CHAPTER 1
INTRODUCTION

The theory of differential equations began its development as a branch of modern mathematics only in nineteenth and twentieth centuries through the pioneering work of a number of mathematicians, notably Birkhoff, Cauchy, Lyapunov, Picard, Poincaré and Riemann. This theory is essential for an understanding of many important physical and mathematical problems occurring in immense areas of science and technology. In many physical phenomena, the future state of physical system depends not only on the present state but also on its past history. Differential equations with a deviating argument or delay - differential equations or more generally functional differential equations provide a mathematical model for such physical systems. Although functional differential equations were first encountered in the eighteenth century by Bernoulli, Condorcet and Laplace but a systematic study of these equations was accelerated in the twentieth century by the demands of various branches of science and technology. From the beginning of this century, various qualitative aspects of these equations and their applications have been investigated by a number of researchers. An excellent account on this subject may be found in the monographs by R.Bellman and K.L. Cooke [4], T.A.Burton [12], C.Corduneanu [13], L.E. El'sgol'ts and S.D. Norkin [18],
A. Halanay [25], J. Hale [26], N.N. Krasovskii [41],
R.K. Miller [55], A.D. Myshkis [56] and the references given therein:

The theory of nonlinear ordinary differential equations in
arbitrary Banach spaces is an extremely important and interesting
area of research in differential equations. It is well known that
the theory of semigroups of operators is closely related to the
solutions of nonlinear ordinary differential equations in Banach
spaces. Many problems in the theory of abstract systems and
semigroup of operators have been studied extensively in the lite-
rature over the last forty years. As a consequence there now exist
several monographs, see, V. Barbu [2], N. Dunford and J. Schwartz
[16], A. Friedman [20], J.A. Goldstein [22], E. Hille and
R.H. Phillips [31], T. Kato [40], S.G. Krein [42], G.E. Ladas
and V. Lakshmikantham [44], V. Lakshmikantham and S. Leela [46],
R.H. Martin [54], A. Pazy [68], H. Tanabe [72], K. Yosida [80]
and others. Many problems of partial differential equations
arising from physical models such as heat flow in materials, study
of viscoelasticity and several other areas of applied mathematics,
can be represented in the form of ordinary differential equations
in arbitrary Banach spaces. However, the study of abstract versions
of partial differential and integrodifferential equations with
functional arguments is substantially more difficult than that of
nonlinear abstract ordinary differential equations without func-
tional argument. During the past few years, many investigators
have studied the abstract versions of such equations by using various methods, see for example the recent papers by A. Belleni-Morante [3], W.E. Fitzgibbon [19], M.L. Heard [27, 28], S. Heikkila [29, 30], A.G. Kartsatos [33], A.G. Kartsatos and M.E. Parrott [34-36], J.H. Lightbourne III and S.M. Rankin III [50], J.A. Nohel [58], B.G. Pachpatte [60, 62, 63, 67], C.C. Travis and G.F. Webb [73, 74], G.F. Webb [78, 79] and some of the references given therein. Although this sophisticated area of research has, in recent years, developed significantly, the field of abstract functional integrodifferential equations of the more general type awaits for its development. This motivates the author to study abstract functional integrodifferential equations of the more general type.

The aim of the present thesis entitled "Some problems in the theory of nonlinear integral equations" is to investigate various results pertaining to existence, uniqueness, continuation, asymptotic behavior and other properties of the solutions of abstract functional integrodifferential equations of more general type. Banach's and Schauder's fixed point theorems, Comparison theorems, variation of parameters formula, successive approximation method, $\epsilon$-approximation method, Kato's method, semigroup theory, accretive operator theory, evolution operator theory and the integral inequalities established by B.G. Pachpatte are the main tools employed in our analysis.
Problems investigated in the present thesis are as follows.

In chapter 2, we establish the existence, uniqueness and stability of the solutions of a class of general abstract nonlinear functional integrodifferential equations of the form

\[ x'(t) = f(t, x(t), x(h_1(t)), \ldots, x(h_n(t))), \]

\[ \int_0^t k(t, s, x(s), x(h_1(s)), \ldots, x(h_n(s))) \, ds, \quad t \geq 0, \]

\[ x(t) = \phi(t), \quad t \leq 0, \]

where the function \( \phi(t) : \mathbb{R}^- \to X \) (\( X \) is a Banach space) is given and the functions \( h_i(t) : \mathbb{R}^+ \to \mathbb{R} \) (\( i = 1, 2, \ldots, n \)) are delay or advance. The main tool employed in our analysis is based on abstracting the problem to an operator equation in a function space and the applications of the theory of accretive operators.

Chapter 3 devotes to study the abstract functional integrodifferential equations of the form
\[ x'(t) + Ax(t) = \int_0^t \left[ a(t,s)g(s,x_s) + h(t,s,x_s) \right] ds \]

\[ + f(t,x_t), \quad t \in [0,T], \]

(2)

\[ x_0 = \phi(t), \quad -r \leq t \leq r, \]

and

\[ (Mx(t))' + Bx(t) = \int_0^t \left[ b(t,s)G(s,x_s) + H(t,s,x_s) \right] ds \]

\[ + F(t,x_t), \quad t > r, \]

(3)

\[ x_0 = \psi(t), \quad -r \leq t \leq r, \]

where \(-A\) is an infinitesimal generator of a strongly continuous semigroup in a Banach space \(X\), the operators \(M\) and \(B\) are linear and closed, \(\phi\) and \(\psi\) are given functions. The main results concerning the existence, uniqueness, continuation, and asymptotic behavior of solutions of the equations (2) and (3) are obtained by using applications of semigroup theory, the method of successive approximations, Schauder's fixed point theorem, variation of parameters formula, \(\epsilon\)-approximation method and integral inequalities established by Pachpatte.
In Chapter 4, we study the existence, uniqueness, stability, boundedness and asymptotic behavior of solutions of abstract functional integrodifferential equations of the form

$$x'(t) + A(t)x(t) = f(t, x_t, \int_0^t k(t, s, x_s) \, ds), \quad t \in [0, T],$$

$$x_0 = \phi(t), \quad -r \leq t \leq 0,$$

where $-A(t)$, for each $t \in [0, T]$, is the infinitesimal generator of an analytic semigroup and $\phi(t)$ is a given function. The method of successive approximation, comparison theorems and the integral inequality recently established by B.G. Pachpatte are the key instruments to develop our results.

Chapter 5 deals with the existence and uniqueness of a strong solution of a nonlinear abstract functional integrodifferential equation of the form

$$x'(t) + A(t)x(t) = \int_0^t \left[ a(t, s)g_0(s, x_s) + g_1(t, s, x_s) \right] ds$$

$$+ f_1(t, x_t) + f_0(t), \quad t \in [0, T],$$

$$x_0 = \phi(t), \quad -r \leq t \leq 0,$$
where the operator $A(t)$, for each $t \in [0,T]$, is $m$-accretive and $\phi$ is a given function. Kato's approach, Yosida approximations and integral inequalities established by B.G. Pachpatte are the main tools used to establish our results.

Here it is to be noted that several interesting problems in the field of abstract functional nonlinear differential and integro-differential equations are not yet investigated in the literature. For example, initial and boundary value problems for abstract functional nonlinear differential and integro-differential equations of higher order are not studied to a satisfactory level and there is a considerable scope to study these problems. We hope that these problems will be solved in near future and will open new fields of applications.