TWO INDEPENDENT VARIABLE GENERALIZATIONS
OF CERTAIN INTEGRAL INEQUALITIES

1. INTRODUCTION

One of the most useful techniques used in the theory of ordinary and partial differential and integral equations consists in applying so called Gronwall type inequalities. Since the appearance of Gronwall's inequality in 1919 in [22], a great deal of attention has been given in the literature to this inequality, although a special case of it had occurred in Peano [48] as early as 1885 (see [7]). A two independent variable generalization of this inequality given by Wendroff [2, p. 154] has evoked, in recent time, a considerable interest, as may seen from the recent papers of Snow [53], Ghosal and Masood [20], Young [60], Chandra and Davis [12] which are motivated by certain applications in the theory of partial differential and integral equations. Our objective is to establish two independent variable generalizations of the integral inequalities established by Wendroff [2, P. 154], Gollwitzer [21] and Pachpatte [33, 36, 39] which can be used in the analysis of various problems in the theory of partial differential and integral equations.
2. MAIN RESULTS

In this section we state and prove some interesting
and useful nonlinear two independent variable generalizations
of the integral inequalities of Wendroff \([2, \text{P. 154}]\) and
Fachpatte \([33, 36]\) which can be used as handy tools in the
theory of partial differential and integral equations.

A useful nonlinear generalization of Wendroff's
inequality is embodied in the following theorem.

**Theorem 1.** Let \(\phi(x,y), a(x,y), b(x,y)\) and \(c(x,y)\) be real-
valued nonnegative continuous functions defined on \(I \times I\),
where \(I = [0, \infty)\), let \(W(x)\) be a real-valued, continuous,
monotonic, nondecreasing, subadditive and submultiplicative,
function for \(x \geq 0\), \(W_y(x) \geq 0\); let \(H(x)\) be a real-valued,
continuous, monotonic, nondecreasing function which is never zero
for \(x \geq 0\); and suppose further that the inequality

\[
(3.1) \quad \phi(x,y) \leq a(x,y) + b(x,y) \int_0^y \int_0^x c(s,t)W(\phi(s,t))dsdt, \\
\]

is satisfied for all \(x, y \in I\), then for \(0 \leq x \leq x_1, 0 \leq y \leq y_1\),

\[
(3.2) \quad \phi(x,y) \leq a(x,y) + b(x,y) \int_0^1 \int_0^1 \left[ \int_0^x \int_0^y c(s,t)W(a(s,t))dsdt + \int_0^x \int_0^y c(s,t)W(b(s,t))dsdt \right],
\]
where

\[ \Omega(x) = \int_{x_0}^{x} \frac{ds}{W(s)}, \quad x \geq x_0 > 0, \]

\[ \Omega^{-1} \] is the inverse function of \( \Omega \), and

\[ \Omega \left( \int_{0}^{x} c(s,t)W(a(s,t))dsdt + \int_{0}^{y} c(s,t)W(b(s,t))dsdt \in \text{Dom}(\Omega^{-1}) \right) \]

for all \( x, y \) lying in the subintervals \( 0 \leq x \leq x_1, 0 \leq y \leq y_1 \) of real numbers in \( I \).

**PROOF.** Define

\[ u(x,y) = \int_{0}^{x} \int_{0}^{y} c(s,t)W(\phi(s,t))dsdt, \quad u(x,0) = u(0,y) = 0, \]

then from (3.1) we have

\[ W(\phi(x,y)) \leq W(a(x,y)) + W(b(x,y)) W(H(u(x,y))), \]

since \( W \) is subadditive and submultiplicative. Using (3.5) in (3.4) we see that the inequality

\[ u(x,y) \leq \int_{0}^{x} \int_{0}^{y} c(s,t)W(a(s,t))dsdt \]

\[ + \int_{0}^{x} \int_{0}^{y} c(s,t)W(b(s,t))W(H(u(s,t)))dsdt, \]
is satisfied for all \(x, y \in I\). Now fix \(\alpha\) and \(\beta\) such that \(0 \leq x \leq \alpha \leq x_1, 0 \leq y \leq \beta \leq y_1\), and set

\[
\Lambda(x, y) = \int_0^x \int_0^y c(s, t) W(a(s, t)) \, ds \, dt,
\]

then

\[
(3.7) \quad u(x, y) \leq \Lambda(\alpha, \beta) + \int_0^x \int_0^y c(s, t) W(b(s, t)) W(H(u(s, t))) \, ds \, dt,
\]

for \(0 \leq x \leq \alpha, 0 \leq y \leq \beta\). Define

\[
(3.8) \quad V(x, y) = \Lambda(\alpha, \beta) + \int_0^x \int_0^y c(s, t) W(b(s, t)) W(H(u(s, t))) \, ds \, dt,
\]

\[
V(x, 0) = V(0, y) = \Lambda(\alpha, \beta),
\]

then

\[
V_{xy}(x, y) = c(x, y) W(b(x, y)) W(H(u(x, y))),
\]

which in view of (3.7) implies

\[
V_{xy}(x, y) \leq c(x, y) W(b(x, y)) W(H(V(x, y))),
\]

i.e.

\[
(3.9) \quad \frac{V_{xy}(x, y)}{W(H(V(x, y)))} \leq c(x, y) W(b(x, y)) .
\]
From (3.9) we observe that

\[
\frac{W(H(V(x,y)))V_x(x,y)}{W^2(H(V(x,y)))} \leq C(x,y)W(b(x,y)) + \frac{V_x(x,y)W_y(H(V(x,y)))}{W^2(H(V(x,y)))}
\]

i.e.

\[
\frac{\partial}{\partial y} \left( \frac{V_x(x,y)}{W(H(V(x,y)))} \right) \leq C(x,y)W(b(x,y)).
\]

By keeping \( x \) fixed in the above inequality, set \( y = t \) and then integrating with respect to \( t \) from \( 0 \) to \( \beta \) we have

\[
(3.10) \quad \frac{V_x(x,\beta)}{W(H(V(x,\beta)))} - \frac{V_x(x,0)}{W(H(V(x,0)))} \leq \int_0^\beta C(x,t)W(b(x,t))dt.
\]

From (3.3) and (3.10) we observe that

\[
\Omega_x(V(x,\beta)) = \Omega_x(V(x,0)) \leq \int_0^\beta C(x,t)W(b(x,t))dt.
\]

Now setting \( x = s \) and then integrating with respect to \( s \) from \( 0 \) to \( \alpha \) we have

\[
(3.11) \quad \Omega(V(s,\beta)) \leq \Omega(V(0,\beta)) + \int_0^\alpha \int_0^\beta C(s,t)W(b(s,t))dsdt,
\]

for \( 0 \leq x \leq \alpha, 0 \leq y \leq \beta \). Since \( \alpha \) and \( \beta \) are arbitrary we have
(3.12) \( V(x,y) \leq \Omega^{-1} \left[ \Omega \left( \int \int C(s,t)W(a(s,t)) \, ds \, dt \right) \right. \\
+ \left. \int \int C(s,t)W(b(s,t)) \, ds \, dt \right] \\
for 0 \leq x \leq x_1, \ 0 \leq y \leq y_1. \) The desired bound in (3.12) follows from (3.1), (3.7) and (3.12).

We next establish the following two independent variable generalization of the integral inequality recently established by Pachpate [33, Theorem 1].

**THEOREM 2.** Let \( \phi(x,y), \ a(x,y), \) and \( b(x,y) \) be real-valued nonnegative continuous functions defined on \( I \times I \); let \( g(x) \) be a positive, continuous, monotonic, nondecreasing and subadditive function for \( x \geq 0 \); \( g_y(x) \geq 0 \); and suppose further that the inequality

\[
(3.13) \quad \phi(x,y) \leq a(x,y) + \int \int b(s,t) (\phi(s,t) + \int \int b(m,n) g(\phi(m,n)) \, dm \, dn) \, ds \, dt,
\]

is satisfied for all \( x, y \in I. \) Then for \( 0 \leq x \leq x_2, \ 0 \leq y \leq y_2 \),

\[
(3.14) \quad \phi(x,y) \leq a(x,y) + \int \int b(s,t) (a(s,t) + \int \int b(m,n) g(a(m,n)) \, dm \, dn) \, ds \, dt
\]

\[
+ \int \int b(s,t) \left\{ -1 + \int \int b(m,n) (a(m,n) + \int \int b(\xi, \eta) g(a(\xi, \eta)) \, d\xi \, d\eta) \, dm \, dn \right\} \, ds \, dt,
\]
where

\[(3.15) \quad E(x) = \int_{x_0}^{x} \frac{ds}{s + g(s)}, \quad x \geq x_0 > 0.\]

\(E^{-1}\) is the inverse function of \(E\), and

\[E(\int \int b(m,n)(a(m,n) + \int \int b(\xi, \eta)g(a(\xi, \eta))d\xi d\eta)dm\alpha n)\]

\[+ \int \int b(m,n)dm\alpha n \in \text{Dom}(E^{-1}),\]

for all \(x, y\) lying in the subintervals \(0 \leq x \leq x_2, 0 \leq y \leq y_2\) of real numbers in \(I\).

\textbf{Proof.} Define

\[(3.16) \quad u(x, y) = \int \int b(s,t)(\phi(s,t) + \int \int b(m,n)g(\phi(m,n))dm\alpha n)dsdt,\]

\[u(x,0) = u(0,y) = 0,\]

then using the facts that \(\phi(x,y) \leq a(x,y) + u(x,y)\) from (3.13) and the subadditive property of \(g\) in (3.16) we have

\[u(x, y) \leq \int \int b(s,t)(a(s,t) + \int \int b(m,n)g(a(m,n))dm\alpha n)dsdt\]

\[+ \int \int b(s,t)(u(s,t) + \int \int b(m,n)g(u(m,n))dm\alpha n)dsdt\]

for all \(x, y \in I\). Now fix \(a\) and \(b\) such that \(0 \leq x \leq a \leq x_2, 0 \leq y \leq b \leq y_2\), and set
\[ B(x, y) = \int_0^x \int_0^y b(s, t) (a(s, t) + \int_0^s \int_0^t b(m, n) g(a(m, n)) dmdn) dsdt, \]

then

(3.17) \[ u(x, y) \leq B(x, y) + \int_0^x \int_0^y b(s, t) (u(s, t) + \int_0^s \int_0^t b(m, n) g(u(m, n)) dmdn) dsdt, \]

for \( 0 \leq x \leq \alpha, \ 0 \leq y \leq \beta \). If we put

(3.18) \[ V(x, y) = B(x, y) + \int_0^x \int_0^y b(s, t) (u(s, t) + \int_0^s \int_0^t b(m, n) g(u(m, n)) dmdn) dsdt \]

\[ V(x, 0) = V(0, y) = B(\alpha, \beta), \]

then

\[ V_{xy}(x, y) = b(x, y)(u(x, y) + \int_0^x \int_0^t b(m, n) g(u(m, n)) dmdn), \]

which in view of (3.17) implies

(3.19) \[ V_{xy}(x, y) \leq b(x, y)(V(x, y) + \int_0^x \int_0^t b(m, n) g(V(m, n)) dmdn). \]

Define

(3.20) \[ r(x, y) = V(x, y) + \int_0^x \int_0^y b(m, n) g(V(m, n)) dmdn, \]

\[ r(x, 0) = r(0, y) = B(\alpha, \beta), \]
\[(3.21) \quad r_{xy}(x,y) = v_{xy}(x,y) + b(x,y)g(v(x,y)).\]

Using the facts that \(v_{xy}(x,y) \leq b(x,y)r(x,y)\) from (3.19) and \(v(x,y) \leq r(x,y)\) from (3.20) in (3.21) we see that the inequality

\[r_{xy}(x,y) \leq b(x,y) \left[ r(x,y) + g(r(x,y)) \right].\]

Now by following the similar argument as in the proof of Theorem 1 in view of the definition of \(B(r)\) given in (3.15) we obtain the estimate for \(r(\alpha,\beta)\) such that

\[r(\alpha,\beta) \leq B^{-1} \left[ B(B(\alpha,\beta)) + \int_0^\alpha \int_0^\beta b(m,n)dm\,dn \right],\]

for \(0 \leq x \leq \alpha, 0 \leq y \leq \beta\). Using this bound on \(r(\alpha,\beta)\) in (3.19) we see that

\[v_{xy}(\alpha,\beta) \leq b(\alpha,\beta) \left\{ B^{-1} \left[ B(B(\alpha,\beta)) + \int_0^\alpha \int_0^\beta b(m,n)dm\,dn \right] \right\},\]

for \(0 \leq x \leq \alpha, 0 \leq y \leq \beta\). In this inequality, keeping \(\alpha\) fixed, set \(\beta = t\) and then integrate with respect to \(t\) from \(0\) to \(\beta\). Now keeping \(\beta\) fixed, set \(\alpha = s\) and then integrate with respect to \(s\) from \(0\) to \(\alpha\), and using the bound on \(V(\alpha,\beta)\) in (3.19) we see that the inequality

\[u(\alpha,\beta) \leq B(\alpha,\beta) + \int_0^\alpha \int_0^\beta b(s,t) \left\{ B^{-1} \left[ B(B(s,t)) + \int_0^s \int_0^t b(m,n)dm\,dn \right] \right\} ds\,dt,\]
is satisfied for \( 0 \leq x \leq \alpha \), \( 0 \leq y \leq \beta \). Since \( \alpha \) and \( \beta \) are arbitrary we have

\[
u(x,y) \leq B(x,y) + \int \int b(s,t) \{ \frac{B(B(s,t))}{B(s,t)} + \int \int b(m,n) dmdn \} \, ds \, dt.
\]

Substituting this bound on \( u(x,y) \) in (3.13) and using the definition of \( B(x,y) \) we obtain the desired bound in (3.14).

In the following two theorems we establish the two independent variable generalizations of the integral inequalities recently established by Pachpatte in [33, Theorem 3] and [36, Theorem 1].

**Theorem 7.** Let \( \phi(x,y) \), \( a(x,y) \), \( b(x,y) \) and \( c(x,y) \) be real-valued nonnegative continuous functions defined on \( I \times I \); and let \( g(x), g'(y) \) be the same functions as defined in Theorem 2; and suppose further that the inequality

\[
(3.22) \quad \phi(x,y) \leq a(x,y) + \int \int b(s,t) \left[ \phi(s,t) + \int \int c(m,n) \phi(m,n) \, dmdn \right] + \int \int (b(m,n) + c(m,n)) g(\phi(m,n)) \, dmdn \, ds \, dt,
\]

is satisfied for all \( x, y \in I \). Then for \( 0 \leq x \leq x_2 \), \( 0 \leq y \leq y_2 \)

\[
(3.23) \quad \phi(x,y) \leq a(x,y) + B(x,y) + \int \int b(s,t) \left\{ \frac{B(B(s,t))}{B(s,t)} + \int \int (b(m,n) + c(m,n)) \, dmdn \right\} \, ds \, dt,
\]
where \( E, E^{-1} \) are as defined in Theorem 2, and
\[
D(x,y) = \int_0^x \int_0^y b(s,t) \left[ a(s,t) + \int_0^s \int_0^t a(m,n) e(m,n) \, dm \, dn \right] ds \, dt \\
+ \int_0^y \int_0^x (b(m,n) + e(m,n)) g(a(m,n)) \, dm \, dn \Bigg] \, ds \, dt,
\]
and
\[
E(D(x,y)) + \int_0^x \int_0^y (b(m,n) + e(m,n)) \, dm \, dn \in \text{Dom}(E^{-1}).
\]

for all \( x, y \) lying in the subintervals \( 0 \leq x \leq x_3, 0 \leq y \leq y_3 \) of real numbers in \( I \).

**Theorem 4.** Let \( \phi(x,y), a(x,y), b(x,y), e(x,y), \) and \( k(x,y) \) be real-valued nonnegative continuous functions defined on \( I \times I \); let \( W(x), W_y(x) \) be the same functions as defined in Theorem 1; and suppose further that the inequality
\[
(3.24) \quad \phi(x,y) \leq a(x,y) + b(x,y) \left[ \int_0^x \int_0^y c(s,t) W(\phi(s,t)) \\
+ \int_0^x \int_0^y k(m,n) W(\phi(m,n)) \, dm \, dn \right] ds \, dt,
\]
is satisfied for all \( x, y \in I^2 \). Then for \( 0 \leq x \leq x_4, 0 \leq y \leq y_4 \)
\[
(3.25) \quad \phi(x,y) \leq a(x,y) + b(x,y) \left[ L(x,y) + \int_0^x \int_0^y c(s,t) W(b(s,t)) \\
+ \int_0^x \int_0^y (c(m,n) + k(m,n)) W(b(m,n)) \, dm \, dn \right] ds \, dt,
\]

\[ \{ \int_0^1 \left[ W(L(s,t)) + \int_0^s \int_0^t (c(m,n) + k(m,n)) W(b(m,n)) \, dm \, dn \right] \} ds \, dt \]
where

\[ M(x) = \int_{x_0}^{x} \frac{ds}{W(s)} \quad , \quad x \geq x_0 > 0 , \]

\( M^{-1} \) is the inverse function of \( M \), and

\[ L(x, y) = \int_{0}^{x} \int_{0}^{y} c(s, t) W(a(s, t) + b(s, t)) \int_{0}^{S} \int_{0}^{T} k(m, n) W(a(m, n) + b(m, n)) \, dm \, dn \, ds \, dt , \]

and

\[ M(L(x, y)) = \int_{0}^{x} \int_{0}^{y} \left( c(m, n) + k(m, n) \right) W(b(m, n)) \, dm \, dn \in \text{Dom}(M^{-1}) \]

for all \( x, y \) lying in the subintervals \( 0 \leq x \leq x_4, 0 \leq y \leq y_4 \) of real numbers in \( I \).

The details of the proofs of Theorem 3 and 4 follow by arguments similar to those in the proofs of Theorems 1 and 2, in view of the proofs of the corresponding theorems given in [33, Theorem 3] and [36, Theorem 1], and we leave the details to the reader.

3. FURTHER INEQUALITIES

In this section we give some further results related to Wendroff's integral inequality. These results are the two independent variable generalizations of some of the integral inequalities established by Gellwitzer [21] and Rachpate [36, 39]. An elementary method used by Snow [33] to obtain a generalization of Gronwall's inequality in two independent variables will be used to establish our results.
First result in this section deals with the two independent variable generalization of the integral inequality established by Cullwitzer [21, Theorem 1].

**Theorem 5.** Suppose $\phi(x,y)$, $a(x,y)$, $b(x,y)$, and $c(x,y)$ are real-valued nonnegative continuous functions defined on domain $D$. Let $G(x)$ be continuous strictly increasing, convex and submultiplicative function for $x \geq 0$; $G(0) = 0$, $\lim_{x \to \infty} G(x) = \infty$ for all $(x,y)$ in $D$; $\alpha(x,y)$, $\beta(x,y)$ be positive continuous functions defined on a domain $D_1$ and $\alpha(x,y) + \beta(x,y) = 1$. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in $D$ such that $(x-x_0)(y-y_0) > 0$ and $R$ be the rectangular region whose opposite corners are the points $P_0$ and $P$. Let $V(s,t;x,y)$ be the solution of the characteristic initial value problem

$$ (3.27) \quad I[V] = V_{st} - c(s,t)\beta(s,t)G(b(s,t)\beta(s,t))V(s,t) = 0, $$

$$ V(x,t) = V(s,y) = 1, $$

and let $D_1$ be a connected subdomain of $D$ which contains $P$ and on which $V > 0$. If $R \subset D_1$ and $\phi(x,y)$ satisfies

$$ (3.28) \quad \phi(x,y) \leq a(x,y) + b(x,y)G(\int_{x_0}^{x} \int_{y_0}^{y} c(s,t)G(\phi(s,t)) ds dt), $$

then $\phi(x,y)$ also satisfies

$$ (3.29) \quad \phi(x,y) \leq a(x,y) + b(x,y)G(\int_{x_0}^{x} \int_{y_0}^{y} c(s,t)a(s,t) \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)a(s,t) \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)a(s,t) ds dt))V(s,t;x,y) ds dt), $$

$$ G(a(s,t)a'(s,t))V(s,t;x,y) ds dt), $$

$$ G(a(s,t)a'(s,t))V(s,t;x,y) ds dt), $$
PROOF. Rewrite (3.28) as

\[
\phi(x,y) \leq a(x,y)a(x,y)^{-1}(x,y) + \beta(x,y)b(x,y)^{-1}(x,y)
\]

\[
\cdot \mathcal{G} \left( \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)G(\phi(s,t))dsdt \right).
\]

Since \( \mathcal{G} \) is convex, submultiplicative and monotonic we have

\[
(3.30) \quad \mathcal{G}(\phi(x,y)) \leq a(x,y)\mathcal{G}(a(x,y)^{-1}(x,y))
\]

\[
+ \beta(x,y)\mathcal{G}(b(x,y)^{-1}(x,y)) \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)\mathcal{G}(\phi(s,t))dsdt.
\]

Define

\[
(3.31) \quad u(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)\mathcal{G}(\phi(s,t))dsdt, u(x,y_0) = u(x_0,y) = c,
\]

then

\[
\mathcal{G}(x,y) = \mathcal{G}(\phi(x,y)),
\]

which in view of (3.30) implies

\[
(3.32) \quad \mathcal{L}[u] = \mathcal{G}(x,y) - \beta(x,y)\mathcal{G}(b(x,y)^{-1}(x,y))u(x,y)
\]

\[
\leq \mathcal{G}(x,y) \left[ a(x,y)\mathcal{G}(a(x,y)^{-1}(x,y)) \right].
\]
The operator $L$ is self-adjoint and hyperbolic. For any twice continuously differentiable $u$ and $V$ the operator $L$ satisfies the identity

$$VL[u] - uL[V] = -(uv_y)_x + (Vu_x)_y.$$  

Integrating (3.32) on $\mathcal{R}$, and using the Green's Theorem and the conditions on $u$ and $V$, we obtain (see Appendix for details)

$$u(x,y) = \iint_{\mathcal{R}} VL[u] \, ds \, dt$$

$$\leq \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)a(s,t)c(a(s,t)c^{-1}(s,t))V(s,t;x,y) \, ds \, dt,$$

since $V > c$, and (3.32) holds. The conclusion (3.29) of the theorem follows from (3.30) and (3.33). The continuity of $c$, $\beta$ and $G$ on $\mathcal{D}$ guarantee the existence and continuity of $V$.

We note that the generalisation in (3.29) of the exponential function in [21, p.643] is the Riemann function $V(s,t;x,y)$ relative to the point $P(x,y)$ for the self-adjoint operator $L$ and the existence and continuity of the Riemann function is well known (see, Courant and Hilbert [17]).

In Theorem 6 given below we establish the following two independent variable generalisation of the integral inequality recently established by Pachpatte [39, Theorem 2].
THEOREM 6. Suppose $\phi(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ and $k(x,y)$ be real-valued nonnegative continuous functions defined on a domain $D$. Let $G(x)$, $a(x,y)$, $b(x,y)$ be the same functions as defined in Theorem 5. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in $D$ such that $(x-x_0) - (y-y_0) > 0$ and $R$ be the rectangular region whose opposite corners are the points $P_0$ and $P$. Let $V(s,t;x,y)$ be the solution of the characteristic initial value problem

$$(3.34) \quad L[V] = V_{st} - \beta(s,t)\beta(b(s,t)\beta^{-1}(s,t)) [c(s,t) + k(s,t)]V(s,t) = 0,$$

$$V(x,t) = V(s,y) = 1,$$

and let $D^+$ be a connected subdomain of $D$ which contains $P$ and on which $V > 0$. If $R \subset D^+$ and $\phi(x,y)$ satisfies

$$(3.35) \quad \phi(x,y) \leq a(x,y) + b(x,y)G^{-1} \left[ \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)G(\phi(s,t)) \, ds \, dt \right. + \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)\beta(s,t)G(b(s,t)\beta^{-1}(s,t)) \left. \right] \left[ \int_{x_0}^{x} \int_{y_0}^{y} d(m,n)G(\phi(m,n)) \, dm \, dn \right] \, ds \, dt,$$

then $\phi(x,y)$ also satisfies

$$(3.36) \quad \phi(x,y) \leq a(x,y) + b(x,y)G^{-1} \left[ \int_{x_0}^{x} \int_{y_0}^{y} Q(s,t) \, ds \, dt \right. + \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)\beta(s,t)G(b(s,t)\beta^{-1}(s,t)) \left. \right] \left[ \int_{x_0}^{x} \int_{y_0}^{y} Q(m,n) \, dm \, dn \right] \, ds \, dt.$$
where

\[(3.37) \quad G(x,y) = 0(x,y)α(x,y)G(a(x,y)α^{-1}(x,y)) + 0(x,y)β(x,y)G(b(x,y)β^{-1}(x,y)) \]

\[\cdot \int \int_{x_0y_0}^X \int_{x_0}^Y \int_{m,n}^α(m,n)G(a(m,n)α^{-1}(m,n))dm\,dn\,ds\,dt.\]

**PROOF.** Rewrite (3.35) as

\[\phi(x,y) \leq a(x,y)α(x,y)α^{-1}(x,y) + β(x,y)β(x,y)β^{-1}(x,y) \]

\[\cdot [ \int \int_{x_0y_0}^X \int_{x_0}^Y \int_{s,t}^α(\phi(s,t))ds\,dt \]

\[+ \int \int_{x_0y_0}^X \int_{x_0}^Y \int_{s,t}^β(\phi(s,t))β^{-1}(s,t))ds\,dt \]

\[\cdot (\int \int_{x_0y_0}^X \int_{x_0}^Y \int_{m,n}^α(m,n)G(\phi(m,n))dm\,dn)ds\,dt \].

Since \( G \) is convex, submultiplicative and monotonic we have

\[(3.38) \quad G(\phi(x,y)) \leq a(x,y)G(a(x,y)α^{-1}(x,y)) + β(x,y)G(b(x,y)β^{-1}(x,y)) \]

\[\cdot [ \int \int_{x_0y_0}^X \int_{x_0}^Y \int_{s,t}^α(\phi(s,t))ds\,dt \]

\[+ \int \int_{x_0y_0}^X \int_{x_0}^Y \int_{s,t}^β(\phi(s,t))β^{-1}(s,t))ds\,dt \]

\[\cdot (\int \int_{x_0y_0}^X \int_{x_0}^Y \int_{m,n}^α(m,n)G(\phi(m,n))dm\,dn)ds\,dt \].
Define

\[
\begin{align*}
  u(x,y) &= \int_{x_0}^{x} \int_{y_0}^{y} c(s,t) \phi'(s,t) ds dt + \int_{x_0}^{x} \int_{y_0}^{y} \beta(s,t) \phi'(s,t) \beta^{-1}(s,t) \\
  &\quad \cdot \left( \int_{x_0}^{x} \int_{y_0}^{y} k(m,n) \phi(m,n) \phi(m,n) \phi(m,n) \phi(m,n) d\mu d\nu \right),
\end{align*}
\]

then

\[
\begin{align*}
  u_{xy}(x,y) &= c(x,y) [c(\phi(x,y)) + \beta(x,y) \phi'(x,y) \beta^{-1}(x,y)] \\
  &\quad \cdot \left( \int_{x_0}^{x} \int_{y_0}^{y} k(m,n) \phi(m,n) \phi(m,n) d\mu d\nu \right),
\end{align*}
\]

which in view of (3.38) implies

\[
\begin{align*}
  (3.39) \quad u_{xy}(x,y) &\leq c(x,y) + c(x,y) \beta(x,y) \beta^{-1}(x,y) \\
  &\quad \cdot \left\{ u(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} k(m,n) \beta(m,n) \beta^{-1} (m,n) u(m,n) d\mu d\nu \right\}.
\end{align*}
\]

Define

\[
\begin{align*}
  (3.40) \quad r(x,y) &= u(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} k(m,n) \beta(m,n) \beta^{-1} (m,n) u(m,n) d\mu d\nu, \\
  r(x,y) &= r(x_0,y) = 0,
\end{align*}
\]
then

(3.41) \( u_{xy}(x,y) = u_{xy}(x,y) + k(x,y)\beta(x,y)\gamma(b(x,y)\beta^{-1}(x,y))u(x,y) \).

Using the facts that

\[ u_{xy}(x,y) \leq Q(x,y) + e(x,y)\beta(x,y)\gamma(b(x,y)\beta^{-1}(x,y))R(x,y) \]

from (3.39) and \( u(x,y) \leq R(x,y) \) from (3.40) in (3.41) we have

\[ L[R] = u_{xy}(x,y) - \beta(x,y)\gamma(b(x,y)\beta^{-1}(x,y))[e(x,y) + k(x,y)]R(x,y) \leq Q(x,y). \]

Now by following the last argument as in the proof of Theorem 5 we obtain the bound on \( R(x,y) \) such that

\[ R(x,y) \leq \int \int_{x_0}^{x} \int_{y_0}^{y} Q(s,t)\gamma(s,t;x,y)dsdt. \]

Using this bound on \( R(x,y) \) in (3.39) we have

\[ u_{xy}(x,y) \leq Q(x,y) + e(x,y)\beta(x,y)\gamma(b(x,y)\beta^{-1}(x,y)) \]

\[ \int \int_{x_0}^{x} \int_{y_0}^{y} Q(s,t)\gamma(x,y)dsdt. \]

In this inequality, keeping \( x \) fixed, set \( y = t \) and then integrate with respect to \( t \) from \( y_0 \) to \( y \). Now keeping \( y \) fixed, set \( x = s \) and then integrating with respect to \( s \) from \( x_0 \) to \( x \) we have
\[ (3.42) \quad u(x,y) \leq \int_0^y \int_0^x Q(s,t) \, ds \, dt + \int_0^y \int_0^x \phi(s,t) \, g(b(s,t) \beta^{-1}(s,t)) \, ds \, dt \cdot (\int_0^s \int_0^m v(m,n) \, dn \, dn) \, ds \, dt. \]

The desired bound in (3.36) follows from (3.38) and (3.42).

To this end we establish the two independent variable generalization of the integral inequality established by Pachpatte in [36, Theorem 2].

**THEOREM 7.** Suppose \( \phi(x,y), a(x,y), b(x,y), c(x,y), \) and \( k(x,y) \) be real-valued nonnegative continuous functions defined on a domain \( D. \) Let \( N(x) \) be a positive, continuous, strictly increasing, subadditive and submultiplicative function for \( x > 0, \)

\( N(0) = 0 \) for all \( (x,y) \in D; \) and \( N^{-1} \) is the inverse function of \( N. \)

Let \( P_0(x_0,y_0) \) and \( P(x,y) \) be two points in \( D \) such that \( (x-x_0)(y-y_0) > 0 \) and \( R \) be the rectangular region whose opposite corners are the points \( P_0 \) and \( P. \) Let \( e(s,t;x,y) \) be the solution of the characteristic initial value problem

\[ (3.43) \quad \begin{bmatrix} e \\ s \end{bmatrix} = e_{st}(s,t) - N(b(s,t)) \begin{bmatrix} c(s,t) + k(s,t) \\ e(s,t) \end{bmatrix} e(s,t) = 0, \quad e(x,t) = e(s,y) = 1, \]

and let \( D^+ \) be a connected subdomain of \( D \) which contains \( P \) and on which \( e > 0. \) If \( R \subset D^+ \) and \( \phi(x,y) \) satisfies
\[(3.44) \quad \phi(x,y) \leq a(x,y) + b(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)N(b(s,t)) dsdt \]
\[+ \int_{x_0}^{x} \int_{y_0}^{y} c(s,t)N(b(s,t)) \left( \int_{x_0}^{s} \int_{y_0}^{t} k(m,n)N(m,n) dm dn \right) dsdt, \]

then \( \phi(x,y) \) also satisfies

\[(3.45) \quad \phi(x,y) \leq N^{-1} \left[ N(a(x,y)) + N(b(x,y)) \int_{x_0}^{x} \int_{y_0}^{y} c(s,t) \right] N(a(s,t)) \]
\[+ N(b(s,t)) \int_{x_0}^{s} \int_{y_0}^{t} \left[ c(m,n) + k(m,n) \right] \]
\[. N(b(m,n)) c(m,n; s, t) dm dn \] dsdt. \]

The proof of this theorem follows by the similar argument as in the proofs of Theorems 5 and 6 with suitable modifications (see, also [36, Theorem 2]). We omit the details.

4. SOME APPLICATIONS

In this section we indicate some applications of our results to obtain the bounds on the solutions of some partial differential and integral equations. We believe that the integral inequalities established in this chapter can be used in the theory of partial differential and integral equations, in essentially the same capacity as the inequalities of Gronwall [22], Gollwitzer [21] and Pachpatte [33, 36, 39] are used in the theory of ordinary differential and integral equations. To illustrate the application of our Theorem 2 we establish the bound on the solution of a partial integro-differential equation of the form
(3.46) \[ u_{xy}(x,y) = f\left[ x,y,u(x,y),\int \int k(x,y,s,t,u(s,t))dsdt \right], \]

with the given boundary conditions \( u(x,0) = g(x), \) \( u(0,y) = h(y), \)
\( u(0,0) = a, \) where all the functions are continuous on their
respective domains of their definitions and

(3.47) \[ |g(x)+h(y)| \leq a(x,y). \]

(3.48) \[ |k(x,y,s,t,u(s,t))| \leq b(s,t)g(|u(s,t)|). \]

(3.49) \[ |f\left[ x,y,u(x,y),v \right]| \leq b(x,y)\left[ |u(x,y)| + |v| \right], \]

where \( a, b, \) and \( g \) are as defined in Theorem 2. By using
the given boundary conditions any solution \( u(x,y) \) of equation
(3.46) can be represented by the equivalent Volterra integral
equation

(3.50) \[ u(x,y) = g(x)+h(y)+\int \int k(s,t,u(s,t)), \]

\[ \int \int \int b(s,t,u(s,t),u(m,n))dm\,dn\,dsdt. \]

Using (3.47) - (3.49) in (3.50) we have

\[ |u(x,y)| \leq a(x,y)+\int \int b(s,t)(|y(s,t)|+\int \int b(m,n)g(|u(m,n)|)dm\,dn)dsdt. \]
Now an application of Theorem 2 yields the desired bound on the solution $u(x,y)$ of (3.46).

We note that the special form of the integral inequality established in Theorem 1 can be used to establish the bound on the solution $u(x,y)$ of the equation (3.46), when $k \neq c$ and $|f[x,y,u(x,y)]| \leq c(x,y)W(\|u(x,y)\|)$ where $c$ and $W$ are as defined in Theorem 1, under the same boundary conditions as given in equation (3.46).

In concluding this chapter we also note that the integral inequality established in Theorem 5 can be used to obtain the bound on the solution of a partial Volterra integral equation of the form

$$(3.51) \ u(x,y) = g(x,y) + P[ x,y, \int_{x_0}^{x} \int_{y_0}^{y} k_0(x,y,s,t,u(s,t)) ds dt ],$$

under some suitable conditions on the functions involved in (3.51).
Let \( P_0 \) and \( P \) be any two points as in Theorem 5 of Chapter 5 and label the directed sides and corners of the rectangle \( R \) as shown in Fig. 2. Using \( s \) and \( t \) as the independent variables, we integrate the identity
\[
V_L[u] - uL[V] = - (uV_{u,s})_s + (V_{u,t})_t
\]
over \( R \) and use Green's Theorem to obtain
\[
(1) \quad \iint_R (V_L[u] - uL[V]) \, ds \, dt = - \int_{c_1}^{c_2} \int_{c_3}^{c_4} (V_{u,s} ds + uV_{u,t} dt)
\]
\[
= - \int_{c_1}^{c_4} V_{u,s} ds - \int_{c_2}^{c_3} uV_{u,t} dt.
\]
This holds for any functions in \( C^2 \).

For the particular function \( u \) defined in (3.5.1) we have \( u = 0 \) on \( c_3 \) and \( u = u_3 = 0 \) on \( c_4 \); so the right-hand side of the above relation reduces to
\[
(2) \quad - \int_{c_1}^{c_4} V_{u,s} ds - \int_{c_2}^{c_3} uV_{u,t} dt
\]
Now suppose that \( V \) satisfies
\[
(3) \quad L[V] = V_{st} - \alpha(s,t)\beta(s,t)\gamma(s,t)\beta^{-1}(s,t)V(s,t) = 0
\]
\[
(4) \quad V = 1 \text{ on } c_4
\]
and
\[
(5) \quad V_{t} = 0 \text{ on } c_2
\]
Then (4) and (5) imply that

\[ V = 0 \quad \text{on} \quad G_2. \]

Since \( V \geq 0 \) on \( R \) and \( u(P_1) = 0 \) by using (3.31), identity (4) becomes (3.35).