \land b$ or $a$. Hence, $c = d \lor b = d \lor (a \land b) \lor b$ which is equal to either $b$ or $a \lor b$.

q.e.d.

We are now in a position to introduce concepts of covering property and exchange property in join-semilattices.

Definition (3.2): Let $L$ be a join-semilattice with $0$.

The following property is called the covering property.

If $p$ is an atom and $p \nmid a$ then $a \leq a \lor p$.

The following property is called the exchange property.

If $p$ and $q$ are atoms and if $p \nmid a$ then $p \leq a \lor q$ implies $q \leq a \lor p$.

In the next result we explore the relationship between covering property, exchange property and covering relation in a join-semilattice. Incidentally we show that covering
property and exchange property are all equivalent in \( L \) when \( L \) is atomistic.

**Theorem (3.3)**: For a join-semilattice \( L \) with \( 0 \), we consider the following conditions.

(i) If \( p \) is an atom of \( L \) then \((p,x)^*\) for every \( x \) in \( L \).

(ii) \( L \) has the covering property.

(iii) \( L \) has the exchange property.

(iv) If \( a \land b \) exists and \( a \land b \leq a \) in \( L \) then \( b \leq a \lor b \).

(v) If \( a \land b \) exists and \( a \land b \leq a \) in \( L \) then \((a,b)^*\).

Then, (i) \( \iff \) (ii), (iv) \( \iff \) (v) and (iv) \( \implies \) (i) \( \implies \) (iii). Moreover, these five statements are all equivalent if \( L \) is atomistic.

**Proof**: (i) \( \implies \) (ii) Let \( p \) be an atom and \( p \not\leq a \). Then, \( p \land a = 0 \leq p \). Since \((p,a)^*\) by (i), it follows from (ii) of Lemma (3.2) that \( a \leq a \lor p \).

(ii) implies (i). Let \( p \) be an atom. If \( p \leq x \), then evidently \((p,x)^*\). If \( p \not\leq x \), then it follows from (ii) that \( x \leq p \lor x \). Hence \((p,x)^*\) holds by (i) of Lemma (3.2).

The equivalence of (iv) and (v) follows from Lemma (3.2).
(iv) implies (ii): Since \( p \vdash a \) implies \( p \land a = c \vdash p \), by (iv) we have \( a \vdash a \lor p \). Hence \( L \) has covering property.

(ii) implies (iii): Let \( p, q \) be atoms in \( L \), \( p \vdash a \) and \( p \leq a \lor q \). Then \( q \nvdash a \), since otherwise \( p \leq a \lor q = a \), a contradiction. Hence, \( a \nvdash a \lor q \) by (ii). Since \( a \vdash a \lor p \leq a \lor q \), we have \( a \lor p = a \lor q \geq q \).

Finally, if we assume that \( L \) is atomistic, then we prove (iii) implies (iv). Suppose \( a \land b \) exists and \( a \land b \leq a \). \( a \land b \leq a \) implies that \( b \leq a \lor b \) and by atomisticity of \( L \), there exists an atom \( p \) such that \( p \nvdash a \land b \) and \( p \leq a \). Now \( a \land b \leq (a \land b) \lor p \leq a \) together with \( a \land b \leq a \) implies that \( (a \land b) \lor p = a \). Hence \( b \vdash (a \land b) \lor p = b \lor p = a \lor b \).

Suppose \( b \vdash c \leq a \lor b \), then there exists an atom \( q \) such that \( q \nvdash b \) and \( q \leq c \leq a \lor b \). Hence \( q \leq b \lor p \). By (iii) we have \( p \leq b \lor q \leq c \). Therefore \( c \geq b \lor p = a \lor b \). Thus \( c = a \lor b \), i.e. \( b \vdash a \lor b \).

q.e.d.

Theorem (3.3) generalizes theorem 7.10 of Maeda and Maeda [5].

As in the case of a lattice, we can introduce trivially the notion of finite element in a join-semilattice. While doing so we also single out some special subsets of \( L \).
Definition (3,3): Let $L$ be a join-semilattice with $0$, and let $\mathcal{A}$ be the set of all atoms of $L$. We define $\mathcal{A}^n = \{ p_1 \vee \ldots \vee p_n; \ p_i \in \mathcal{A} \}$ for $n=1,2,\ldots$. Evidently $\mathcal{A}^1 = \mathcal{A}$ and $\mathcal{A}^n \subseteq \mathcal{A}^{n+1}$. An element of the set $F = \bigcup_{n=1}^{\infty} \mathcal{A}^n \cup \{ 0 \}$ is called finite.

Let $A$ be a non-empty subset of $L$. The following property is called the $A$-covering property.

If $a \in A$, $p \notin \mathcal{A}$ and $p \nmid a$ then $a \leq ap$.

This is a generalization of covering property. Let us state a characterization of $A$-covering property.

Proposition (3.4): Let $L$ be a join-semilattice and $A$ be a subset of $L$. The following conditions are equivalent.

(i) $L$ has the $A$-covering property.

(ii) If $p \in \mathcal{A}$ then $(p,x)^*M$ for every $x \in A$.

Proof: (i) implies (ii): Suppose that $L$ has $A$-covering property. Let $p \in \mathcal{A}$, $x \in A$. If $p \leq x$ then $(p,x)^*M$ evidently. Suppose $p \nmid x$ and $x \leq c \leq x \vee p$. Then by the $A$-covering property $x \nleq x \vee p$. Hence $x = c$ or $x \vee p = c$. If $c = x \vee p$ then put $d = p$. If $x = c$ then put $d = 0$. In both cases, $c = d \lor x$, $d \leq p$. Thus $(p,x)^*M$ holds.
(ii) implies (i): Suppose $a \in A$ and $p \leq a$, $p \nmid a$. If $a \leq c \leq avp$, then by $(p, a)^* M$ there exists $d \leq p$ such that $d \cdot a = c$. Since $p$ is atom, $d = 0$ or $d = p$. If $d = 0$ then $a = c$. If $d = p$ then $c = avp$, which proves $a \leq avp$.

q.e.d.

On similar lines, we now introduce the concept of $A$-exchange property.

If in a join-semilattice $L$ with $0$ the following condition holds then we say that $L$ has the $A$-exchange property.

If $a \in A$, $p$, $q \leq a$, and $p \nmid a$ then $p \leq q \cdot a$ implies $q \leq p \cdot a$.

As for covering property and exchange property, both $A$-covering property and $A$-exchange property coincide in an atomistic join-semilattice.

**Proposition (3.5):** If $L$ is a join-semilattice and $A \subseteq L$ then $A$-covering property implies the $A$-exchange property. The converse is true if $L$ is atomistic.

**Proof:** Suppose $L$ has the $A$-covering property. Let $a \in A$, $p$, $q \leq a$, and $p \nmid a$, $p \leq q \cdot a$. From this we have $q \nmid a$, since otherwise $q \leq a \Rightarrow p \leq q \cdot a = a$, a contradiction.
Therefore by the A-covering property we have \( a \prec avq \). But \( a \prec avp \leq avq \), which implies that \( avp = avq \), i.e. \( q \preceq avp \). Thus \( L \) has the A-exchange property.

Now suppose that \( L \) is atomistic and suppose \( L \) has the A-exchange property. Let \( a \in A \), \( p \in \mathcal{A} \) and \( p \nmid a \). If \( a \prec x \preceq avp \) then by atomisticity of \( L \), there exists an atom \( q \) such that \( q \preceq x \) and \( q \nmid a \). Also \( q \preceq avp \).

Now \( a \in A \), \( p \in \mathcal{A} \), \( q \in \mathcal{A} \), \( q \nmid a \) and \( q \preceq avp \). Therefore, by the A-exchange property we have \( p \preceq avq \preceq x \), also \( a \prec x \).

Which shows that \( avp \preceq x \), i.e. \( x = avp \). Thus \( L \) has the A-covering property.

q.e.d.

Instead of \( A \) being any arbitrary subset of \( L \) let us select the special subsets \( \mathcal{A}^n \) which we introduced earlier.

In that direction, we have

**Lemma (3.6)**: Let \( L \) be a join-semilattice with 0, and assume that \( L \) has the \( \mathcal{A}^n \)-covering property.

(i) Let \( p_i \), \( q_i \in \mathcal{A} \) for \( i = 1, 2, \ldots, n \). If \( p_i \preceq q_i \vdots \vdots q_n \) (i=1,2,...,n), and \( p_i \nmid p_1 \vdots p_{i-1} \) (i=2,...,n) then \( p_1 \vdots p_n \preceq q_1 \vdots q_n \).

(ii) If \( a \in \mathcal{A}^n \), \( p_i \in \mathcal{A} \), \( p_i \preceq a \) (i=1,...,m) and \( p_i \nmid p_1 \vdots p_{i-1} \) (i=2,...,m) then \( m \preceq n \).
Proof: We prove this Lemma by mathematical induction. For $n = 1$, proof is trivial. Assume that the Lemma holds for $n = m-1$.

Let $p_i \not\preceq p_1 v \ldots v p_{i-1}$ for $i = 2, \ldots, m$ and let

$$p_i \preceq q_1 v \ldots v q_m \quad \text{for} \quad i = 1, \ldots, m.$$  

If $p_i \preceq q_1 v \ldots v q_{m-1}$ for every $i = 1, \ldots, m$ by the assumption for $n = m-1$ we would have $p_1 v \ldots v p_{m-1} = q_1 v \ldots v q_{m-1}$ which contradicts $p_m \not\preceq p_1 v \ldots v p_{m-1}$.

Hence there exists $p_i(1)$ such that $p_i(1) \equiv q_1 v \ldots v q_{m-1}$. Then since $q_1 v \ldots v q_{m-1} \preceq q_1 v \ldots v q_{m-1} v p_i(1) \preceq q_1 v \ldots v q_m$, by the covering property we have $q_1 v \ldots v q_{m-1} v p_i(1) = q_1 v \ldots v q_m$. Hence

$$p_i \preceq q_1 v \ldots v q_{m-1} v p_i(1) \quad \text{for} \quad i = 1, \ldots, m.$$  

Starting from (2) instead of (1), there exists $p_i(2)$ such that $p_i(2) \not\preceq q_1 v \ldots v q_{m-2} v p_i(1)$. Then we have $i(1) \not\equiv i(2)$ and

$$q_1 v \ldots v q_{m-2} v p_i(1) v p_i(2) = q_1 v \ldots v q_{m-1} v p_i(1) = q_1 v \ldots v q_m.$$
Continuing this process we obtain \( p_i(m) \) such that
\[
p_i(m) \nmid p_1(m-1) \quad \ldots \quad p_{i-1}(m-1).
\]
We have \( i(m) \neq i(1), \ldots, i(m-1) \)
and \( p_1(m-1) \ldots p_i(m) = q_1 \ldots p_1(m-1) \ldots p_i(m-1) = \ldots = q_1 \ldots q_m. \)
Hence \( p_1 \ldots p_m = q_1 \ldots q_m. \)

q.e.d.

We now relate the subsets \( \mathcal{R}^n \) with \( \mathcal{R}^{n-1}. \)

**Lemma (3.7):** Let \( L \) be an atomistic join-semilattice with the \( \mathcal{R}^{n-1} \) covering property.

(i) If \( a \in \mathcal{R}^n \) and \( 0 < b < a \), then \( \exists b \in \mathcal{R}^{n-1}. \)

(ii) If \( \forall a \in \mathcal{R}^n \) then \( a \lor x \) exists for every \( x \in L. \)

**Proof:** (i) Let \( 0 < b < a \in \mathcal{R}^n. \) If \( b \in \mathcal{R}^{n-1}, \) then since \( L \) is atomistic, there exist \( p_1, \ldots, p_n \in \mathcal{R} \) such that \( p_i \leq b \) and \( p_i \nmid p_1 \ldots p_{i-1}. \) Since \( p_i \leq a \in \mathcal{R}^n, \) it follows from Lemma (3.6) (i) that \( a = p_1 \ldots p_n \leq b, \) a contradiction.

(ii) Let \( B \) be the set of all lower bounds of \( \{a, x\}. \)
If \( B = \{0\}, \) we have \( a \lor x = 0. \) If \( B \neq \{0\}, \) then \( B \lor a \) is not empty since \( L \) is atomistic. For any subset \( \{p_1, \ldots, p_n\} \) of \( B \lor a \) such that

\[(*) \quad p_i \nmid p_1 \ldots p_{i-1} \quad \text{for} \quad i \geq 2,
\]
we have $m \leq n$ by Lemma (3.6)(ii). Since $p_i \leq a$ for all $i$. Hence, there exists a maximal subset $\{p_1, \ldots, p_k\}$ satisfying (*) \footnote{(*)}. Then, $q \leq p_1 \vee \ldots \vee p_k$ for every $q \in \mathcal{B} \cap \mathcal{A}$, and hence $b \leq p_1 \vee \ldots \vee p_k$ for every $b \in \mathcal{B}$ since $L$ is atomistic. Thus, $p_1 \vee \ldots \vee p_k$ is the greatest element of $\mathcal{B}$.

\textit{q.e.d.}

Let us now discuss the lattice theoretic position in $L$ of $F$, the set of all finite elements in $L$. In this context, let us mention that if $A$ is replaced by $F$, then resulting $F$-covering property is called finite-covering property.

\textbf{Theorem (3.8)}: Let $L$ be an atomistic join-semilattice.
If $L$ has the finite-covering property, then $F$ is an ideal of $L$ and $F$ forms a lattice by the same order as $L$.

\textbf{Proof}: Evidently, $a, b \in F$ implies $a \vee b \in F$. If $b \leq a \in F$ then $b \in F$ by Lemma 3.7 (i). Hence, $F$ is an ideal of $L$.
If $a, b \in F$ then by Lemma 3.7 (ii), $a \wedge b$ exists. Therefore $F$ is a lattice.

\textit{q.e.d.}

In order to show that $F$ forms a special type of lattice under some restriction, we recall many known concepts.
Definition (3.4). Let \( a \) and \( b \) be elements in a lattice \( L \) with \( 0 \). An element \( b_1 \) is called a left complement within \( b \) of \( a \) in \( a \lor b \) when \( a \lor b = a \lor b_1 \), \( a \land b_1 = 0 \), \( (b_1, a) \in M \) and \( b_1 \leq b \).

We call \( L \) a left complemented lattice when for every pair of elements \( a \) and \( b \) in \( L \) there exists such a left complement.

Definition (3.5): A lattice \( L \) is called \( M \)-symmetric if for \( a, b \) in \( L \), \( (a, b) \in M \) implies \( (b, a) \in M \).

Corollary (3.9): Let \( L \) be an atomistic join-semilattice. Then the set \( F \) of all finite elements of \( L \) is left complemented and \( M \)-symmetric.

Proof: If we show that the interval \( L[0, a] \) is left complemented for every \( a \in F \), then proof is complete.

Since \( L[0, a] \) is an \( Ac \)-lattice of finite length, it follows from Lemma (6.10) and Theorem (7.15) of [5] that \( L[0, a] \) is left complemented. Hence \( F \) is left-complemented. By Theorem (7.9) of [5] \( F \) is also \( M \)-symmetric.

q.e.d.

The notion of height is well known in lattices. In the same vein, we have
Definition (3.6): Let $L$ be an atomistic join-semi-lattice with the finite-covering property. It follows from Lemma (3.6) (ii) that, for $a \in F$, the number of atoms $\{p_i\}$ satisfying $p_i \leq a$ and $p_i \not\leq p_{i+1}$ is uniquely determined. This number is called the height (or dimension) of $a$ and is denoted by $h(a)$ (see [5], (8.5)).

Evidently, $h(a) \leq n$ if and only if $a \in \bigcup_{n=0}^{n} \{p\}$, and $a \leq b$ implies $h(a) < h(b)$ $(a, b \in F)$.

In the proof of the next proposition, we need the concept of semiorthogonality in a lattice.

Definition (3.7): In a lattice $L$ with $0$, if there exists a binary relation "⊥" which satisfies the following axioms:

1) $a \perp a$ implies $a = 0$,
2) $a \perp b$ implies $b \perp a$,
3) $a \perp b, a \perp a$ imply $a \perp b$,
4) $a \perp b, a \lor c \perp c$ imply $a \perp a \lor c$,

then two elements $a$ and $b$ of $L$ are said to be semi-orthogonal when $a \perp b$, and $L$ is called a semi-ortholattice.

If $F$ is an Ac-lattice, then by Theorem (8.11) and Theorem (3.1) of [5], $F$ has a semi-orthogonal relation
a \perp b defined by the conditions; \( a \land b = 0 \) and \( (a,b)M \).

**Proposition (3.10):** For \( a,b \in F \),

\[
h(\overline{a \lor b}) + h(\overline{a \land b}) \leq h(a) + h(b),
\]
equality holds if and only if \( (a,b)M \).

**Proof:** (I) Suppose \( a,b \in F \). Since \( F \) is left complemented, there exists \( a_1 \) such that

\[
a = (a \land b) \lor a_1 , \quad \text{and} \quad a \land b \perp a_1 .
\]

Now \( a_1 \lor b = a_1 \lor (a \land b) \lor b = a \lor b \).

Hence by Lemma (3.13) of \([5]\) we have

\[
h(\overline{a \lor b}) \leq h(a_1) + h(b) \quad \text{and}
\]

\[
h(a) = h(a \land b) + h(a_1),
\]

Hence \( h(\overline{a \lor b}) + h(\overline{a \land b}) \leq h(a) + h(b) \).

(II) Assume that \( (a,b)M \) holds. If \( b_1 \) is a left complement within \( b \) of \( a \) in \( a \lor b \) then by Lemma (3.8) of \([5]\) \( b_1 \) is a left complement of \( a \land b \) in \( b \). Hence

\[
h(\overline{a \lor b}) + h(\overline{a \land b}) = h(a) + h(b_1) + h(a \land b)
\]

\[= h(a) + h(b).\]
Now assume that \( (a,b)_M \) does not hold. Then there exists \( c \in F \) with \( c < b \) such that

\[(cva) \land b > cv(a \land b). \]

Put \( x = (cva) \land b \) and \( y = cv(a \land b) \). Since \( a \land b \leq a \land y \leq a \land x = a \land b \), we have \( a \land y = a \land b \). By (I) we have

\[h(a) + h(y) \geq h(a \lor y) + h(a \land y) = h(cva) + h(a \land b)\]

and

\[h(b) + h(cva) \geq h(a \lor b) + h(x).\]

Hence

\[h(a) + h(b) \geq h(cva) + h(a \land b) - h(y) + h(a \lor b) + h(x) - h(cva)\]

\[= h(a \lor b) + h(a \land b) + h(x) - h(y) > h(a \lor b) + h(a \land b).\]

q.e.d.

An atomistic join-semilattice with the covering property ( = L-covering property) may be called an Ac-join-semilattice. For Ac-join-semilattice we discuss the necessary and sufficient condition for an element to be an atom of an interval of \( L \).

Theorem (3.17): Let \( L \) be an Ac-join-semilattice.

(i) If \( a < b \) in \( L \), then the interval \( [a,b] \) is also an Ac-join-semilattice, and an element \( c \in L \) is an atom of \( [a,b] \) if and only if there exists an atom \( p \) of \( L \) such that
(*) \( c = \text{avp}, \ p \nmid a \ \text{and} \ p \leq b. \)

The same statement holds for the interval \([a, \rightarrow]\).

(ii) For \(a, b \in L\), \(a \wedge b\) exists if there exists a lower bound \(c\) of \(\{a, b\}\) such that \(a\) (or \(b\)) is a finite element of \([c, \rightarrow]\).

Proof: (I) In this first part we shall prove that an element \(c\) is an atom of \([a, b]\) if and only if (*) holds. Firstly suppose that (*) holds. Then by the covering property of \(L\), \(p \nmid a\) implies \(a < \text{avp} = c \leq b\), i.e., \(a < c\). Thus \(c\) is an atom of \([a, b]\).

Conversely, if \(c\) is an atom of \([a, b]\) then choose an atom \(p \nmid a\) but \(p \leq c\). Then we have \(a < \text{avp} \leq c\). But \(a < c\), hence \(c = \text{avp}\), therefore (*) holds.

(II) In the second part we shall prove that for \(a < b\), the interval \([a, b]\) is also an \(Ac\)-join-semilattice. Assume that \(x \nmid y\) in \([a, b]\). Then there exists an atom \(p\) of \(L\) such that \(p \leq x\) but \(p \nmid y\). Since \(p \nmid a\), and \(p \leq b\), \(c = \text{avp}\) is an atom of \([a, b]\). Then \(c \nmid y\) and \(c \leq x\). Thus \([a, b]\) is atomistic. Since the statement (iv) of Theorem (3.3) holds in \(L\), it also holds in \([a, b]\). Hence \([a, b]\) has the covering property. Thus \([a, b]\) is an \(Ac\)-join-semilattice.
(ii) Suppose $c$ is a lower bound of $\{a,b\}$ and let $a$ be a finite element of $[c, \rightarrow]$. By (i) of this theorem, $[c, \rightarrow]$ is also an $\mathbb{A}$-join-semilattice. Hence by Lemma (3.7), since $a$ is a finite element of $[c, \rightarrow]$, $a \wedge b$ exists.

q.e.d.

Now we supply an example of an $\mathbb{A}$-join-semilattice which is not a lattice. This gives weightage to the necessity of building up the study of theory of symmetricity in general join-semilattices.

Example (3.12): Let $L$ be a matroid lattice with infinite length (for instance, the lattice of all linear subspaces of an infinite dimensional linear space). Put

$$L_0 = \{1\} \cup \{\text{dual atoms}\} \cup F.$$

Then $L_0$ is an $\mathbb{A}$-join-semilattice by the same order as $L$, and it is not a lattice. There exists a non-dual-modular pair in $L_0$, even if $L$ is modular. In fact, if $a$ is a dual atom and $b \uparrow a$, then $(a,b)$ is not dual-modular; because if we take a dual atom $c$ with $c \geq b$, then we have $b < c < \top = a \vee b$ and $d \vee b \neq c$ for every $d \leq a$. 
Let us also observe that the dual statements of the results obtained in this section give some lemmas and theorems about modular pairs, dual atoms, dual covering property and dual exchange property for meet-semilattices. However, we shall not state them explicitly.

4. *Dual-modular pairs of subsets in atomistic join-semilattices.*

Now, let us extend the concept of dual-modular pairs to that of dual-modular subsets of a join-semilattice.

Let \( L \) be a join-semilattice and let \( A \) and \( B \) be subsets of \( L \). We write \((A, B)^*\) when \((a, b)^*\) for every \( a \in A \) and \( b \in B \).

The concept of dual modular subsets makes it possible to obtain a nice characterization of \( A \)-covering property, \( (A \leq L) \).

**Proposition (4.1)** If \( A \leq L \), then \((-\omega, A)^*\) holds if and only if \( L \) has the \( A \)-covering property.

**Proof:** Suppose \((-\omega, A)^*\) holds and let \( p \in L \), \( a \in A \), \( p \nleq a \). We shall show that \( a \leq a \wedge p \). Let \( a \leq x \leq a \wedge p \).

Since \((-\omega, A)^*\) holds, \((p, a)^*\) holds. Hence there
exists \( d \leq p \) such that \( x = d \vee a \). But \( p \) is atom.

Therefore, either \( d = p \) or \( d = 0 \). If \( d = p \) then \( x = p \cdot a \) and if \( d = 0 \) then \( x = a \). Thus \( a \not\leq \text{avp} \). Conversely, suppose \( L \) has the \( A \)-covering property. Suppose \( p \not\in L \) and \( a \in A \). If \( p \leq a \) then trivially \( (p,a)M^* \) holds. Let \( p \not\leq a \) and \( a \leq x \leq \text{avp} \). By \( A \)-covering property we have \( a = x \) or \( x = p \cdot a \). If \( a = x \) then take \( d = 0 \) and if \( x = p \cdot a \) then choose \( d = p \) so that \( d \leq p \) and \( d \cdot a = x \). Hence \( (p,a)M^* \) holds.

q.e.d.

It is easy to see that if \( A_1 \subseteq A_2 \) and \( B_1 \subseteq B_2 \), then \( (A_2,B_2)^* \Rightarrow (A_1,B_1)^* \).

Let us also explicitly state some notations that we shall adopt hence afterwards.

\( \text{av}_0 \) will denote the atom space \( \text{av} \) along with the least element 0 of \( L \). Similarly, \( A \vee B = \{ avb; a \in A \) and \( b \in B \} \).

In what follows \( L \) always denotes an atomistic join-semilattice.

**Lemma (4.2):** \( (A \text{av} \text{av}_0, \text{av}^{n-1})M^* \Rightarrow (A, \text{av}^n)M^* \) for any \( A \subseteq L \) and for \( n > 2 \).
Proof: Let \( a \in A \) and \( b \in A^m \). Then, \( b = p \cap c \) with \( p \in \mathfrak{A} \) and \( c \in \mathfrak{A}^{-1} \). By assumption we have \((a, p)^* M^*\) and \((a, p, c)^* M^*\), and hence \((a, p, c)^* M^*\) by Lemma (2.4). Therefore, \((A, \mathfrak{A}^m)^* M^*\) holds.

q.e.d.

We now state one more concept that was first introduced by Macca [6].

Definition (4.1): Let \( a \) and \( b \) be non-zero elements of \( L \). We write \((a, b)^P\) if for any \( p \in \mathfrak{A} \) with \( p \leq a \cap b \) there exist \( q, r \in \mathfrak{A} \) such that \( q \leq a \), \( r \leq b \) and \( p \leq q \cap r \).

For subsets \( A \) and \( B \) of \( L \), we write \((A, B)^P\) if \((a, b)^P\) for every \( 0 \neq a \in A \) and \( 0 \neq b \in B \). Note that \((A, B)^P\) is symmetric and that \( a \leq b \) implies \((a, b)^P\).

Let us interrelate the notions of dual-modular pairs and \( P \)-relation in the following.

Lemma (4.3): Assume that \((A, \mathfrak{A}^m)^* M^*\) holds for \( L \) (that is, \( L \) has the \( \mathfrak{A}^m \)-covering property). For \( p \in \mathfrak{A} \) and \( 0 \neq a \in L \), we have the following implications:

\[(a, p)^* M^* \implies (a, p)^P \implies (p, a)^* M^*\]
Proof: We shall prove \((a, p)M^* \Rightarrow (a, p)P\). We may assume \(p \not\models a\). Take \(q \in \mathcal{N}\) with \(q \sqsubseteq avp\). If \(q = p\), then \(q \sqsubseteq rvp\) for any \(r \sqsubseteq a\). If \(q \neq p\) then since \(p \sqsubseteq pvq \sqsubseteq avp\) and \((a, p)M^*\), there exists \(d \sqsubseteq a\) with \(dvp = pvq\). Then, \(d \neq 0\), since otherwise \(p = q\), a contradiction. Hence there exists \(r \in \mathcal{N}\) with \(r \sqsubseteq d\). Then \(r \not\models p\) since \(r \sqsubseteq a\) and \(p \not\models a\). Now \(r \sqsubseteq pvq\) implies \(q \sqsubseteq rvp\) by the \(-\mathcal{N}\)-exchange property. This shows that \((a, p)P\) holds.

Next, assume that \((a, p)P\) and let \(a \sqsubseteq c \sqsubseteq pva\). If \(c = a\) then \(c = ova\). If \(c > a\) then there exists \(q \in \mathcal{N}\) such that \(q \sqsubseteq c\) and \(q \not\models a\). Since \(q \sqsubseteq avp\), there exists \(r \in \mathcal{N}\) such that \(r \sqsubseteq a\) and \(q \sqsubseteq rvp\). Now \(r \not\models q\). Hence by the \(-\mathcal{N}\)-exchange property we have \(p \sqsubseteq rvq \sqsubseteq avc = c\). Therefore, \(p \sqsubseteq c\), \(a \sqsubseteq c\) imply \(pva \sqsubseteq c\). Thus \(c = avp\). Which shows \((p, a)M^*\).

\[\text{q.e.d.}\]

More generally, we have

**Lemma (4.4):** Let \(A\) and \(B\) be subsets of \(L. (\neg \mathcal{N}, \mathcal{N})M^*, (B, \neg \mathcal{N})M^*\) and \((A, B)M^*\) together imply \((A, B)P\).
**Proof**: We shall prove \((a,b)p\) for \(0 \neq a \in A\) and \(0 \neq b \in B\). Take \(p \in \Lambda^*\) with \(p \leq ab\). Putting \(c = bvp\), we have \(b \leq c \leq ab\). Since \((a,b)m^*\), there exists \(d \leq a\) with \(dvp = c\). If \(d \leq b\), then since \(p \leq c = b\), taking any atom \(q \leq a\) and putting \(r = p\) we get \(p \leq qvr\). If \(d \nleq b\), then there exists \(q \in \Lambda^*\) such that \(q \leq d\) and \(q \nleq b\). Then \(q \leq a\), and \(q \leq c \leq bvp\). Since \((b,p)m^*\) implies \((b,p)p\) by lemma (4.3), there exists \(r \in \Lambda^*\) such that \(r \leq b\), \(q \leq rvp\). Since \(r \neq q\), we have \(p \leq qvr\) by the \(\Lambda\)-exchange property. Hence \((a,b)p\) holds.

q.e.d.

In the same vein, we obtain

**Lemma (4.5)**: Let \(A\) be a subset of \(L. (\Lambda^*, \Lambda^{n-1})^\)** and \((\Lambda^*, A)p\) together imply \((\Lambda^n, A)^*\) \((n \geq 2)\).

**Proof**: We shall prove \((u,a)m^*\) for \(u \in \Lambda^n\) and \(a \in A\). We may assume \(u \neq 0\), \(a \neq 0\). Let \(a \leq c \leq uva\). From Lemma (3.7) (ii), it follows that \(u \uparrow c = d\) exists. It suffices to show that \(dva = c\). Evidently, \(dva \leq c\). If \(dva < c\), then there would exist \(p \in \Lambda^*\) such that \(p \leq c\) and \(p \nleq dva\). Since \(p \leq c \leq uva\) and \((u,a)p\), there are \(q, r \in \Lambda^*\) such that \(q \leq u, r \leq a\) and \(p \leq qvr\). Then, \(p \nleq r\) since \(p \nleq a\). Hence, by the \(\Lambda\)-exchange property we have \(q \leq pvr \leq cva = c\), and hence \(q \leq u \uparrow c = d\). From this, \(p \leq qvr \leq dva\), a contradiction.

q.e.d.
The next property of P-relation needs to be noted explicitly.

Lemma (4.6): (i) Let $A$ and $B$ be subsets of $L$, each of which contains $\bot$. Then, $(A, B v L)P$ implies $(A v \bot, B)P$.

(ii) $(L, L)P$ implies $(F, L)P$.

Proof: (i) Let $0 \notin a \Delta$, $0 \notin b \Delta$ and $q \notin \bot$. We shall prove $(avq, b)P$. If $p \notin \bot$ and $p \leq avqb$, then, by $(A, B v \bot)P$, there exist $r, s \notin \bot$ such that $r \leq a$, $s \leq qvb$ and $p \leq r v s$. Moreover, since $\bot \subseteq A$, we have $(q, b)P$. Hence, there exists $t \notin \bot$ such that $t \leq b$ and $s \leq qvt$. Then, $p \leq r v q v t$. Since $\bot \subseteq B v \bot$, we have $(t, rvq)P$. Therefore there exists $\bot \subseteq B v \bot$ such that $t \leq rvq$ and $p \leq t v t$. Then $t \leq avq$. Thus $(avq, b)P$ holds.

(ii) Putting $A = \bot \subseteq \bot$ and $B = L$ in (i), we obtain that $(\bot \subseteq \bot, L)P$ implies $(\bot \subseteq \bot, L)P$. Hence $(\bot \subseteq \bot, L)P$ implies $(\bot \subseteq \bot, L)P$ for every $n$, i.e. $(F, L)P$ holds.

q.e.d.

Note the symmetry of dual modularity of $P$ and $L$ in Proposition (4.7) below.
Proposition (4.7): \((L,F)^*\) implies \((F,L)^*\).

Proof: Suppose \((L,F)^*\). Since \(\mathcal{N} \subseteq L\), \(\mathcal{N} \subseteq F\) and \(F \subseteq L\), \((L,F)^*\) implies \((\mathcal{N}, \mathcal{N})^*\) and \((F, \mathcal{N})^*\).

Putting \(A = L\) and \(B = F\) in Lemma (4.4) we have \((\mathcal{N}, \mathcal{N})^*\).

Putting \(A = L\) in Lemma (4.5) we get \((\mathcal{N}, \mathcal{N}^{n-1})^*\) and \((\mathcal{N}^n, L)^*\) which together imply \((L,F)P = (F,L)P\).

We give more characterizations involving dual modularity of some special subsets of \(L\) including \(L\) itself.

Proposition (4.8): (i) \((L, \mathcal{N}^n)^*\) if and only if \((L,F)^*\) for \(n \geq 1\).

(ii) \((F, \mathcal{N}^n)^*\) if and only if \((F,F)^*\) for \(n \geq 1\).

Proof: (i) If \((L,F)^*\) then since \(\mathcal{N} \subseteq F\) for every \(n\), we have \((L, \mathcal{N}^n)^*\).

Now let \((L, \mathcal{N}^n)^*\) for every \(n \geq 1\). If we put \(A = L\) in Lemma (4.2) then \(NW_{\emptyset} = A\). Hence \((A, \mathcal{N}^n)^*\) implies \((A, \mathcal{N}^{n+1})^*\) for every \(n \geq 1\). Thus \((L,F)^*\).
(ii) If \((F, F)^*\) then since \(\omega_n \subseteq F\) for every \(n \geq 1\),
\((F, \omega_n)^*\) holds.

Conversely, if \((F, \omega_n)^*\) for every \(n \geq 1\) then putting
\(A = F\) in Lemma (4.2) we have \(A \cap \omega_0 = A\) and hence
\((A, \omega_n)^*\) implies \((A, \omega_{n+1})^*\) for every \(n \geq 1\).
Therefore \((A, F)^*\) i.e. \((F, F)^*\).

q.e.d.

In the same vein one has

Proposition (4.9) : \((\omega_n, L)^*\) if and only if
\((F, L)^*\) for \(n \geq 2\).

Proof: \((F, L)^* \Rightarrow (\omega_n, L)^*\), since \(\omega_n \subseteq F\) for
every \(n\). Suppose \((\omega_n, L)^*\) for \(n \geq 2\). We shall
prove \((\omega_{n+1}, L)^*\). Let \(u \in \omega_n, a \in L\). We put
\(u = p_0 \vee p_1 v \ldots \vee p_n\) where \(p_i \in \omega\). If

\(p_i \leq a \vee p_0 v \ldots \vee p_{i-1}\) for some \(i(0 \leq i \leq n)\),

then putting \(v = p_0 \vee p_1 v \ldots \vee p_{i-1} v \vee p_{i+1} v \ldots \vee p_n\) we have
\(v \in \omega_n\) and \(a \vee v = a \vee u\). Since \((v, a)^*\) by assumption
and since \(u \in [v, vv a]\), we have \((u, a)^*\) by Lemma 2.4(ii).
Hence, we may assume

\((*)\) \(p_i \leq a \vee p_0 v \ldots \vee p_{i-1}\) for every \(i = 0, 1, \ldots, n\).

Since \((\omega_n, L)^*\) implies the covering property, \(L\) is
an Ac-join-semilattice and hence \([a,avu]\) is also an Ac-join-semilattice by Theorem (3.11). For \(x \in [a,avu]\), we denote by \(h(x)\) the height of \(x\) in \([a,avu]\). It follows from \((*)\) that \(h(ayu) = n+1\).

Let \(c \in [a,avu]\), and we shall show the existence of \(d \in L\) with \(d \leq u\) and \(dva = c\). If \(c = a\), then \(d = 0\), which is desired. Next, we assume \(h(c) = 1\). Putting \(v = p_1v...v_p\), we have

\[avp_0 \leq cvp_0 \leq avu = vvavp_0.\]

Since \((v,avp_0)^n^*\), there exists \(x \leq v\) such that \(xvavp_0 = cvp_0\). If \(x = v\), then we would have \(cvp_0 = vvavp_0 = avu\), and then

\[n+1 = h(avy) = h(cvp_0) \leq h(c) + 1 = 2 \leq n,\]
a contradiction. Hence \(x < v\). Since \(v \in \Omega^n\), we have \(x \in \Omega^{n-1} \{c\}\) by Lemma (3.7) (1), and hence \(xvp_0 \in \Omega^n\). Since \(a \leq c \leq xvap_p\), it follows from \((xvp_0,a)^n^*\) that there exists \(d \leq xvp_p\) such that \(dva = c\). Then \(d \leq vvp_p = u\).

If \(h(c) = k \geq 2\), then there exist \(c_1, \ldots, c_k \in [a,avu]\) such that \(h(c_i) = 1\) and \(c = c_1v...vc_k\). As above, there exist \(d_1 \leq u\) such that \(d_1va = c_1\). Putting \(d = d_1v...vd_k\), we have \(d \leq u\) and \(dva = c\).

q.e.d.
Next equivalence is also worth noting.

**Proposition (4.10)**: \((\mathcal{L}_n, \mathcal{F})^M\) if and only if
\[(F, F)^M\text{ for } n \geq 2.\]

**Proof**: Trivially \((F, F)^M\) implies \((\mathcal{L}_n, F)^M\).
By the same way as above, we can prove \((\mathcal{L}_n, F)^M\) implies \((\mathcal{L}_{n+1}, F)^M\) for \(n \geq 2\). In fact, if \(u \in \mathcal{L}_{n+1}\) and \(a \in F\), then since \((\mathcal{L}_n, F)^M\) implies the \(F\)-covering property, \([a, au]\) is an \(Ac\)-join-semilattice. The remaining proof is the same as in Proposition 4.9.

q.e.d.

One shows that dual modularity of subsets \(\mathcal{L}_m, \mathcal{L}_p\) for \(m \geq 2, p \geq 1\) \((m+p = n+1, n \geq 3)\) are all equivalent to each other.

**Proposition (4.11)**:
\[(\mathcal{L}_n, \mathcal{L}_m)^* \iff (\mathcal{L}_{n-1}, \mathcal{L}_2)^* \iff \ldots \iff (\mathcal{L}_2, \mathcal{L}_{n-1})^*\]
for \(n \geq 3\).

**Proof**: Let \(n \geq 3\). First we shall prove that \((\mathcal{L}_n, \mathcal{L}_m)^*\) implies \((\mathcal{L}_n, \mathcal{L}_m)^*\). If \((\mathcal{L}_n, \mathcal{L}_m)^*\), then putting \(A = \mathcal{L}_{n-1}\) and \(B = \mathcal{L}_2\) in Lemma (4.4) we get \((\mathcal{L}_{n-1}, \mathcal{L}_2)^*\).
Hence by Lemma (4.6), \((-\Omega^n, \Omega)_P\) holds and by Lemma (4.3),
\((-\Omega, \Omega^n)_M^*\) holds. Now, it follows from Lemma (4.5) that
\((-\Omega^n, \Omega)_M^*\) holds.

Since the implications \((-\Omega^n, \Omega)_M^* \implies (-\Omega^{n-1}, -\Omega^2)_M^* \implies \ldots \implies (-\Omega^2, -\Omega^{n-1})_M^*\) hold by
Lemma (4.2), it suffices to show that

\[ (*) \quad (-\Omega^2, -\Omega^{n-1})_M^* \text{ implies } (-\Omega^{n-1}, -\Omega^2)_M^*. \]

When \(n = 3\), \((*)\) is trivial.
Assume that \((*)\) is true for \(n = k\),
i.e. \((-\Omega^2, -\Omega^{k-1})_M^* \implies (-\Omega^{k-1}, -\Omega^2)_M^*\).

To prove that \((*)\) is true for \(n = k+1\). Let \((-\Omega^2, -\Omega^k)_M^*\) hold. Then \((-\Omega^2, -\Omega^{k-1})_M^*\) holds. Hence, by assumption
\((-\Omega^{k-1}, -\Omega^2)_M^*\) holds. Therefore, \((-\Omega^k, -\Omega)_M^*\) holds as shown above. Now \((-\Omega, -\Omega)_M^*\) is also true. In Lemma (4.4),
putting \(A = -\Omega^2\), \(B = -\Omega^k\) we have \((-\Omega, -\Omega)_M^*, (-\Omega^k, -\Omega)_M^*\)
and \((-\Omega^2, -\Omega^k)_M^*\) together imply \((-\Omega^2, -\Omega^k)_P\) i.e.
\((-\Omega^k, -\Omega^2)_P\). By Lemma (4.5), \((-\Omega^k, -\Omega^2)_M^*\) holds. Thus
\((*)\) is true for \(n = k+1\) and hence by induction \((*)\) is
true for every \(n \geq 3\).

q.e.d.
As a final result of this chapter, we invite attention, in the light of Proposition (4.11), to symmetry of dual modularity of subsets $\Omega$ and $-\Omega^n \ (n \geq 2)$.

Proposition (4.12): $(\Omega^2, \Omega^{n-1})^*_M \Rightarrow (\Omega, \Omega^n)^*_M$ for $n \geq 2$.

Proof: In Lemma (4.2), putting $A = \Omega$ we get

$$(\Omega^2 \vee \Omega^{n-1})^*_M \Rightarrow (\Omega, \Omega^n)^*_M,$$

which is required.

q.e.d.

Remark (4.13): It needs to be explicitly pointed out that the reverse implications in Proposition (4.7) and Proposition (4.12), and the following implications are not reversible.

$$(\Omega^2, L)^*_M \Rightarrow (\Omega^2, F)^*_M \Rightarrow \ldots \Rightarrow (\Omega, \Omega^n)^*_M \Rightarrow \ldots \Rightarrow (\Omega^2, \Omega)^*_M,$$

$$(\Omega, L)^*_M \Rightarrow (\Omega, F)^*_M \Rightarrow \ldots \Rightarrow (\Omega, \Omega^n)^*_M \Rightarrow \ldots \Rightarrow (\Omega, \Omega)^*_M,$$

$$(\Omega^2, L)^*_M \Rightarrow (\Omega, L)^*_M,$$

$$(\Omega^2, F)^*_M \Rightarrow (\Omega, F)^*_M.$$
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